The Elementary Theory of Dedekind Cuts

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1. Introduction. Let X be a totally ordered set. A (Dedekind) cut p of X is a tuple (p^L, p^R) of subsets p^L, p^R of X such that $p^L \cup p^R = X$ and $p^L < p^R$, i.e. $a < b$ for all $a \in p^L,\ b \in p^R$. In this article we do the model theoretic groundwork of the first order theory of cuts, relative to a given o-minimal structure expanding real closed fields. This means the following. Let M be an o-minimal structure in the language $\mathscr{L}$, expanding a real closed field. Let $\mathscr{D}$ be a new unary predicate and let p be a cut of (the underlying ordered set of) M. Then we study the $\mathscr{L}(\mathscr{D})$-structure (M, p^L) relative to the $\mathscr{L}$-structure M. The problem in question of this paper is the following:

Let $A \prec M, N$ be models of T and let p, q be cuts of M, N respectively.

Give a smallest possible set Φ of $\mathscr{L}(\mathscr{D})$-sentences with parameters in A, such that $(M, p^L) \equiv_A (N, q^L)$ if and only if for each $\varphi \in \Phi$ we have $(M, p^L) \models \varphi \Leftrightarrow (N, q^L) \models \varphi$.

We mention the parameter set A here, since we have a rather satisfactory answer in the case where T is polynomially bounded with archimedean prime model (c.f. the main result Theorem (9.4) and section 10 for optimality); especially in the case of real closed fields.

Theorem (9.4) is of technical nature and is formulated for all choices of $(M, p^L), (N, q^L)$ and A. In all cases the conditions from (9.4) can be reduced further. Those cases where this reduction is not obvious are formulated in the corollaries (9.6)-(9.8). In section 10 we explain what we mean with "optimality", i.e. in which way one can not improve the conditions in (9.4).

Due to the optimality demand the sets Φ are not elementary in the sense that they actually vary in a non definable way if (M, p^L), (N, q^L) and A vary. This is desirable for quantifier elimination results. In order to get quantifier elimination we assume now that T has quantifier elimination and a universal system of axioms, which can always be achieved by adding names for 0-definable functions to the language $\mathscr{L}$. We put together all the sets Φ which appear as (M, p^L),(N, q^L) and A vary. For this, we take a smallest possible set Ψ of $\mathscr{L}(\mathscr{D})$-formulas, independent of $(M, p^L), (N, q^L)$ and A, such that each Φ is contained in $\{\psi(\bar{a}) \mid \psi \in \Psi, \bar{a} \subseteq A\}$. Finally, if we add new predicates for each formula from Ψ we get quantifier elimination results (Corollary (9.9)) for the theories $Th(M, p^L)$, actually for much smaller theories.

A pretty rough consequence of (9.4) is Corollary (9.5):

There is some $a \in A$ such that $(M, p^L) \equiv_A (M, q^L)$ iff for each $\mathscr{L}(\mathscr{D})$-formula φ in one free variable with parameter a the subset of M defined by φ in (M, p^L) has the same trace over A as the subset of N defined by φ in (N, q^L).

Theorem (9.4) describes how one has to choose the element a. The first example of section 10 shows that even in the case of real closed fields we actually need to pick one parameter in A, i.e. $(M, p^L) \equiv_A (N, q^L)$ is not implied by $(M, p^L) \equiv_b (N, q^L)$ $(b \in A)$. Hence in this example the theory of (M, p^L) does not have quantifier elimination if we add a new predicate for each unary $\mathscr{L}(\mathscr{D})$-formula $\varphi(x)$.

On the other hand (9.5) implies that the $\mathscr{L}(\mathscr{D})$-theory which expands T and says that $\mathscr{D}$ is the set p^L of a cut p has quantifier elimination, if we add new predicates for each $\mathscr{L}(\mathscr{D})$-formula in 2 variables.

Finally, example (10.2) shows that in general the theory of (M, p^L) does not have quantifier elimination if we add new predicates to the language $\mathscr{L}(\mathscr{D})$ for finitely many $\mathscr{L}(\mathscr{D})$-formulas.

In the proof of (9.4) we first do the case where $A = M$ and we are looking for reasonable model completeness results for the theories $Th(M, p^L)$ in a suitable language. This is the content of the first half of the paper until Theorem (7.4), which also covers the case, where T is an arbitrary o-minimal expansion of real closed fields and the structure (M, p^L) defines a so called T-convex valuation ring. These are subrings of M, which are the convex hull of an elementary restriction of M.

In some cases the question above has already been solved. We prove (9.4) by reducing to theses cases. To be more precise we need some notations: if Z is a nonempty subset of M we write Z^+ for the cut p with $p^R = \{a \in M \mid a > Z\}$ and Z^- for the cut q with $q^L = \{a \in M \mid a < Z\}$. We call Z^+ the **upper edge** of Z and Z^- the **lower edge** of Z. Suppose p, q are the upper edges of proper T-convex valuation rings of M, N respectively. Then $(M, p^L) \equiv_A (N, q^L)$ if and only if $p \restriction A = q \restriction A$. This has first been proved by Cherlin and Dickmann in the case of real closed fields and in full generality by van den Dries and A. Lewenberg in [vdD-Lew],(3.10).

If G is a convex subgroup of a real closed field, such that G is neither a convex valuation ring nor the maximal ideal of a convex valuation ring and $p = G^+$, then one can show that the $\mathscr{L}(\mathscr{D})$-theory (where $\mathscr{L}$ is the language of ordered rings) of (M, p^L) is not model complete (c.f [T], §16). So for model completeness we actually have to extend the language $\mathscr{L}(\mathscr{D})$. We do the following. Let M be again an o-minimal expansion of a real closed field and let p be a cut of M. Let $G(p) := \{a \in M \mid a + p = p\}$, where $a + p := (a + p^L, a + p^R)$. $G(p)$ is a convex subgroup of $(M, +, <)$, called the invariance group of p. Moreover $V(p) := \{b \in M \mid b \cdot G(p) \subseteq G(p)\}$ is a convex valuation ring of M, called the invariance ring of p. Suppose $V(p)$ is T-convex (so if M is a pure real closed field, then this is no restriction). Then Theorem (3.3) and Theorem (7.4) imply that we get model completeness of $Th(M, p^L)$ if we expand the language $\mathscr{L}(\mathscr{D})$ by a unary predicate for the group $G(p)$, a unary predicate for the ring $V(p)$ and unary predicates for the sets $\{a \in M \mid a \pm G(p)^+ = p\}$ and $\{a \in M \mid a \cdot V(p) = G(p) \text{ or } a \cdot \mathfrak{m}(p) = G(p)\}$, where $\mathfrak{m}(p)$ denotes the maximal ideal of $V(p)$.

Let $\mathscr{L}^*$ denote the new language and let T be polynomially bounded (i.e. each unary definable map is less than a polynomial for large x, for example real closed fields are polynomially bounded) with archimedean prime model. In this case every convex valuation ring of a model of T is T-convex and our question is reduced to give conditions under which the $\mathscr{L}^*$-structures above have the same universal $\mathscr{L}^*$-theory over A. This is carried out in sections 8 and 9, where we first do the case of convex subgroups and from this the general case.

2. Heirs. Let X be a totally ordered set. The Dedekind completion $X \cup \{\text{the cuts of X}\}$ is always regarded as a totally ordered set. A cut p of X is called **definable** if p is of the form $a_X^+ := \{a\}^+$, $a_X^- := \{a\}^-$ $(a \in X)$, $+\infty_X := X^+$ or $-\infty_X := X^-$. An **interval** of X is a convex subset I of X such that I^+ and I^- is definable. A **neighborhood** I of the cut p is an interval of X with $I^- \leq p \leq I^+$; we write $p \in I$ if I is a neighborhood of p. Observe that there exist an open neighborhood for each cut.

Let p be a cut of X and let $Y \supseteq X$ be a totally ordered set. If q is a cut of Y we say q **extends** p or q **lies over** p and write $p \subseteq q$ or $p = q \restriction X$ if $q^L \cap X = p^L$. A **realization** α of p in Y is an element $\alpha \in Y$ such that $p^L < \alpha < p^R$. In particular $\alpha \notin X$.

Let T be an o-minimal expansion of the theory of dense linear ordered sets without endpoints in the language $\mathscr{L}$. If $M \prec N$ are models of T and $Z \subseteq N$, then $M\langle Z\rangle$ denotes the definable closure of $M \cup Z$ in N. Note that if T expands the theory $DOAG$ of divisible, ordered abelian groups and $M \cup Z$ contains an element different from 0, then $M\langle Z\rangle$ is an elementary substructure of N (c.f. [vdD-Lew], (2.4) and (2.5)).

(2.1) Definition. Let M, N be models of T and let $A \prec M, N$ be a common elementary substructure. Let $p_1, ..., p_n$ be mutually distinct cuts of M and let $q_1, ..., q_n$ be mutually distinct cuts of N. Let $\mathcal{D}_1, ..., \mathcal{D}_n$ be new unary predicates. We say the tuple $(q_1, ..., q_n)$ is an **heir** of $(p_1, ..., p_n)$ over A if the following condition holds: if $\varphi(x_1, ..., x_k)$ is an $\mathscr{L}$-formula with parameters from A, $\psi(x_1, ..., x_k)$ is a quantifier free $\mathscr{L}(\mathcal{D}_1, ..., \mathcal{D}_n)$-formula with parameters from A and if

$$(N, q_1^L, ..., q_n^L) \models \exists \bar{x}\ \varphi(\bar{x}) \wedge \psi(\bar{x})$$

then

$$(M, p_1^L, ..., p_n^L) \models \exists \bar{x}\ \varphi(\bar{x}) \wedge \psi(\bar{x})$$

We write $(q_1, ..., q_n) \equiv\rangle_{\exists,A} (p_1, ..., p_n)$ if $(q_1, ..., q_n)$ is an heir of $(p_1, ..., p_n)$ over A. Of course we have $p_i \restriction A = q_i \restriction A$ $(1 \leq i \leq n)$ in this case.

(2.2) Remark. If $A = M$ in (2.1), then we say 'heir' instead of 'heir over M'. It is easy to see that a single cut q of N is an heir of $p := q \restriction M$ in the sense of (2.1) if and only if the 1-type corresponding to the cut q is an heir of the 1-type corresponding to the cut p in the sense of [Poi] 11.a. More generally the tuple $(q_1, ..., q_n)$ is an heir of $(p_1, ..., p_n)$ if and only if for each n-type p over M with projections $p_1, ..., p_n$ there is an n-type q over N with projections $q_1, ..., q_n$ which is an heir of p in the sense of [Poi]. Moreover a compactness argument as in [Poi],11.01 implies that for given $A \prec M, N$ and cuts $p_1, ..., p_n$ of M there is always an heir $(p_1, ..., p_n)$ over A on N.

(2.3) LEMMA. *Let $p_1, ..., p_n$ be mutually distinct cuts of M and let $q_1, ..., q_n$ be extensions of $p_1 \restriction A, ..., p_n \restriction A$ on N respectively with $q_i \notin \{\pm\infty_N\}$. Then $(q_1, ..., q_n)$ is an heir of $(p_1, ..., p_n)$ over A if and only if the following condition holds:*

if $\varphi(u_1, ..., u_n, v_1, ..., v_n)$ is an $\mathscr{L}$-formula with parameters from A and $\bar{\alpha}, \bar{\beta} \in N^n$ such that $\alpha_1 < q_1 < \beta_1, ..., \alpha_n < q_n < \beta_n$ and $N \models \varphi(\bar{\alpha}, \bar{\beta})$ then there are $\bar{a}, \bar{b} \in M^n$ such that $a_1 < p_1 < b_1, ..., a_n < p_n < b_n$ and $M \models \varphi(\bar{a}, \bar{b})$

PROOF. If $(q_1, ..., q_n)$ is an heir of $(p_1, ..., p_n)$ over A then certainly the condition holds. The converse follows from a straight forward simplification of the condition in (2.1). $\square$

(2.4) DEFINITION. If $M \prec N$ are o-minimal structures, $\bar{\alpha} \in N^n$, then we write $t(\bar{\alpha}/M)$ for the n-type p of $\bar{\alpha}$ over M. Hence p is the set of all $\mathscr{L}$-formulas in fixed n-variables with parameters from M. The tupel $\bar{\alpha}$ is called a realization of p. If $\alpha \in N \setminus M$, then by o-minimality we may identify the 1-type $t(\alpha/M)$ with the cut of α over M.

Let p be an n-type of $M \models T$ and let $f : M^n \longrightarrow M^k$ be M-definable. Then $\{\varphi(y_1, ..., y_k) \mid \varphi(f(x_1, ..., x_n)) \in p\}$ is obviously a k-type of M, which we denote by $f(p)$. If $k = 1$ and $f(p)$ is not realized in M, then $f(p)$ denotes the cut determined by $f(p)$, too.

So if $\bar{\alpha}$ is a realization of p from some elementary extension of M, then $f(\bar{\alpha})$ is a realization of $f(p)$.

The application of a definable map $f : M \longrightarrow M$ to a cut p does the obvious thing: by the monotonicity theorem [vdD1] 3,(1.2) there is an open interval I of M with $p \in I$ such that f is continuous and strictly monotonic or constant on I. Suppose f is strictly increasing on I. Then $f(p)$ is the cut $f(p^L \cap I)^+$ if $p^L \cap I \neq \emptyset$ and $f(p^R \cap I)^-$ otherwise.

Furthermore - by our definition of "cut" - $f(p)$ is a cut if and only if f is nonconstant in each neighborhood of p.

Observe also that $f(p) = p$ implies that f is strictly increasing on some interval containing p; otherwise the graph of f meets the diagonal $y = x$ in a point a which would be a realization of p.

(2.5) PROPOSITION. *Let $M, N \models T$ and $A \subseteq M, N$. Let q be a cut of N and let p be a cut of M with $p \restriction A = q \restriction A$, such that p is not definable. Then the following are equivalent:*
(i) q is an heir of p over A
(ii) If $\beta \in N$ is a realization of $q \restriction A$, then $(q \restriction A\langle\beta\rangle)$ is an heir of p over A.
(iii) For each A-definable map $f : M \longrightarrow M$ with $f(p) = p$ we have $f(q) = q$.

PROOF. (i)$\Rightarrow$(ii)$\Rightarrow$(iii) clearly holds without any assumption on p. Conversely suppose (iii) holds. Let $\varphi(x, y)$ be an $\mathscr{L}$-formula with parameters from A and let $\alpha, \beta \in N$ with $\alpha < q < \beta$ and $N \models \varphi(\alpha, \beta)$. By (2.3) it is enough to find elements $a, b \in M$ with $a < p < b$ and $M \models \varphi(a, b)$.

Case 1. β is not a realization of $p \restriction A$. Then there is some $c \in A$ with $p < c < \beta$. Let $\psi(x)$ be the formula $\exists y\ c < y \wedge \varphi(x, y)$. It is enough to find some $a \in M$ with $a < p$ and

$M \models \psi(a)$. Suppose there is no such element. Then p^R is defined by $\exists x\ x < z \wedge \psi(x)$, which contradicts our assumption that p is not definable.
Case 2. α is not a realization of p. Analog to case 1.
Case 3. α and β realizes $p \restriction A$. Suppose for all $a, b \in M$ with $a < p < b$ we have $M \models \neg\varphi(a, b)$. We may assume that there is some $c \in A$ with $p < c$ (otherwise we proceed with $-p$ and $-q$). Because $\beta < c$ we can suppose $M \models \forall u, v\ \varphi(u, v) \rightarrow u < v < c$. We define $f : A \longrightarrow A$ by

$$f(a) := \begin{cases} \sup\{b \in A \mid A \models \varphi(a, b)\} & \text{if } A \models \exists y\ \varphi(a, y) \\ a & \text{otherwise} \end{cases}$$

f is well defined and A-definable. We have $f(p) = p$: if $a \in M$ with $a < p$, then $f(a) < p$ by assumption and because of $p \neq d^-$ ($d \in A$). Let $I \subseteq A$ be an A-definable open interval with $p \restriction A \in I$, such that f is continuous and monotonic on I. From $a \leq f(a) < p$ ($a < p$, $a \in I_M$) we know that f must be strictly increasing on I. By continuity it follows $f(p) = p$. Since α and β realizes $p \restriction A$ we have $\alpha, \beta \in I_N$, so $f(\alpha) \geq \beta$ implies $f(q) \neq q$ in contradiction to (iii). $\square$

(2.6) COROLLARY. *Let $M \prec N$ be models of T and let p be a nondefinable cut of M. Then*
(i) The least and the largest extension of p on N are heirs of p.
(ii) If $\dim N/M = 1$ then the least and the largest extension of p on N are the only heirs of p on N.

PROOF. (i). If $f : M \longrightarrow M$ is M-definable with $f(p) = p$ then f is strictly increasing in some open M-definable neighborhood containing p. Thus the least and the largest extension of p on N are fixed by f as well. By (2.5) this proves (i).
(ii). Let q be an extension of p, different from the least and the largest extension of p. Thus there are realizations α, β of p with $\alpha < q < \beta$. As $\dim N/M = 1$ we have $\beta \in M\langle\alpha\rangle$. Let $f : M \longrightarrow M$ be M-definable with $f(\alpha) = \beta$. Since α and β realizes p we have $f(p) = p$ and f is strictly increasing in some open M-definable neighborhood containing p. This shows $f(q) \neq q$, hence q is not an heir of p. $\square$

Another Consequence of (2.5) is that we always have $p \equiv\rangle_{\exists,A}\, q$ or $q \equiv\rangle_{\exists,A}\, p$ if $p \restriction A = q \restriction A$. We don't need this and omit the proof.

3. The invariance group. From now on we work with an o-minimal expansion T of divisible, ordered, abelian groups in the language $\mathscr{L}$.

(3.1) COROLLARY. *If p is a nondefinable cut of M, then the following conditions are equivalent:*
(i) p has a definable heir.
(ii) All one dimensional extensions of p are heirs of p.
(iii) If $f : M \longrightarrow M$ is M-definable and $f(p) = p$, then $f(x) = x$ in some open interval containing p.
(iv) If α is a realization of p, then α is the unique realization of p in $M\langle\alpha\rangle$.

(v) If α is a realization of p, then M is dense in $M\langle\alpha\rangle$, i.e. for all $\beta<\gamma$ from $M\langle\alpha\rangle$ we have $(\beta,\gamma)\cap M\neq\emptyset$.
(vi) If $a\in M$ with $a+p=p$ then $a=0$.

PROOF. (iii)$\Rightarrow$(ii) follows immediately from (2.5) and (ii)$\Rightarrow$(i) is obvious.

(i)$\Rightarrow$(iii) Let $N\succ M$, $\beta\in N$ and suppose β^+ is an heir of p. Let f be strictly monotonic on the M-interval I and $p\in I$. Since p is not definable we have $\beta\in I$ and $\beta\models p$. From $f(p)=p$ we get $\beta^+=f(\beta^+)=f(\beta)^+$, hence $f(\beta)=\beta$ and $f(x)=x\in p$.

As (iv) is just a reformulation of (iii) we already know that (i)-(iv) are equivalent. Certainly (v) implies (iv).

(i)$\Rightarrow$(v) Let $\gamma<\beta$ be elements of $M\langle\alpha\rangle$ and suppose $\gamma\notin M$. There is $f:M\longrightarrow M$ definable in M with $f(\alpha)=\gamma$. The cut $t(\gamma/M)$ is not definable and fulfills (i), hence (iv) again. (iv) says, that γ is the unique realization of $t(\gamma/M)$ in $M\langle\gamma\rangle=M\langle\alpha\rangle$, so we can find an element $m\in M$ with $\gamma<m<\beta$.

Clearly (iii) implies (vi).

(vi)$\Rightarrow$(iii). Suppose $f(p)=p$ and $f(x)=x\notin p$ say $f(x)>x\in p$. Let I be an interval with $p\in I$ such that f is strictly increasing on I. There is a closed interval $I'\subseteq I$ with $p\in I'$ such that $f(x)-x$ is monotonic or constant. In any case there is some $a\in M$ such that $f(x)-x>a>0$ for all $x\in I'$. Certainly we have $a+p=p$ in this case. $\square$

(3.2) DEFINITION. A cut p is called **dense** if one of the equivalent conditions of (3.1) holds for p.

Let $\mathcal{D}$ be a new unary predicate and let T^{dense} be the $\mathscr{L}(\mathcal{D})$-theory T together with the axiom

$\mathcal{D}$ is the set of left options of a dense cut

By (3.1)(iii) this can indeed be expressed by a set of $\mathscr{L}(\mathcal{D})$ sentences. By (3.1)(vi) this can be expressed by a single $\forall\exists$-sentence.

The following theorem has been proved for a special case in [MMS],section 2.3. Note also that the theory of dense paires of o-minimal structure is quite accessible. For example the theory of paires (N,M), where $M\prec N$ is proper and dense in N is a complete theory, cf.[vdD2].

(3.3) THEOREM. *Let $A\prec M,N$ be models of T and let p,q be dense cuts of M,N respectively. Then $(M,p^L)\equiv_A(N,q^L)$ if and only if $p\restriction A=q\restriction A$. Hence if T has quantifier elimination and a universal system of axioms, then T^{dense} has quantifier elimination.*

PROOF. In order to prove the Theorem , we may assume that T has quantifier elimination and a universal system of axioms. By the Blum criterion for quantifier elimination and our assumptions on T it is enough to prove the following. Let (M,D) and (N,E) be models of T^{dense} and let (A,U) be a common $\mathscr{L}(\mathscr{D})$-substructure such that $A\prec M,N$ and (N,E) is $|A|^+$-saturated. Let $\alpha\in M$. Then there is some $\beta\in N$ such that the $\mathscr{L}(\mathscr{D})$-structures $(A\langle\alpha\rangle,D\cap A\langle\alpha\rangle)$ and $(A\langle\beta\rangle,E\cap A\langle\beta\rangle)$ are isomorphic over A. Since (N,E) is $|A|^+$-saturated it is enough to prove for A-definable maps $f_1,...,f_k$: there is some $\beta\in N$ such that $p:=t(\alpha/A)=t(\beta/A)$ and $f_i(\alpha)\in D\Leftrightarrow f_i(\beta)\in E$. If $f_i(\alpha)\le a\in U$, then $f(x)\le a\in p$ and we can cross out f_i from the list. Similar if $f_i(\alpha)\ge a\in A\setminus U$.

Furthermore we can suppose that α realizes the cut U^+ of A, that is $p = U^+$ and $f_i(p) = p$ for all i. In a small neighbourhood of p the maps f_i are comparable and this is witnessed in the type p. Thus it remains to show:

If $\alpha \models p = U^+$ and $f, g : A \longrightarrow A$ are A-definable maps such that $f(p) = g(p) = p$, $f(\alpha) \in D$ and $g(\alpha) \notin D$, then there is some $\beta \in N$ such that $p = t(\beta/A)$, $f(\beta) \in E$ and $g(\beta) \notin E$.

Because E^+ is a dense cut, $g \circ f^{-1}(x) > x \in p$ and $g \circ f^{-1}(p) = p$ we get some $\gamma \in N$ such that $\gamma \models p$, $\gamma \in E$ and $g \circ f^{-1}(\gamma) \notin E$. We choose $\beta = f^{-1}(\gamma) \models p$. Then $f(\beta) = \gamma \in E$ and $g(\beta) = g \circ f^{-1}(\gamma) \notin E$. Since β realizes p the Theorem is proved. $\square$

(3.4) DEFINITION. Let M be an ordered abelian group. If p is a cut of M we define the **invariance group** of p as

$$G(p) := \{a \in M \mid a + p = p\}$$

Furthermore we define $\hat{p}$ to be the cut $G(p)^+$.

Clearly $G(p)$ is a convex subgroup of $(M, +, <)$. If M is an o-minimal expansion of an ordered abelian group then by (3.1)(iv) we know that $G(p)$ is definable in M (i.e. equal to M or to 0) if and only if p is dense or definable.

If K is an ordered field and p is a cut of K with $p > 0$, then p is a cut of the ordered abelian groups $(K, +, <)$ and $(K^{>0}, \cdot, <)$. Hence we can associate two invariance groups with p. The group $G(p)$ denotes the invariance group of p with respect to $(K, +, <)$.

(3.5) DEFINITION. Let p be a cut of an ordered field K. We define the **multiplicative invariance group** of p as

$$G^*(p) = \{a \in K \mid a \cdot p = p\}$$

Hence $G^*(p)$ is the invariance group of $|p|$ with respect to $(K^{>0}, \cdot, <)$.

The following proposition shows how both invariance groups are linked if p is not an edge of a convex subgroup of $(K, +, <)$.

(3.6) PROPOSITION. *Let K be an ordered field and let p be a cut of K with $|p| > \hat{p}$. There is some $c \in K$ such that*

$$G^*(p) = c \cdot G(p) + 1 \ (= \{c \cdot a + 1 \mid a \in G(p)\})$$

PROOF. We may assume that $p > \hat{p}$. Let $H := G^*(p)$. Since $p > \hat{p}$ we have $2 \notin H$.

Claim 1. $H - 1$ is a convex subgroup of $(K, +, <)$.

Obviously $H - 1$ is convex. Hence we only have to show that $2 \cdot (H - 1) \subseteq H - 1$ and $H - 1 = -(H - 1)$. Let $\varepsilon \in H - 1, > 0$. Then $0 < 2\varepsilon < (1+\varepsilon)^2 - 1 \in H - 1$, hence $2\varepsilon \in H - 1$. Since $2 \notin H$ we have $\frac{\varepsilon^2}{1-\varepsilon} < \varepsilon$, thus $1 < \frac{1}{1-\varepsilon} = 1 + \varepsilon + \frac{\varepsilon^2}{1-\varepsilon} < 1 + 2\varepsilon < (1+\varepsilon)^2 \in H$. We get $\frac{1}{1-\varepsilon} \in H$, therefore $-\varepsilon \in H - 1$.

If $\varepsilon > 0$ with $-\varepsilon \in H - 1$, then $1 \leq 1 + \varepsilon \leq \frac{1}{1-\varepsilon} \in H$, that is $\varepsilon \in H - 1$.

Claim 2. $H - 1 = \{a \in K \mid |a| \cdot p < \hat{p}\} = \{a \in K \mid |a| \cdot p \leq \hat{p}\}$.

The second equality holds since $p > \hat{p}$. To see the first equality we assume that $a > 0$. If $a{\cdot}p < \hat{p}$, then obviously $(1+a){\cdot}p = p$. Conversely if $a > 1$, $a{\cdot}p = p$ and $(a-1)p > \hat{p}$, then there is some $0 < a_1 < p$ with $(a-1)a_1 \notin G(p)$. If $0 < a_1 \leq a_2 < p$ with $a_2 + (a-1)a_1 > p$, then $p = ap > aa_2 = a_2 + (a-1)a_2 \geq a_2 + (a-1)a_1 > p$, a contradiction. So $a > 1$ and $a{\cdot}p = p$ implies $(a-1){\cdot}p \leq \hat{p}$, thus $(a-1){\cdot}p < \hat{p}$. If $0 < a < 1$ and $a{\cdot}p = p$ then by claim 1 we have $(2-a){\cdot}p = p$ and $2-a > 1$. By what we have just proved it follows $(1-a){\cdot}p < \hat{p}$.

Now we prove the Proposition. Let $q := H^+ - 1$. By claim 1 it is enough to find some $c \in K$ with $q = c{\cdot}\hat{p}$. Let L be an ordered field containing K and realizations α, γ of $\hat{p}$ and p respectively. By claim 2 we know that $\beta := \frac{\alpha}{\gamma}$ realizes q. Let G' be the convex hull of $G(p)$ in $K(\alpha,\gamma)$ and let α' be a realization of G'^+ from an ordered field extension of $K(\alpha,\gamma)$. Certainly $\frac{\alpha'}{\beta}$ is a realization of $\frac{1}{\beta}{\cdot}G'^+$, hence of U^+, where $U := K \cap \frac{1}{\beta}{\cdot}G'$. Since U is a convex subgroup of $(K,+,<)$ and $p > \hat{p}$, the element $\gamma = \frac{\alpha}{\beta}$ is not a realization of U^+, say $\frac{\alpha}{\beta} \leq a \leq \frac{\alpha'}{\beta}$ for some $a \in K$. As α and α' realize $\hat{p}$ it follows that $a{\cdot}\beta$ realizes $\hat{p}$. Since β realizes q this means $q = \frac{1}{a}{\cdot}\hat{p}$. □

(3.7) LEMMA. *Let M be a divisible ordered abelian group, let G, H be convex subgroups of M and let $a, b \in M$. Then $a + G^+ \neq b + H^-$.*

PROOF. We may assume that $a = 0$. Suppose $b + H^- = G^+$. We write p for this cut. As M is divisible the pair $(2{\cdot}p^L, 2{\cdot}p^R)$ is a again a cut, which we denote by $2{\cdot}p$. Since $p = G^+$ we have $2{\cdot}p = G^+ = b + G^+$. Since $p = b + H^-$ we have $2{\cdot}p = 2b + H^-$. Hence $2b + H^- = b + H^-$, which implies $b \in H$. But this contradicts $b + H^- = G^+$. □

(3.8) DEFINITION. Let M be a divisible ordered abelian group and let p be a cut of M. By (3.7) we may define the **signature** of p as

$$\operatorname{sign} p := \begin{cases} 1 & \text{if there is a convex subgroup } G \text{ of } M \text{ and some } a \in M \text{ with } p = a + G^+ \\ -1 & \text{if there is a convex subgroup } G \text{ of } M \text{ and some } a \in M \text{ with } p = a - G^+ \\ 0 & \text{otherwise} \end{cases}$$

Furthermore if K is real closed, we define the **multiplicative signature** $\operatorname{sign}^* p$ **of** p as the signature of $|p|$ with respect to $(K^{>0}, \cdot, <)$ (e.g. $\operatorname{sign}^* p = 1$ means that $p = a{\cdot}G^*(p)^+$ for some $a \in K^*$.)

Note that if $p - a$ is an edge of a convex subgroup, then this group is $G(p)$.

Let again T denote an o-minimal expansion of the theory of divisible, ordered abelian groups.

(3.9) PROPOSITION. *Let p be a cut of $M \models T$. Then we have*
(i) $\operatorname{sign} p \geq 0$ *if and only if for all $N \succ M$ and the largest extension q of p on N the cut $\hat{q}$ is the largest extension of $\hat{p}$ on N.*
(ii) $\operatorname{sign} p \leq 0$ *if and only if for all $N \succ M$ and the least extension q of p on N the cut $\hat{q}$ is the largest extension of $\hat{p}$ on N.*
(iii) If q is the least or the largest extension on some $N \succ M$, then $\hat{q}$ is the least or the largest extension of $\hat{p}$ on N.

(iv) If $\operatorname{sign} p = 0$ *and* α *is a realization of* p *in some* $N \succ M$*, then the map* $2\alpha - x$ *sends the least extension of* p *on* N *to the largest extension of* p *on* N.
(v) If $\operatorname{sign} p = 0$ *and* p *is omitted in* $N \succ M$ *then the unique extension of* p *on* N *has again signature 0.*

PROOF. Obviously we may assume that T is the theory of ordered abelian groups.
(i). Let q be the largest extension of p on $N \succ M$. If there is some $\beta \in N$ such that $\beta + q > q$ and $\beta \models \hat{p}$, then there is some $a \in M$ with $p < a < \beta + q$ and $a - \beta \models a - \hat{p}$. Thus $p \leq a - \hat{p} \leq q \restriction M = p$ and $p = a - \hat{p}$. Conversely if $p = a - \hat{p}$, then the largest extension of p on $M\langle\alpha\rangle$, $\alpha \models p$, is $a - q$, where q is the least extension of $\hat{p}$ on $M\langle\alpha\rangle$.
(ii) is (i) applied to $-p$.
(iii). Say q is the largest extension of p. If $\operatorname{sign} p \geq 0$, then we know (iii) from (i). If $p = a - \hat{p}$, then $a - q$ is the least extension of $\hat{p}$ on N, hence this extension is $\hat{q}$.
(iv). Let q denote the largest extension of p on N. Because of $\operatorname{sign} p \geq 0$ and (i), the cut $\hat{q}$ is the largest extension of $\hat{p}$ on N. Hence $\alpha + \hat{q} = q$. As $\operatorname{sign} p \leq 0$ the same argument shows that $\alpha - \hat{q}$ is the least extension of p on N.
(v). Suppose $q = \alpha + \hat{q}$, where q is the unique extension of p on N. Take $a \in M$ with $\alpha < a < p$. Then $a < q$, hence $q = a + \hat{q}$, which implies $p = a + \hat{p}$. □

(3.10) DEFINITION. Let p and q be cuts of a model M of T. We define

$$p \sim q \;:\Leftrightarrow\; \text{there is some } M\text{-definable map } f : M \longrightarrow M \text{ with } f(p) = q$$

It follows easily from the monotonicity theorem that $p \sim q$ implies $q \sim p$ (see also [Ma], Lemma (3.1)). So $\sim$ is an equivalence relation among the cuts of M.

(3.11) LEMMA. *Let* p *be a cut and let* q *be a cut extending* p *on* $N \succ M$*. Then*
(i) If $f : M \longrightarrow M$ *is* M*-definable and* q *is an heir of* p*, then* $f(q)$ *is an heir of* $f(p)$.
(ii) If $p = a \pm \hat{p}$ *for some* $a \in M$ *then* q *is an heir of* p *if and only if* $q = a \pm \hat{q}$ *and* $\hat{q}$ *is an heir of* $\hat{p}$.
(iii) If $p \not\sim \hat{p}$*, then* q *is an heir of* p *if and only if* $\hat{q}$ *extends* $\hat{p}$.

PROOF. (i) follows easily from (2.3). (ii). Say $p = a + \hat{p}$. If $\hat{q}$ is an heir of $\hat{p}$ and $q - a = \hat{q}$, then clearly $q = a + \hat{q}$ is an heir of $p = a + \hat{p}$. Conversely, if q is an heir of p, then $q - a$ is an heir of $p - a = \hat{p}$. Since $\hat{p}$ is the upper edge of a convex subgroup it follows that $q - a$ is the upper edge of a convex subgroup as well. Hence $q - a = \hat{q}$.

In order to see (iii) we have to show that q is an heir of p if $\hat{p} \not\sim p$ and $\hat{p} \subseteq \hat{q}$. As $\operatorname{sign} p = 0$ the cut p is not definable and we may use (2.5). So let $f : M \longrightarrow M$ be M-definable with $f(p) = p$. We have to show that $f(q) = q$. From $f(p) = p$ we know that f is strictly increasing in some open interval with endpoints in M, containing p. Therefore we may assume that $f : M \longrightarrow M$ is bijective. By replacing f with f^{-1} if necessary we may furthermore assume that $f(x) > x$ in some open M-definable interval I containing p. Let α be a realization of q. Since $f(\alpha)$ and α realize p it follows from the definition of $\hat{p}$ that $t(f(\alpha) - \alpha/M) \leq \hat{p}$. Since $p \not\sim \hat{p}$ the cut $\hat{p}$ is not realized in $M\langle\alpha\rangle$, hence there is some $a \in M$ with $0 < f(\alpha) - \alpha < a < \hat{p}$. By assumption $a < \hat{q} = G(q)^{+}$, thus $a + q = q$. From $\alpha < f(\alpha) < a + \alpha$ we get $f(q) = q$. □

Hence in the cases $\operatorname{sign} p \neq 0$ or $p \not\sim \hat{p}$ the question if q is an heir of p can be reduced to the question if $\hat{q}$ is an heir of $\hat{p}$ or if $\hat{q}$ extends $\hat{p}$. A reformulation of the alternative "$\operatorname{sign} p \neq 0$ or $p \not\sim \hat{p}$" is (3.12)(i).

(3.12) REMARK. *Let p be a cut of M, let $f : M \longrightarrow M$ be M-definable with $f(p) = \hat{p}$ and let $N \succ M$.*
(i) If p is realized in N, then the following are equivalent:
(a) $\operatorname{sign} p \neq 0$.
(b) If q is an heir of p on N, then $f(q) = \hat{q}$.
(ii) If p is omitted in N, then $f(q) = \hat{q}$ for the unique extension q of p on N.

PROOF. (i) (a)$\Rightarrow$(b). Say $p = a + \hat{p}$. By (3.11)(ii) we have $q = a + \hat{q}$ and the cut $\hat{q}$ is an heir of $\hat{p}$. Since the map $f_1(x) := f(a + x)$ fixes the cut $\hat{p}$ we know $f_1(\hat{q}) = \hat{q}$, i.e $f(q) = f_1(\hat{q}) = \hat{q}$.
(i) (b)$\Rightarrow$(a). Suppose $\operatorname{sign} p = 0$. By (3.9)(i),(ii) the least and the largest extension q_1 and q_2 of p on N have the same invariance group. On the other hand we have $f(q_1) \neq f(q_2)$, since f is strictly monotonic in an M-definable neighborhood of p. By (b) the cut $f(q_i)$ is the upper edge of the invariance group of q_i, a contradiction.
(ii) holds since q is an heir of p, so $\hat{q}$ is the unique extension of $\hat{p}$ on N. □

(3.13) REMARK AND DEFINITION. If p is a cut of M with $\operatorname{sign} p = 0$ and p is omitted in $N \succ M$ then the unique extension q of p on N has again signature 0. To see this suppose there is some $\gamma \in N$ with $q = \gamma \pm \hat{q}$, say $q = \gamma + \hat{q}$. Since p is omitted in N there is some $c \in M$ with $\gamma < c < p$. This implies $q = c + \hat{q}$, so $q - c$ is the upper boundary of a convex group. But $q - c$ lies over $p - c$, thus $p - c$ is the upper boundary of a convex subgroup, in contradiction to $\operatorname{sign} p = 0$.
We say that p has the **signature alternative** if $\operatorname{sign} p \neq 0$ or if $p \not\sim \hat{p}$ and $q \not\sim \hat{q}$ where q denotes the unique extension of p on $M\langle\alpha\rangle$ $(\alpha \models \hat{p})$.

(3.14) LEMMA. *Let $M \prec N$ be models of T, $p_1, ..., p_n$ be cuts of M and let $(q_1, ..., q_n)$ be an heir of $(p_1, ..., p_n)$ on N. Let $\varphi(\bar{x}, \bar{y})$ be an $\mathscr{L}(\mathscr{D}_1, ..., \mathscr{D}_n)$-formula, where $\bar{x}$ and $\bar{y}$ are n-tuple of variables. If there are*
$\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in N^n$ with $(N, q_1^L, ..., q_n^L) \models \varphi(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$, $\hat{q}_i < \alpha_i$ and $\beta_i < q_i < \gamma_i$ $(1 \leq i \leq n)$,
then there are
$\bar{a}, \bar{b}, \bar{c} \in M^n$ with $(M, p_1^L, ..., p_n^L) \models \varphi(\bar{a}, \bar{b})$, $\hat{p}_i < a_i$ and $b_i < p_i < c_i$ $(1 \leq i \leq n)$

PROOF. Choose $\gamma_1, ..., \gamma_n \in N$ with $\gamma_i < q_i < \gamma_i + \alpha_i$. Since $(q_1, ..., q_n)$ is an heir of $(p_1, ..., p_n)$, there are $\bar{b}, \bar{c} \in M^n$ such that
$(M, p_1^L, ..., p_n^L) \models \exists \bar{u}, \bar{v}\ \varphi(\bar{u}, \bar{b}, \bar{c}) \wedge \bigwedge v_i \in \mathscr{D}_i < u_i + v_i$ and $b_i < p_i < c_i$ $(1 \leq i \leq n)$. This proves the Lemma. □

(3.15) LEMMA. *Let $M \prec N$ be models of T, let p be a nondefinable cut of M and let q be a cut of N extending p. Assume $\hat{q}$ is an heir of $\hat{p}$ and $\alpha \in N$ is a realization of $\hat{p}$ with $\alpha < \hat{q}$. Then $G(q) \cap M\langle\alpha\rangle = G(q \restriction M\langle\alpha\rangle)$.*

PROOF. We only have to show $\supseteq$. We may assume that $p > 0$ and that $\hat{p}$ is not definable. Let $f(\alpha) \in M\langle\alpha\rangle$, $f(\alpha) > 0$ such that $f(\alpha) + q \restriction M\langle\alpha\rangle = q \restriction M\langle\alpha\rangle$. If $t(f(\alpha)/M) > \hat{p}$

then there is some $m \in M$ with $f(\alpha) \geq m \notin G(p)$ and we get $f(\alpha)+q \restriction M\langle\alpha\rangle > q \restriction M\langle\alpha\rangle$. So by the choice of $f(\alpha)$ we have $t(f(\alpha)/M) \leq \hat{p}$.

If $t(f(\alpha)/M) < \hat{p}$ there is some $m \in M$ with $f(\alpha) \leq m \in G(p) \subseteq G(q)$, hence $f(\alpha)+q = q$. So we may suppose that $f(\alpha)$ realizes $\hat{p}$. Then $f(\hat{p}) = \hat{p}$, so f is strictly increasing in some open M-definable interval I containing $\hat{p}$. Since $\hat{q}$ is an heir of $\hat{p}$ we have $f(\hat{q}) = \hat{q}$. As $\alpha < \hat{q}$ and f is strictly increasing on I we have $f(\alpha) < \hat{q}$ too, i.e. $f(\alpha) + q = q$. □

(3.16) Proposition. *Let $M \prec N$ be models of T, let p be a cut of M with* $\operatorname{sign} p = 0$ *and let q be a cut of N extending p such that $\hat{q}$ is an heir of $\hat{p}$. Suppose p has the signature alternative. Then $(q,\hat{q})$ is an heir of $(p,\hat{p})$.*

Proof. Because $p \not\sim \hat{p}$, q is an heir of p by (3.11)(iii). By (2.3) we have to show: if $\alpha_1, \alpha_2, \beta_1$ and β_2 are elements of N and $\varphi(v_1,v_2,v_3,v_4)$ is an $\mathscr{L}$-formula with parameters from M, such that $\alpha_1 < \hat{q} < \alpha_2$, $\beta_1 < q < \beta_2$ and $N \models \varphi(\alpha_1,\alpha_2,\beta_1,\beta_2)$ then there are $a_1,a_2,b_1,b_2 \in M$ such that $a_1 < \hat{p} < a_2$, $b_1 < p < b_2$ and $M \models \varphi(a_1,a_2,b_1,b_2)$.

By (3.14) we may assume that α_1 is a realization of $\hat{p}$. Since $p \not\sim \hat{p}$ the cut p has a unique extension $q_1 = q \restriction M\langle\alpha_1\rangle$ on $M\langle\alpha_1\rangle$. Since p has the signature alternative we get $q_1 \not\sim \hat{q}_1$. By (3.15) we have $G(q) \cap M\langle\alpha_1\rangle = G(q \restriction M\langle\alpha_1\rangle)$. Hence $\hat{q}$ extends $\hat{q}_1$ and $\hat{q}_1$ extends $\hat{p}$. By (3.11)(iii) q is an heir of q_1. By (3.14) applied to q_1 and q, there are M-definable maps f, g_1, g_2 such that

$$\alpha_1 < \hat{q}_1 < f(\alpha_1),\ g_1(\alpha_1) < q_1 < g_2(\alpha_1) \text{ and } M\langle\alpha_1\rangle \models \varphi(\alpha_1, f(\alpha_1), g_1(\alpha_1), g_2(\alpha_1))$$

Since p is omitted in $M\langle\alpha_1\rangle$ there are $c,d \in M$ with $g_1(\alpha_1) < c < p < d < g_2(\alpha_1)$. Let $\psi(v_1,v_2)$ be the formula $\exists v_3 v_4\ v_3 < c \wedge d < v_4 \wedge \varphi(v_1,\ldots,v_4)$. As $\hat{q}_1$ is an heir of $\hat{p}$ and $\alpha_1 < \hat{q}_1 < f(\alpha_1)$, $M\langle\alpha_1\rangle \models \psi(\alpha_1, f(\alpha_1))$ we get some $a_1 \in M$ with

$$a_1 < \hat{p} < f(a_1) \text{ and } M \models \psi(a_1, f(a_1))$$

Hence there are b_1, b_2 with $b_1 < c$, $d < b_2$ and $M \models \varphi(a_1, f(a_1), b_1, b_2)$. So with $a_2 := f(a_1)$ we found the required elements. □

4. Review of T-convex valuation rings. For the rest of this article we work with a complete, o-minimal expansion T of the theory of real closed fields in the language $\mathscr{L}$. Basic valuation theory for models of these theories has been developed in the paper [vdD-Lew]:

(4.1) Definition. Let M be a model of T. A convex valuation ring V of M is called **T-convex**, if V is closed under all $\emptyset$-definable, continuous maps $M \longrightarrow M$.

For example, if $M_0 \prec M$ and if V is the convex hull of M_0 in M, then V is T-convex.

(4.2) Proposition. *Let V be a T-convex valuation ring of a model M of T and let $M_0 \prec M$ with $M_0 \subseteq V$.*
(i) If $f : M^n \longrightarrow M$ is a continuous, M_0-definable map, then $f(V^n) \subseteq V$.
(ii) If $\alpha \in V$ and $\alpha > M_0$, then $M_0\langle\alpha\rangle \subseteq V$.

Proof. (i) is [vdD-Lew], (2.9) and (2.10). (ii) is [vdD-Lew], (2.13). □

(4.3) THEOREM. *Let $V \subseteq M \models T$ be T-convex and $M_0 \prec M$ with $M_0 \subseteq V$. Then M_0 is maximal definably closed in V if and only if the composed map $M_0 \longrightarrow V \longrightarrow \kappa_V := V/\mathfrak{m}(V)$ is onto. In this case we have:*

(i) V is the convex hull of M_0 in M and M_0 is tame in M, i.e. each M_0-bounded element in M is infinitely close to an element of M_0.

(ii) If $M_0' \subseteq V$ is maximal definably closed, then there is a unique $\mathscr{L}$-isomorphism $M_0 \longrightarrow M_0'$, which commutes with the residue maps.

PROOF. [vdD-Lew], (2.12) and (2.15). □

Thus $M_0 \longrightarrow \kappa_V$ induces an $\mathscr{L}$-structure on κ_V, which is independent of the choice of M_0. κ_V is again a model of T. We always consider κ_V to be this expansion of the field κ_V. (4.3) implies

(4.4) COROLLARY. *Let $V \subseteq M \models T$ be T-convex and $M \prec N$. If $W \subseteq N$ is a T-convex ring of N lying over V, then the canonical morphism $\kappa_V \longrightarrow \kappa_W$ is an elementary map of models of T.* □

Let $\mathscr{L}_{convex}$ be the language $\mathscr{L}$ expanded by a new unary predicate $\mathscr{O}$ for the T-convex valuation ring and let T_{convex} be the $\mathscr{L}_{convex}$-theory of pairs (M, V), where $M \models T$ and V is a proper T-convex valuation ring of M.

(4.5) THEOREM. *T_{convex} is a complete $\mathscr{L}_{convex}$-theory. T_{convex} is weakly o-minimal (i.e. each M-definable subset of a model (M, V) of T_{convex} is a finite union of convex subsets of M). If T has quantifier elimination and a universal system of axioms, then T_{convex} has quantifier elimination.*

PROOF. By [vdD-Lew], (3.10) and (3.13). □

(4.6) COROLLARY. *Let $(M, V) \models T_{convex}$ and let $A \subseteq M$.*

(i) Each subset of M^n, which is A-definable in (M, V) is a boolean combination of sets of the form $f^{-1}(V)$ or $f^{-1}(0)$ with some A-definable map $f : M^n \longrightarrow M$

(ii) If p is a nondefinable cut of M such that p^L is A-definable in (M, V) then $f(V^+) = p$ for some A-definable homeomorphism $f : M \longrightarrow M$.

(iii) If $V \neq M$, $N \succ M$ and q is a cut of N extending V^+, then q is an heir of V^+ if and only if q is the upper edge of a T-convex valuation ring of N.

PROOF. It is enough to prove the Corollary in the case $A = \emptyset$. We work in the expanded language $\mathscr{L}^{df}$ with the theory T^{df} (c.f. [vdD-Lew], (2.3) and (2.4)), where we have a function symbol for each $\emptyset$-definable function $M^n \longrightarrow M$. Since T has definable Skolem functions the theory T^{df} has quantifier elimination and a universal system of axioms.

(i). By (4.5) the theory $(T^{df})_{convex}$ has quantifier elimination. This implies claim (i).

(ii). By (i) the set p^L is a boolean combination of sets of the form $f^{-1}(0)$ or $f^{-1}(V)$ with some $\emptyset$-definable map $f : M \longrightarrow M$. Since T_{convex} is weakly o-minimal and p is not definable, there are $a, b \in M$, $a < p < b$ such that $f^{-1}(V) \cap (a, b) = p^L \cap (a, b)$ or $f^{-1}(V) \cap (a, b) = p^R \cap (a, b)$. In any case $f(V^+) = p$. Since f is continuous inside some A-definable neighborhood of p, we can change f to a $\emptyset$-definable homeomorphism.

(iii). Since $(T^{df})_{convex}$ is model complete we have that W^+ is an heir of V^+ for each T-convex valuation ring of N, lying over V. Conversely if q is an heir of p, then it is easy to see that $W := q^L \cap (-q)^R$ is a T-convex valuation ring of N. □

(4.7) THEOREM. *Let $M \prec N$ be tame, let B be the convex hull of M in N. If $X \subseteq N^n$ is definable in (N, B) with parameters from N, then $X \cap M^n$ is definable in M.*

PROOF. [vdD-3], Theorem (A). □

5. The invariance ring.

(5.1) DEFINITION. Let K be an ordered field and let G be a convex subgroup of $(K, +, <)$. We define the **invariance ring** $V(G)$ of G as

$$V(G) := \{a \in K \mid a \cdot G \subseteq G\}$$

As G is a convex subgroup of K, the set $V(G)$ is a convex valuation ring of K and G is a $V(G)$-submodule of K. The maximal ideal of $V(G)$ is denoted by $\mathfrak{m}(G)$. If p is a cut of K we define the invariance ring $V(p)$ of p as $V(p) := V(G(p))$. The maximal ideal of $V(p)$ is denoted by $\mathfrak{m}(p)$.

Note that in the case $1 \notin G$, the ring $V(G)$ is in general not the largest convex subring W of K, such that G is contained in the maximal ideal of W.

The group $V(G)^{*>0}$ of positive units is the invariance group of the cut G^+ with respect to the multiplicative, ordered group of positive elements of K.

Let $R \subseteq S$ be real closed and let V denote the intersection of all $V(p)$, where p runs through the cuts of R, which are realized in S. One can show that the convex hull W of V in S is the largest convex valuation ring of S, such that the residue field of $W \cap R$ is dense in the residue field of W. Hence this valuation ring is the same as defined by Macintyre ([Mac]) and Baur ([Ba]). We don't use this and omit the proof.

(5.2) THEOREM. *Let M be a model of T, let V be a T-convex valuation ring of M and let A be a subset of M. If p is a nondefinable cut of M, such that p^L is A-definable in (M, V), then there is an A-definable map $f : M \longrightarrow M$ and elements $a, b \in \mathrm{dcl}(A)$, $b \neq 0$ such that $f(V^+) = V(p)^+$ and such that $p = a + b \cdot V(p)^+$ or $p = a + b \cdot \mathfrak{m}(p)^+$. Moreover $V \subseteq G(p)$ or $G(p) \subseteq \mathfrak{m}(V)$.*

PROOF. Claim 1. If G is a convex subgroup of $(M, +, <)$, which is M-definable in (M, V), then $G \subseteq \mathfrak{m}(V)$ or $V \subseteq G$.

Suppose $\mathfrak{m}(V) \subsetneqq G \subsetneqq V$. Let $M_0 \prec M$ be tame such that V is the convex hull of M_0 and let $G_0 := G \cap M_0$. Since $\mathfrak{m}(V) \subsetneqq G \subsetneqq V$, the set G_0 is a convex subgroup of $(M_0, +, <)$ with $G_0 \neq 0$ and $G_0 \neq M_0$. Hence G_0 is not M-definable in contradiction to (4.7).

Claim 2. The theorem holds in the case $A = M$.

Since p^L, $G(p)$ and $V(p)$ are definable in (M, V) with parameters we have $p \sim V^+, \hat{p} \sim V^+$ and $V(p)^+ \sim V^+$ by (4.6)(ii). Let α be a realization of p and let $N := M\langle\alpha\rangle$. Let V_1 be the convex hull of V in N and let V_2 be the largest convex valuation ring of N with $V_2 \cap M = V$. Easily we have that V_2^+ is the largest extension of V^+ on N and V_2 is

T-convex. Since $\mathfrak{m}(V_2) \subsetneqq V_1 \subsetneqq V_2$, the set V_1 is not definable in (N, V_2) by claim 1. In particular $V_1^+ \not\sim V_2^+$.

Let r_1, r_2 be the least and the largest extension of p on N. Since $p \sim V^+$ and $V_1^+ \not\sim V_2^+$ we get $r_1 \not\sim r_2$. By (3.9)(iv) we have $\operatorname{sign} p \neq 0$.

The same argument applied to $\hat{p}$ and $(M^{>0}, \cdot, <)$ shows that $\operatorname{sign}^* \hat{p} \neq 0$. So we know $\operatorname{sign} p, \operatorname{sign}^* \hat{p} \neq 0$, which gives claim 2.

Let M_0 be the definable closure of A, $V_0 := V \cap M_0$ and let p_0 be the cut $(p^L \cap M_0, p^R \cap M_0)$. Since p^L is A-definable in (M, V) we know from (4.6) that there is an A-definable homeomorphism $h : M \longrightarrow M$ with $h(V^+) = p$. Clearly $h(V_0^+) = p_0$ and p_0 is M_0-definable in (M_0, V_0).

Claim 3. The theorem holds if $M_0 \subseteq V$ and $\dim M/M_0 = 1$.

As $\dim M/M_0 = 1$, the cut V^+ is the least extension of $V_0^+ = +\infty_{M_0}$ on M ((4.2)(ii)). Thus $p = h(V^+)$ is the least or the largest extension of $p_0 = h(V_0^+)$ on M. Since $V_0^+ = +\infty_{M_0}$ is a definable cut, the cut p_0 is definable as well. Since p is not definable and p is the least or the largest extension of the definable cut p_0, we have $p = a + b \cdot V^+$ or $p = a + b \cdot \mathfrak{m}(V)^+$ for some $a, b \in M_0$, $b \neq 0$. This proves claim 3.

For the rest of the proof we assume $\operatorname{sign}^* \hat{p} = 1$, the case $\operatorname{sign}^* \hat{p} = -1$ is analogue.

Claim 4. If $M_0 \not\subseteq V$, $\varepsilon \in \{\pm 1\}$ and if $a, b \in M_0$ with $p_0 = a + b \cdot (V(p_0)^+)^\varepsilon$, then $p = a + b \cdot (V(p)^+)^\varepsilon$.

This is so, since V_0 is a proper T-convex valuation ring of M_0, hence $(M_0, V_0) \prec (M, V)$. As p^L is A-definable in (M, V) the valuation ring $V(p)$ is A-definable in (M, V) as well, hence there is an A-definable homeomorphism $h_1 : M \longrightarrow M$ with $h_1(V^+) = V(p)^+$. As $p = h(V^+)$, the equation $h_1(V^+) = V(p)^+$ can be expressed by an $\mathscr{L}_{convex}$-sentence with parameters from A. Since $(M_0, V_0) \prec (M, V)$ we get $h_1(V_0^+) = V(p_0)^+$. Consequently $h(V_0^+) = a + b \cdot h_1(V_0^+)^\varepsilon$ and $(M_0, V_0) \prec (M, V)$ imply $p = a + b \cdot (V(p)^+)^\varepsilon$.

Now we prove the theorem. The moreover-part has been proved in claim 1.

Case 1. $M_0 \not\subseteq V$.

By claim 2 applied to M_0 there are $a, b \in M_0$ and $\varepsilon \in \{\pm 1\}$ with $p_0 = a + b \cdot (V(p_0)^+)^\varepsilon$. Hence by claim 4 we have $p = a + b \cdot (V(p)^+)^\varepsilon$.

Case 2. $M_0 \subseteq V$.

Let $\gamma \in M$, $\gamma > V$, $M_1 := M_0\langle\gamma\rangle$, $p_1 := (p^L \cap M_1, p^R \cap M_1)$ and $V_1 := V \cap M_1$. Then p_1^L is definable in (M_1, V_1), so by claim 3 there are $a, b \in M_0$ and $\varepsilon \in \{\pm 1\}$ with $p_1 = a + b \cdot (V(p_1)^+)^\varepsilon$. By claim 4 applied to M_1 we get $p = a + b \cdot (V(p)^+)^\varepsilon$. □

(5.3) REMARK. We'll apply the results of section 3 also for the expansion of the multiplicative group of positive elements of M induced by M. This means the following.

Let $\mathscr{L}^{>0}$ be the signature of ordered groups $\mathscr{L}_0 := \{+, -, <, 0\}$ together with an n-ary predicate R_φ for each $\mathscr{L}$-formula φ with exactly n free variables. If M is a model of T, then $M^{>0}$ is the $\mathscr{L}^{>0}$-structure with universe $\{a \in M \mid M \models a > 0\}$ and the following interpretation of the non logical symbols:

$M^{>0} \restriction \mathscr{L}_0 := ((0, \infty), \cdot, ^{-1}, <, 1))$, that is the underlying divisible, ordered group of $M^{>0}$ is the multiplicative group of positive elements of M. If φ is an $\mathscr{L}$-formula with exactly n free variables, then $R_\varphi^{M^{>0}} := \varphi[M^n] \cap (0, \infty)^n$. If we relativize to $(0, \infty)$ we can inductively assign to each $\mathscr{L}^{>0}$-formula a formula from $\mathscr{L}$. We define $T^{>0} := Th(M^{>0})$. As T is complete, $T^{>0}$ does not depend on the choice of M. $T^{>0}$ is an o-minimal expansion of divisible ordered abelian groups, because $M^{>0}$ is o-minimal. If N is a model of $T^{>0}$, we can construct in an obvious way a model M of T such that $M^{>0} = N$. The transition $M \mapsto M^{>0}$ respects elementary extensions, as well as definable closures: since T is o-minimal, we can recover M-definable subsets of M^n from $M^{>0}$-definable subsets of $(M^{>0})^n$. If M is a model of T then the n-types of M which contains $\bigwedge\!\!\bigwedge x_i > 0$ are in one-to-one correspondence with the n-types of $M^{>0}$. Observe that for these types the notion heir is not touched by switching the point of view. We leave the arguments in this way; the transition $T \mapsto T^{>0}$ in the applications below is always straight forward. For example a consequence of (5.2) is:

(5.4) COROLLARY. *If G is a convex subgroup of $(M,+,<)$, such that $V(G)^+ \sim V^+$ for some T-convex valuation ring V of M, then the signature alternative holds for the cut G^+ with respect to the structure $M^{>0}$.*

PROOF. As $\operatorname{sign}^* G^+ = 0$, there is no M-definable map f with $f(G^+) = V(G)^+$ by (5.2). If $\alpha \models V(G)^+$ and q denotes the unique extension of G^+ on $M\langle\alpha\rangle$, then q is again the upper edge of a convex subgroup H of $(M\langle\alpha\rangle,+)$ with $\operatorname{sign}^* H^+ = 0$ ((3.9)(v) applied to $M^{>0}$ and G^+). Furthermore $V(H)^+$ is the largest extension of $V(G)^+$ on $M\langle\alpha\rangle$. As $V(G)^+ \sim V^+$ we have $V(H)^+ \sim r$, where r is the least or the largest extension of V^+ on $M\langle\alpha\rangle$. Consequently $V(H)^+ \sim W^+$, for some T-convex valuation ring W of $M\langle\alpha\rangle$. Again by (5.2) we have $H^+ \not\sim V(H)^+$. □

(5.5) COROLLARY. *Let $\mathcal{O}$ and $\mathcal{G}$ be new unary predicates. Let $M \prec N$ be models of T and let G be a convex subgroup of $(M,+,<)$ with $\operatorname{sign}^* G^+ = 0$, such that $f(V(G)^+) = V^+$ for some T-convex valuation ring V of M and some M-definable map $f : M \longrightarrow M$. If H is a convex subgroup of $(N,+,<)$ with $H \cap M = G$, $V(H) \cap M = V(G)$ and $f(V(H)^+) = W^+$ for some T-convex valuation ring W of N, then the $\mathscr{L}(\mathcal{O},\mathcal{G})$-structure $(M, V(G), G)$ is existentially closed in $(N, V(H), H)$ relative $\mathscr{L}$.*

PROOF. By (5.4) and (3.16). □

(5.6) COROLLARY. *Suppose T is model complete. Let $\mathcal{O}$ and $\mathcal{G}$ be new unary predicates. Let M be a model of T and let G be a convex subgroup of $(M,+,<)$ with $\operatorname{sign}^* G^+ = 0$, such that $f(V(G)^+,\bar{a}) = V^+$ for some T-convex valuation ring V of M and some $\emptyset$-definable map $f : M^{n+1} \longrightarrow M$, $\bar{a} \in M^n$. Let $c_1, ..., c_n$ be new constants and let T' be the $\mathscr{L}(\mathcal{O},\mathcal{G},c_1,...,c_n)$-theory, which says the following things about a model $(N, W, H, \bar{d})$:*
(a) W is a T-convex valuation ring.
(b) H is a convex subgroup of $(N,+,<)$ with $\operatorname{sign} G^+ = 0$.
(c) $f(W^+, d_1, ..., d_n) = V(G)^+$.
Then T' is model complete.

PROOF. By (5.5) and Robinsons test for model completeness. □

6. A method for producing cuts with signature 0.

(6.1) LEMMA. *Let M be a divisible ordered abelian group, let G be a convex subgroup of M and let $(a_i)_{i\in\mathbb{N}}$ be a sequence in M with $a_{i+1}\le \frac{1}{2}a_i$ for all $i\in\mathbb{N}$ such that $G^+=\{a_i \mid i\in \mathbb{N}\}^-$. Let p be the cut $\{\sum_{i=1}^n a_i \mid n\in\mathbb{N}\}^+$. Then $G(p)=G$ and $\operatorname{sign} p\le 0$. Moreover $p<3a_1-G^+$.*

PROOF. If $g\in G,\ g>0$, then $g+\sum_{i=1}^n a_i\le \sum_{i=1}^{n+1}a_i\le p$, hence $g\in G(p)$ and G is contained in $G(p)$. Conversely if $g>G$ and $k\in\mathbb{N}$ such that $a_k<\frac{g}{2}$, then $p<g+\sum_{i=1}^{k-1}a_i$, since for each $n>k$ we have $\sum_{i=k}^n a_i\le \sum_{i=k}^n(\frac{1}{2})^{i-k}a_k<2a_k<g$. So we know that $G(p)=G$.

We have $p<3a_1-G^+$, since $p<2a_1$ and $G^+<a_1$.

It remains to show that $\operatorname{sign} p\le 0$. Suppose $p=a+G^+$ for some $a\in M$. There is some $k\in\mathbb{N}$ such that $\sum_{i=1}^k a_i\ge a$, hence $p=a+G^+\le\sum_{i=1}^k a_i+G^+<\sum_{i=1}^{k+1}a_i<p$, a contradiction. $\square$

Example. In the situation of (6.1) the signature of p may be -1, for example if $M=\mathbb{R}$, $G=0$ and $a_i=(\frac{1}{2})^i$. Moreover if the cut p constructed in (6.1) has signature 0 and α is a realization of p, then the least extension q of p on $M\langle\alpha\rangle$ is $\{\sum_{i=1}^n a_i\mid n\in\mathbb{N}\}^+$ in $M\langle\alpha\rangle$ and the upper edge of the convex hull H of G in $M\langle\alpha\rangle$ is $\{a_i\mid i\in\mathbb{N}\}^-$. So $\operatorname{sign} q=-1$ and $G(q)=H$.

In particular the signature of p may be -1 even if $\mathbb{N}\cdot\alpha_{i+1}<\alpha_i$ for all i.

(6.2) LEMMA. *Let $M=M_0\subseteq M_1\subseteq\ldots\subseteq N:=\bigcup_{i<\omega}M_i$ be divisible, ordered, abelian groups, let G be a convex subgroup of M and let G_i be the convex hull of G in M_i. Suppose for each $i\in\mathbb{N}$ there is a realization α_i of G_{i-1}^+ in M_i. Let q denote the cut $\{\alpha_1+\ldots+\alpha_n\mid n<\omega\}^+$ of N. Then*

(i) $q\restriction M_i=\alpha_1+\ldots+\alpha_{i-1}+G_i^+\in S_1(M_i)$ for all $i\in\mathbb{N}$, in particular q extends G^+.

(ii) $\operatorname{sign} q=0$ and $G(q)$ is the convex hull of G in N.

(iii) Let p be a cut of M realized in N by β such that $G\subseteq G(p)$.

(a) If $\operatorname{sign} p\ge 0$, then $\beta+q$ extends p.

(b) If $\operatorname{sign} p\le 0$, then $\beta-q$ extends p.

(c) If $G\subsetneqq G(p)$, then $\beta+q$ and $\beta-q$ extend p.

In particular p has an extension with signature 0 and invariance group $co_N(G)$ on N.

PROOF. From the choices of the α_i we have $\alpha_i>2\alpha_{i+1}$ ($i\in\mathbb{N}$).

(i). Let q_i denote the cut $\alpha_1+\ldots+\alpha_{i-1}+G_i^+$ of M_i. Since $\alpha_i\models G_i^+$ we have $q_i\le q\restriction M_i$. On the other hand for each $k>i$ we have $\sum_{j=i}^k a_j\le\sum_{j=i}^k \frac{1}{2}^{j-i}a_i\le 2\alpha_i\models G_i^+$. Hence

$(q-\alpha_1-...-\alpha_{i-1}) \restriction M_i \le G_i^+$ and $q \restriction M_i \le \alpha_1+...+\alpha_{i-1}+G_i^+ = q_i$. This shows $q_i = q \restriction M_i$ and so (i) is proved.
(ii). Let H be the convex hull of G in N. Clearly $H^+ = \{\alpha_i \mid i \in \mathbb{N}\}^- \in S_1(N)$. So by (6.1) we have $G(q) = H$ and $\operatorname{sign} q \le 0$. Suppose $q = \alpha - H^+$ for some $\alpha \in N$. Let $i \in \mathbb{N}$ such that $\alpha \in M_i$. Then $q \restriction M_i = \alpha - (H \cap M_i)^+$. So by (i) we get $\operatorname{sign} q_i = -1$ in contradiction to the definition of q_i.
(iii). We have $q < 2\alpha_1$ and $2\alpha_1 \models G^+$. If $\operatorname{sign} p \ge 0$, then $\beta + 2\alpha_1$ realizes p, hence $\beta + q$ extends p. Similar if $\operatorname{sign} p \le 0$. This shows (a) and (b). Finally, if $G \subsetneqq G(p)$ then $\beta \pm 2\alpha_1 \models p$, which implies (c). □

(6.3) COROLLARY. *Let T be an o-minimal expansion of DOAG, let $\mathscr{C}$ be a set of cuts of M and let G^p be a convex subgroup of $G(p)$ for each $p \in \mathscr{C}$. Then there is an elementary extension $N \succ M$ with $\dim N/M = \aleph_0 + |\mathscr{C}|$, such that each $p \in \mathscr{C}$ has an extension q in $S_1(N)$ with $\operatorname{sign} q = 0$ and such that $G(q)$ is the convex hull of G^p in N.*

PROOF. Let $M' \succ M$ be such that each $p \in \mathscr{C}$ is realized in M' with $\dim M'/M \le |\mathscr{C}|$ and let H^p be the convex hull of G^p in M' ($p \in \mathscr{C}$). Furthermore let α_1^p be a realization of $(H^p)^+$ ($p \in \mathscr{C}$) and let $M_1 := M'\langle \alpha_1^p \mid p \in \mathscr{C}\rangle$. Let G_1^p be the convex hull of G^p in M_1 ($p \in \mathscr{C}$), let α_2^p be a realization of $(G_1^p)^+$ ($p \in \mathscr{C}$) and let $M_2 := M_1\langle \alpha_2^p \mid p \in \mathscr{C}\rangle$. If we continue in this way we get an elementary extension $N := M'\langle \alpha_i^p \mid i \in \mathbb{N},\ p \in \mathscr{C}\rangle$ of dimension $\le \aleph_0 + |\mathscr{C}|$ over M, where we can apply (6.2) for each $p \in \mathscr{C}$: Let q' be the cut $\{\alpha_1^p + ... + \alpha_n^p \mid n \in \mathbb{N}\}^+$ of N. By (6.2) we have $\operatorname{sign} q' = 0$ and $G(q')$ is the convex hull of G^p in N. Since p is realized in N we get the corollary from (6.2)(iii). □

(6.4) LEMMA. *Let T be an o-minimal expansion of DOAG. Let I be a totally ordered index set, let $M_i \prec M_j$ be models of T ($i \le j \in I$) and let p_i be a cut of M_i such that $p_i \subseteq p_j$ whenever $i \le j \in I$. Let $M := \bigcup_i M_i \models T$. There is a unique cut of M extending all p_i. We write $\bigcup p_i$ for this cut. We have:*

(i)
$$W_0(\bigcup p_i) = \bigcup_{k\in I}\bigcap_{j\ge k} W_0(p_j) = \bigcap_{k\in I}\bigcup_{j\ge k} W_0(p_j)$$

(ii) If $\hat{p}_j$ does not extend $\hat{p}_i$ for all $i < j$, then
$$W_0(\bigcup p_i) = \bigcap W_0(p_i)$$

(iii) If $\hat{p}_j$ extends $\hat{p}_i$ finally, then
$$\widehat{\bigcup p_i} = \bigcup \hat{p}_i$$

(iv) If $\hat{p}_0 \subseteq \hat{p}_1 \subseteq \hat{p}_2 \subseteq ...$ and $\operatorname{sign} p_i = 0$ for all i, then $\operatorname{sign} \bigcup p_i = 0$.

(v) If $\operatorname{sign} \bigcup p_i \ne 0$, then there is some $k \in \mathbb{N}$ such that $\hat{p}_k \subseteq \hat{p}_{k+1} \subseteq ...$

PROOF. (i) is obvious. (ii) and (iii) are applications of (i) and (iv) is implied by (iii).
(v) Suppose $\bigcup p_i = \alpha + G^+$ with a convex subgroup G of $\bigcup M_i$. Let $\alpha \in M_k$. Then $p_k = \alpha + \hat{p}_k$ and $p_{k+1} = \alpha + \hat{p}_{k+1}$, hence $\hat{p}_{k+1}$ extends $\hat{p}_k$. □

(6.5) PROPOSITION. *Let T be an o-minimal expansion of RCF, let M be a model of T and let G be a convex subgroup of $(M,+,<)$. Let V be a convex valuation ring of M with $V \subseteq V(G)$. Let $\mathscr{C}$ be a set of cuts of M with $G \subseteq G(p)$ for all $p \in \mathscr{C}$. Then there are an elementary extension N of M with $\dim N/M = \aleph_0 + |\mathscr{C}|$ and a convex subgroup H of $(N,+,<)$, such that $H \cap M = G$, $\operatorname{sign}^* H^+ = 0$, $V(H)$ is the convex hull of V in N and such that for each $p \in \mathscr{C}$ there is a cut q of N extending p with $\operatorname{sign} q = 0$ and $G(q) = H$.*

PROOF. First we reduce to the case where $G(p) = G$ $(p \in \mathscr{C})$ and $V = V(G) \subsetneqq M$. For this we take a model $M' \succ M$ and realizations α_p, β of $p \in \mathscr{C}$, G^+ in M' respectively. Let V' be the convex hull of V in M'. Then $M \cap \beta \cdot V' = G$. Therefore $\alpha_p + \beta \cdot V'^+$ or $\alpha_p - \beta \cdot V'^+$ extends p.

This shows that we may assume $V = V(G)$ and $G = G(p)$ for all $p \in \mathscr{C}$.

We construct a chain $M = M_0 \prec M_1 \prec M_2 \prec \ldots$ of models of T, together with extensions p_i on M_i for each $p \in \mathscr{C}$ such that

(a) For all $p, p' \in \mathscr{C}$ we have $\hat{p}_i = \hat{p}'_i$.

(b) $p_0 \subseteq p_1 \subseteq p_2 \subseteq \ldots$, $\hat{p}_0 \subseteq \hat{p}_1 \subseteq \hat{p}_2 \subseteq \ldots$ and $V(p_i)$ lies over V $(i < \omega,\ p \in \mathscr{C})$; note that $V(p_i)$ is independent of $p \in \mathscr{C}$ by (a).

(c) $\operatorname{sign} p_{2i+1} = 0$ for all i .

(d) $\operatorname{sign}^* \hat{p}_{2i+2} = 0$ and $V(p_{2i+2})^+$ is the convex hull of V in M_{2i+2} for all i

Take $M_0 := M$ and $p_0 := p$. Suppose $M_0, \ldots, M_i$ and $p_0, \ldots, p_i$ are constructed, and (a)-(c) holds up to i. We define M_{i+1} and p_{i+1} in the following way:

<u>Case 1</u> i is even (the case $i = 0$ is included since we do not need that $\operatorname{sign}^* \hat{p}_i = 0$ for this construction step).

By (6.3) applied to $\{p_i \mid p \in \mathscr{C}\}$ and $G(p_i)$ there is some $M_{i+1} \succ M_i$ and cuts p_{i+1} of M_{i+1} extending p_i for all $p \in \mathscr{C}$ such that $\operatorname{sign} p_{i+1} = 0$ and $\hat{p}_{i+1}$ is the least extension of $\hat{p}_i$ on M_i. By (3.9)(iii), $V(p_{i+1})$ lies over $V(p_i)$, hence over V.

<u>Case 2</u> i is odd.

Let α_p be a realization of p_i $(p \in \mathscr{C})$ and let r be the least extension of $\hat{p}_i$ on $M\langle \alpha_p \mid p \in \mathscr{C}\rangle$. Then $\alpha + r$ is an extension of p_i and $V(r)$ lies over $V(p_i)$ ((3.9)(iii)), hence over V. From (6.3) applied to $T^{>0}$, r and the convex hull of V in $M_i\langle \alpha_p \mid p \in \mathscr{C}\rangle$ we get some $M_{i+1} \succ M_i\langle \alpha_p \mid p \in \mathscr{C}\rangle$ and an extension r_1 of r on M_{i+1} with multiplicative invariance group U such that $\operatorname{sign}^* r_1 = 0$ and such that the U is the convex hull of V in M_{i+1}. Since $2 \in U$, we know that r_1 is the upper edge of a convex subgroup of $(M_{i+1}, +, <)$, hence $U = V(r_1)$. Now we take $p_{i+1} = \alpha_p + r_1$ $(p \in \mathscr{C})$, which completes the construction.

Let $N = \bigcup M_i$ and $H := \bigcup\limits_{i<\omega} G(p_i)$; by (a) the group H does not depend on p; by (b) we have $H = \bigcup\limits_{i<\omega} G(p_{2i})$. By (d) and (6.4)(iv) applied to $T^{>0}$ and the sequence $(\hat{p}_{2i+2})_{i<\omega}$ we have $\operatorname{sign}^* H^+ = 0$. By (d) and (6.4)(iii) applied to $T^{>0}$ and the sequence $(\hat{p}_{2i+2})_{i<\omega}$ we have $V(H) = \bigcup\limits_{i \in \mathbb{N}} V(p_{2i+2})$. Since each $V(p_{2i+2})$ is the convex hull of V we get that $V(H)$ is the convex hull of V in N.

It remains to show that for each $p \in \mathscr{C}$ there is a cut q of N with $\operatorname{sign} q = 0$ and $G(q) = H$. We take $q = \bigcup p_i$ and apply (6.4) to the sequence $(p_{2i+1})_{i<\omega}$; by (b) we have $q = \bigcup p_{2i+1}$.

By (b) and (c) we have $\operatorname{sign} q = 0$. By (b) again, we have $\hat{q} = \bigcup_{i \in \mathbb{N}} \hat{p}_{2i+1} = H^+$. $\square$

(6.6) COROLLARY. *Let T be an o-minimal expansion of RCF, let M be a model of T and let G be a convex subgroup of $(M, +, <)$. Let V be a proper convex valuation ring of M with $V \subseteq V(G)$. Let $\mathscr{C}$ be a set of cuts of M with $G \subseteq G(p)$ for all $p \in \mathscr{C}$. Let $\varepsilon, \delta \in \{-1, 0, 1\}$. Then there are an elementary extension N of M with $\dim N/M = \aleph_0 + |\mathscr{C}|$ and a convex subgroup H of $(N, +, <)$, such that $H \cap M = G$, $\operatorname{sign}^* H^+ = \varepsilon$, $V(H)^+$ is an heir of V^+ and such that for each $p \in \mathscr{C}$ there is a cut q of N extending p with $\operatorname{sign} q = \delta$ and $G(q) = H$.*

PROOF. First we apply (6.5) and replace G by H and each $p \in \mathscr{C}$ by a cut q extending p with $\operatorname{sign} q = 0$ and $G(q) = H$. Since $V(H)$ is the convex hull of V in $N \succ M$ and $V \neq M$ we know that $V(H)^+$ is an heir of V^+. So in order to prove the corollary we may assume that $\operatorname{sign} p = \operatorname{sign}^* \hat{p} = 0$, $G = G(p)$ and $V = V(p) \neq M$ $(p \in \mathscr{C})$.

First we do the case $\delta = 0$. If $\varepsilon = 0$ we have nothing to do. So let $\varepsilon \neq 0$.

Let β be a realization of G^+ and for each $p \in \mathscr{C}$ let p_1 be the largest extension of p on $M\langle\beta\rangle$. By (3.9) applied to $Th(M)^{>0}$, the cut $\hat{p}_1$ is the largest extension of G^+ on $M\langle\beta\rangle$. Let G_1 be the convex hull of G in $M\langle\beta\rangle$. Then $\operatorname{sign}^* \hat{p}_1 = 1$ and $\operatorname{sign}^* G_1^+ = -1$ by (3.9)(iv) and since $\operatorname{sign}^* G^+ = 0$. Moreover $V(p_1)$ is the largest convex valuation ring of $M\langle\beta\rangle$ lying over V and $V(p_1) = V(G_1)$ by (3.9)(i),(ii).

Now we apply (6.3) for $G(p_1)$ and p_1 $(p \in \mathscr{C})$. Hence there is some $N \succ M$ and cuts q of N extending p_1 $(p \in \mathscr{C})$ such that $G(q)$ is the convex hull of $G(p_1)$ in N. Since $\operatorname{sign}^*(\hat{p}_1) = 1$ we have $\operatorname{sign}^* \hat{q} = 1$ as well and the cut $V(q)^+$ is an heir of $V(p_1)^+$. Since $\hat{q}$ does not depend on $p \in \mathscr{C}$ we have proved the corollary in the case $\varepsilon = 1$ and $\delta = 0$.

Finally the same argument for G_1 and p_1 $(p \in \mathscr{C})$ yields the corollary in the case $\varepsilon = -1$, $\delta = 0$.

Now we get the Corollary from the case $\delta = 0$ as follows: for $p \in \mathscr{C}$ take an extension q_p of p on N, $\operatorname{sign} q_p = 0$ such that $G(q_p) = H$, $\operatorname{sign}^* H^+ = \varepsilon$ and $V(H)^+$ is an heir of V^+. Let $N_1 \succ N$ such that all q_p $(p \in \mathscr{C})$ are realized and let H_1 be the largest extension of H^+ on N_1. From (3.9) we get that $V(H_1)^+$ is an heir of $V(H)^+$, hence of V^+ and the least and largest extensions of q_p on N_1 have invariance group H_1. These extensions have signature -1,+1 respectively, since $\operatorname{sign} q_p = 0$. $\square$

7. Existentially closed extensions. Again we work with an o-minimal expansion T of the theory of real closed fields in the language $\mathscr{L}$. If p is a cut of a model M of T and $f : M \longrightarrow M$ is M-definable, then by [vdD1] 7,(2.5) the map f is differentiable in a neighborhood I of p and there is an M-definable map $g : M \longrightarrow M$ with $g(x) = f'(x)$ for all $x \in I$. The cut $g(p)$ does not depend on the choice of I and g. We denote this cut by $f'(p)$.

(7.1) PROPOSITION. *If p is a cut of M with $\operatorname{sign} p = 0$ and $f : M \longrightarrow M$ is M-definable with $f(p) = \hat{p}$, then $f'(p) = \mathfrak{m}(p)^+$ or $f'(p) = \mathfrak{m}(p)^-$. In particular $p \sim V(p)^+$.*

PROOF. Fist we prove $|f'(p)| \leq \mathfrak{m}(p)^+$.

Suppose $0 < a < |f'(p)|$ for some $a \in V(p)^*$. Then $\frac{1}{a}\cdot\hat{p} = \hat{p}$, hence the map $g(x) := \frac{1}{a}\cdot f(x)$ fulfills $g(p) = \hat{p}$ and $g'(p) \geq 1$. Let $b, c \in M$ such that $b < p < c$ such that g is differentiable in $[b, c]$, $g(b) \geq 0$ and $g'(x) \geq 1$ for all $x \in [b, c]$. Then $0 \leq g(b) + x - b \leq g(x)$ for all $x \in [b, c]$, hence $0 \leq g(b) + p - b \leq g(p) = \hat{p}$. This implies $g(b) + p - b = \hat{p}$, thus $\operatorname{sign} p \neq 0$ in contradiction to our assumption.

It remains to show $|f'(p)| \geq \mathfrak{m}(p)^+$. We may assume $p > 0$. Suppose there is some $a \in \mathfrak{m}(p)$ with $0 < |f'(p)| < a$.

<u>Claim 1.</u> There is some $c \in M$ with $1 + a \in G^*(p + c)$.

Since $\frac{1}{a} \notin V(p)$ there is some $b \in G(p)$, $b > 0$ with $\frac{b}{a} \notin G(p)$. Let $d < p$ with $d + \frac{b}{a} > p$. We prove $1 + a \in G^*(p + c)$ for $c := \frac{b}{a} - d$. Let $m \in M$ with $d < m < p$. Since $ap < b + ad$ we have $(1 + a)(m + c) = m + c + a(m + \frac{b}{a} - d) = m + c + b + am - ad < m + c + b + b$. As $b \in G(p)$ we get $m + 2b < p$, hence $m + 2b + c < p + c$. This proves claim 1.

With the element c from claim 1 we replace $f(x)$ by $f(x - c)$ and p by $p + c$, hence we may assume that $1 + a \in G^*(p)$. By (3.6) we have $1 - a \in G^*(p)$ as well.

<u>Claim 2.</u> there is a linear map $g : M \longrightarrow M$ with $g(p) = p$ and a neighbourhood (c, d) of p with $g(x) \geq x + f(x)$ in (c, d). For the proof of claim 2 we have two cases.

<u>Case 1.</u> $f'(p) > 0$.

There is a neighbourhood $[c, d]$ of p, $c > 0$, where f is strictly increasing and differentiable, such that $f(c) \in G(p)$ and $f'|_{(c,d)} < a$. Take $g(x) = f(c) + (1 + a)\cdot x$. Then $g(x) > x + f(x)$ in $[c, d]$, because $g(c) > f(c) + c$ and $g' > f' > 0$ in (c, d). We have $g(p) = f(c) + (1 + a)\cdot p = p$ by the choice of c and a.

<u>Case 2.</u> $f'(p) < 0$

There is some neighbourhood $[c, d]$ of p, where f is strictly decreasing and differentiable, such that $f(d) \in G(p)$ and $f'|_{(c,d)} > -a$. Since $(1 + a)\cdot p = p$ and $\operatorname{sign} p = 0$ we have $a\cdot p < \hat{p}$. Let $d > p$ such that $a\cdot d \in G(p)$ and let $g(x) = f(d) + ad + (1 - a)\cdot x$. Then $g(x) \geq x + f(x)$ on $[c, d]$, since $g(d) = f(d) + d$ and $g' < f' < 0$ in (c, d). We have $g(p) = f(d) + ad + (1 - a)\cdot p = p$ by the choice of d and a.

Now claim 2 is proved and we have a linear map g with $g(p) = p$ and $g(x) - x > f(x)$ in (c, d). This means $g_1(p) = \hat{p}$ if $g_1(x)$ denotes $g(x) - x$. Since g_1 is linear it is not possible, that $\operatorname{sign} p = 0$. $\square$

(7.2) THEOREM. *Let p be a cut of a model of T, such that $V(p)^+ \sim V^+$ for some T-convex valuation ring V of M.*

(i) The signature alternative holds for p. In particular $\operatorname{sign} p = 0$ *implies $p \not\sim \hat{p}$.*

(ii) The signature alternative holds for $|p|$ with respect to $M^{>0}$. In particular $\operatorname{sign}^* p = 0$ *implies $p \not\sim G^*(p)^+$.*

PROOF. First a remark. If T is polynomially bounded then (i) follows quickly from the valuation property (c.f. [vdD-S] and (9.3) below). This property even implies that $p \not\sim G^+$ for all convex subgroups G of $(M, +, <)$ and all cuts p of M with $\operatorname{sign} p = 0$.

(i). Suppose $\operatorname{sign} p = 0$ and $p \sim \hat{p}$. By (7.1) we have $p \sim V(p)^+$, in particular p is definable in (M, V), which contradicts (5.2).

Now let $p \not\sim \hat{p}$, let α be a realization of $\hat{p}$ and let q be the unique extension of p on $M\langle\alpha\rangle$. By (3.9) we have $\operatorname{sign} q = 0$ and $\hat{q}$ is the largest extension of $\hat{p}$ on $M\langle\alpha\rangle$. By (3.9) applied

to $\hat{p}$ and $M^{>0}$ the cut $V(q)^+$ is the least or the largest extension of $V(p)^+$ on $M\langle\alpha\rangle$. Since $V(p)^+ \sim V^+$ we have $V(q)^+ \sim W^+$ for some T-convex valuation ring W of $M\langle\alpha\rangle$. By what we have shown before this implies $q \not\sim \hat{q}$.
(ii). Let $\operatorname{sign}^* p = 0$, hence $p \neq a \cdot G^*(p)^\pm$ for all $a \in M$.
Case 1. $\operatorname{sign} p = 0$.
By (i) we have $p \not\sim \hat{p}$ and by (3.6) we have $\hat{p} \sim G^*(p)^+$, hence $p \not\sim G^*(p)^+$.
Case 2. $\operatorname{sign} p \neq 0$, so $p = a \pm \hat{p}$ with some $a \in M$. We may assume that $p > 0$. If $a \leq 0$, then $a \in G(p)$, hence $p = \hat{p}$ and $p \not\sim G^*(p)^+ = V(p)^+$ follows from (5.2). So we may assume that $a > 0$ and we may replace p by $1 + \frac{1}{a} \cdot \hat{p}$. If $1 \notin \frac{1}{a} \cdot G(p)$, then $1 + \frac{1}{a} \cdot G(p)$ is a convex subgroup of $(M^{>0}, \cdot, <)$ in contradiction to $\operatorname{sign}^* p = 0$. So $1 \in \frac{1}{a} \cdot G(p)$, in other words $p = \hat{p}$ and we may use (5.2) again.
So we know that $\operatorname{sign}^* p = 0$ implies $p \not\sim G^*(p)^+$. As in the proof of (i), this implies the the signature alternative for $|p|$ with respect to $M^{>0}$. □

Notation. Let p be a cut of a divisible ordered abelian group M. If $\operatorname{sign} p \neq 0$, then p is an edge of the nonempty set $Z(p) := \{a \in M \mid a + \hat{p} = p \text{ or } a - \hat{p} = p\}$ and we denote the other edge by $z(p)$. If $\operatorname{sign} p = 0$ we define $z(p) := p$.
So $Z(p)$ is the set of all $a \in M$ between p and $z(p)$. Note that $Z(p)$ is 0-definable in the structure (M, p^L). If G is a convex subgroup of a real closed field M we write $z^*(G)$ and $z^*(G^+)$ for the cut $z(G^+)$ build with respect to $(M, <, \cdot)$. Hence

$$z^*(G) := \begin{cases} a \cdot \mathfrak{m}(G)^+ & \text{if } a \cdot V(G) = G,\ a \neq 0 \\ a \cdot V(G)^+ & \text{if } a \cdot \mathfrak{m}(G) = G,\ a \neq 0 \\ G^+ & \text{if } \operatorname{sign}^* G^+ = 0 \end{cases}$$

(7.3) Theorem. *Let M be a model of T and let p be a cut of M such that $f(V(p)^+) = V^+$ for some T-convex valuation ring V of M and some M-definable map $f : M \longrightarrow M$. Suppose* $\operatorname{sign} p = 0$. *Let N be an elementary extension of M and let q be a cut of N extending p such that $G(q) \cap M = G(p)$, $V(q) \cap M = V(p)$ and $f(V(q)^+) = W^+$, where W is a T-convex valuation ring of N.*
(i) If $\operatorname{sign}^* \hat{p} = 0$ *then $(V(q)^+, \hat{q}, q)$ is an heir of $(V(p)^+, \hat{p}, p)$.*
(ii) If $\operatorname{sign}^* \hat{q} = \operatorname{sign}^* \hat{p}$, *$p$ is neither dense nor definable and $Z^*(\hat{q}) \cap M = Z^*(\hat{p})$, then $(V(q)^+, \hat{q}, z^*(\hat{q}), q)$ is an heir of $(V(p)^+, \hat{p}, z^*(\hat{p}), p)$.*

Proof. (i). We may assume that $p > 0$. W lies over V, since $V(q)^+$ extends $V(p)^+$. Hence W^+ is an heir of V^+, so $V(q)^+$ is an heir of $V(p)^+$. By (2.3) we have to show the following: If $\varphi(v_1, ..., v_6)$ is an $\mathscr{L}(M)$-formula and $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in N$ such that

$$\alpha_1 < q < \alpha_2,\ \beta_1 < \hat{q} < \beta_2,\ \gamma_1 < V(q)^+ < \gamma_2,\ N \models \varphi(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2)$$

then there are $a_1, a_2, b_1, b_2, c_1, c_2 \in M$ such that

$$a_1 < p < a_2,\ b_1 < \hat{p} < b_2,\ c_1 < V(p)^+ < c_2,\ M \models \varphi(a_1, a_2, b_1, b_2, c_1, c_2)$$

If γ_1 is not a realization of $V(p)$, we know this from (3.14) applied to the theory $T^{>0}$, because $(q, \hat{q})$ is an heir of $(p, \hat{p})$ ((3.16) and (7.2)(i)).

Hence we assume that γ_1 is a realization of $V(p)^+$.
Since $V(p)$ is definable in (M,V) and $\operatorname{sign} p = \operatorname{sign}^* \hat{p} = 0$, (5.2) implies that p and $\hat{p}$ are omitted in $M\langle\gamma_1\rangle$. Let $p_1 := q \upharpoonright M\langle\gamma_1\rangle$ be the unique extension of p on $M\langle\gamma_1\rangle$. Then $\hat{p}_1$ is the unique extension of $\hat{p}$ on $M\langle\gamma_1\rangle$. Since $V(q)^+$ is an heir of $V(p)^+$ we can apply (3.15) for $T^{>0}$, thus $V(q)$ lies over $V(p_1)$. Because $\operatorname{sign}^* \hat{p_1} = 0$ the cut $\hat{q}$ is an heir of $\hat{p}_1$ ((5.5)). So $(q,\hat{q})$ is an heir of $(p_1,\hat{p}_1)$ by (3.16) and (7.2)(i).

By (3.14) applied to $T^{>0}$, $(q,\hat{q})$ and $(p_1,\hat{p}_1)$, there are M-definable maps h,g_1,g_2, f_1,f_2 with

$$f_1(\gamma_1) < p_1 < f_2(\gamma_1),\ g_1(\gamma_1) < \hat{p}_1 < g_2(\gamma_1),\ \gamma_1 < V(p_1)^+ < h(\gamma_1)$$

and

$$M\langle\gamma_1\rangle \models \varphi(f_1(\gamma_1), f_2(\gamma_1), g_1(\gamma_1), g_2(\gamma_1), \gamma_1, h(\gamma_1))$$

As $V(p_1)^+$ is an heir of $V(p)^+$ it follows from a standard compactness argument (cf. [Poi] 11.01) that there are cuts r_1,r_2,r_3 of $M\langle\gamma_1\rangle$ with $r_1 = V(p_1)^+$ such that (r_1,r_2,r_3) is an heir of $(V(p)^+,\hat{p},p)$. Since p_1, $\hat{p}_1$ are the unique extensions of p, $\hat{p}$ respectively on $M\langle\gamma_1\rangle$ we get $p_1 = r_3$ and $\hat{p}_1 = r_2$. So $(V(p_1)^+,\hat{p}_1,p_1)$ is an heir of $(V(p)^+,\hat{p},p)$ which yields the Theorem .

(ii). If $\operatorname{sign}^* \hat{p} = \operatorname{sign}^* \hat{q} = 0$, then $Z^*(\hat{p}) = Z^*(\hat{q}) = \emptyset$ and (ii) is implied by (i). So we assume that $\operatorname{sign}^* \hat{p} = \operatorname{sign}^* \hat{q} \neq 0$. The assumption that p is neither dense nor definable means that $V(p)$ is proper, in other words $\hat{p}$ is not definable. We do the case $\operatorname{sign}^* \hat{p} = 1$, the case $\operatorname{sign}^* \hat{p} = -1$ is similar. Let $a \in M$ with $a{\cdot}V(p) = G(p)$. Since $Z^*(\hat{q}) \cap M = Z^*(\hat{p}) \ni a$ we get $a \cdot V(q) = G(q)$. Since $V(p)$ is proper we know that $(M,V(p))$ is existentially closed in $(N,V(q))$ (c.f. (4.6)(iii)), hence $(M,G(p))$ is existentially closed in $(N,G(q))$. By (7.2)(i) and (3.16) it follows that $(M,G(p),p^L)$ is existentially closed in $(N,G(q),q^L)$. Since $V(p) = \frac{1}{a}G(p)$ and $V(q) = \frac{1}{a}G(q)$ it follows that $(M,V(p),G(p),p^L)$ is existentially closed in $(N,V(q),G(q),q^L)$. Moreover $Z^*(\hat{p}) = \frac{1}{a}{\cdot}(V(p)^*)^{>0}$ and the maximal ideal $\mathfrak{m}(p)$ of $V(p)$ as well as $M \setminus \mathfrak{m}(p)$ are existentially definable in (M,V). The same is true over N, so $\mathfrak{M}$ is existentially closed in $\mathfrak{N}$. □

(7.4) THEOREM. *Let $f(y,x_1,...,x_n)$ be a 0-definable map. Let $\mathscr{O}$, $\mathscr{G}$, $\mathscr{Z}$, $\mathscr{Z}^*$ and $\mathscr{D}$ be new unary predicates and let $c_1,...,c_n$ be new constants with respect to $\mathscr{L}$. Let $\varepsilon,\delta \in \{-1,0,1\}$ and let $\mathscr{L}^*$ be the language $\mathscr{L}(\mathscr{O},\mathscr{G},\mathscr{Z},\mathscr{Z}^*,\mathscr{D},c_1,...,c_n)$. Let T^ε_δ be the $\mathscr{L}^*$-theory which extends T and says the following things about a model $(M,V,G,Z,Z^*,D,d_1,...,d_n)$:*

(a) $D = p^L$ for some cut p of M, p neither dense nor definable with $\operatorname{sign} p = \delta$.
(b) $V = V(p)$.
(c) $G = G(p)$ and $\operatorname{sign}^* G^+ = \varepsilon$.
(d) $f(V(p)^+,d_1,...,d_n) > 0$ is the upper edge of a T-convex valuation ring of M. This is axiomatizable by the axioms $g(f(V(p)^+,d_1,...,d_n)) \leq f(V(p)^+,d_1,...,d_n)$ for each continuous, 0-definable map g.
(e) $Z = \{a \in M \mid a + G^+ = p$ or $a - G^+ = p\}$ and $Z^* = \{a \in M^* \mid a{\cdot}V = G$ or $a{\cdot}\mathfrak{m} = G\}$.

Then
(i) T^ε_δ is consistent if and only if there is a model M of T, a convex valuation ring V of M and $d_1,...,d_n \in M$ such that $f(V^+,d_1,...,d_n)$ is the upper edge of a T-convex valuation ring.

(ii) If T^ε_δ is consistent then T^ε_δ is model complete relative $\mathscr{L}$.

PROOF. (i). Certainly the condition is necessary for the consistency of T^ε_δ. Conversely suppose we have a model M of T, a convex valuation ring V of M and $d_1, ..., d_n \in M$ such that $f(V^+, d_1, ..., d_n)$ is the upper edge of a T-convex valuation ring. By (6.6) there is some $N \succ M$ and an extension q of V^+ on N with $\operatorname{sign} q = \delta$ and $\operatorname{sign}^* \hat{q} = \varepsilon$ such that $V(q)^+$ is an heir of V^+. Hence $f(V(q)^+, d_1, ..., d_n)$ is an heir of $f(V^+, d_1, ..., d_n)$, thus $f(V(q)^+, d_1, ..., d_n)$ is the upper edge of a T-convex valuation ring of N. This proves that T^ε_δ is consistent.

(ii). We may assume that T is model complete. In order to prove model completeness of T^ε_δ we use Robinsons test. So let $\mathfrak{M} \subseteq \mathfrak{N}$ be an extension of $\mathscr{L}^*$-structures, such that $\mathfrak{M}, \mathfrak{N} \models T^\varepsilon_\delta$. We have to show that $\mathfrak{M}$ is existentially closed in $\mathfrak{N}$. If $\delta \neq 0$ then this follows easily from (5.6) and the assumption that p is not definable. If $\delta = 0$, then $\mathfrak{M}$ is existentially closed in $\mathfrak{N}$ by (7.3)(ii). $\square$

8. The elementary theory of convex subgroups. We start with a model theoretic preparation:

(8.1) LEMMA. *Let $\mathscr{L} \subseteq \mathscr{L}^*$ be languages and let M^*, N^* be $\mathscr{L}^*$-structures. Let A be a common $\mathscr{L}$-substructure of $M := M^* \restriction \mathscr{L}$ and $N := N^* \restriction \mathscr{L}$, such that $Th(M, A) = Th(N, A)$. Then there are $\mathscr{L}^*$-structures $\tilde{M}, \tilde{N}$ with the same universe, such that $\tilde{M} \restriction \mathscr{L} = \tilde{N} \restriction \mathscr{L}$ together with elementary $\mathscr{L}^*$-embeddings $\varphi : M^* \longrightarrow \tilde{M}$, $\psi : N^* \longrightarrow \tilde{N}$, with $\varphi(a) = \psi(a)$ for all $a \in A$.*

PROOF. Since $Th(M, A) = Th(N, A)$, there is an $\mathscr{L}$-structure K and elementary $\mathscr{L}$-embeddings $\varphi_0 : M \longrightarrow K$, $\psi_0 : N \longrightarrow K$ such that $\varphi(a) = \psi(a)$ for all $a \in A$. We may assume that φ and ψ are inclusions, hence we assume that $M, N \prec K$. By further extending K we may assume that K is κ-resplendent, where $\kappa := (2^{|\mathscr{L}|+|M|+|N|})^+$ (see [Poi], 9.c.). Since K is κ-resplendent we can expand K to $\mathscr{L}^*$-structures $\tilde{M}$ and $\tilde{N}$, such that $M^* \prec \tilde{M}$ and $N^* \prec \tilde{N}$. $\square$

(8.2) THEOREM. *Let $A \prec M, N$ be models of T and let $G \subseteq M, H \subseteq N$ be proper convex subgroups (thus $G \neq 0, M$ and $H \neq 0, N$). Suppose there is an A-definable map $f : A \longrightarrow A$ such that $f(V(G)^+)$ and $f(V(H)^+)$ are the upper edges of a T-convex valuation ring of M, N respectively. Then*

$$(M, G) \equiv_A (N, H)$$

if and only if

(a) $V(H) \cap A = V(G) \cap A =: V_0$, $Z^*(G) \cap A = Z^*(H) \cap A$, $H \cap A = G \cap A =: G_0$,

(b) $\operatorname{sign}^* G^+ = \operatorname{sign}^* H^+$ and

(c) If $V_0^+ \sim G_0^+$, then there is some A-definable map $g : A \longrightarrow A$ with $g(V_0^+) = G_0^+$ such that

$$g(V(G)^+) < G^+ \Leftrightarrow g(V(H)^+) < H^+.$$

PROOF. Since $(M, V(G)) \equiv_A (N, V(H))$ we can apply (8.1) . Therefore we may assume that $M = N$ and $V(H) = V(G) =: V$. Let W be the T-convex valuation ring of M with upper edge $f(V^+)$. By redefining f outside an A-definable interval containing V^+ in a

suitable way, we may assume that f is an A-definable homeomorphism. Furthermore we may assume that (M,H,G) is $|A|^+$-saturated and $|A|^+$-strong homogeneous.
Case 1. $Z^*(G)\cap A=Z^*(H)\cap A\neq\emptyset$. Let $a\in Z^*(G)\cap A$ and say $\operatorname{sign}^* G^+=\operatorname{sign}^* H^+=1$. Then $G=a\cdot V$ and $H=a\cdot V$, thus $G=H$.
Case 2. $Z^*(G)\cap A=Z^*(H)\cap A=\emptyset$.

Suppose first that $A\subseteq W$. By (5.2) we have $W\subseteq V$. Since $f^{-1}(W^+)=V^+$ it follows that f^{-1} is a strictly increasing A-definable homeomorphism, so $f^{-1}(W^+)=W^+$. This shows that $V=W$ is T-convex. From $A\subseteq V$ we get $A=V_0=V(G_0)$. Since $G,H\notin\{V,\mathfrak{m}(V)\}$ and (M,V) is $|A|^+$-saturated, there is some $\alpha\in M$ such that $A\langle\alpha\rangle\not\subseteq V$ and $G\cap A\langle\alpha\rangle=H\cap A\langle\alpha\rangle\in\{0,A\langle\alpha\rangle\}$. In particular $Z^*(G)\cap A\langle\alpha\rangle=Z^*(H)\cap A\langle\alpha\rangle=\emptyset$ and conditions $(a),(b)$ hold for $A\langle\alpha\rangle$ instead of A. Since $G\cap A\langle\alpha\rangle$ is definable and $V\cap A\langle\alpha\rangle$ is a proper convex valuation ring of $A\langle\alpha\rangle$ we have $V\cap A\langle\alpha\rangle^+\not\sim G_0^+$ and condition (c) holds, too. Hence it is enough to prove the theorem under the assumption of case 2 and the assumption $W\cap A\subsetneqq A$; note that V_0 is also proper, since $f(G^+)=V_0^+$.

By (5.2) and our assumption in case 2 we have $G^+,H^+\not\sim_A V^+$.
Case 2.1. $\operatorname{sign}^* G^+=\operatorname{sign}^* H^+\neq 0$.
Claim. There are $m,m'\in M$ with $G=m\cdot V$, $H=m'\cdot V$ such that $t^{(M,V)}(m/A)=t^{(M,V)}(m'/A)$.
Case 2.1.1. $V_0^+\not\sim G_0^+$. By saturation there are $m,m'\in G$ with $G=m\cdot V$, $H=m'\cdot V$ such that m and m' realize G_0^+. Since $V_0^+\not\sim G_0^+$, the cut G_0 is not A-definable in (A,V_0) (by (4.6)). Hence there is a unique 1-type ξ of (A,V_0) which extends the cut G_0^+. Since $W\cap A\subsetneqq A$ we have $(A,V_0)\prec(M,V)$. Thus $t^{(M,V)}(m/A)=\xi=t^{(M,V)}(m'/A)$.
Case 2.1.2. $V_0^+\sim G_0^+$. By item (c) there is an A-definable map $g:M\longrightarrow M$ such that $g(V_0^+)=G_0^+$ and such that $g(V^+)<G^+\Leftrightarrow g(V^+)<H^+$. Now there are exactly two 1-types of (A,V_0) extending the cut G_0^+. One contains the statement $x<b\cdot g(V_0^+)^\varepsilon$, the other one contains the statement $x>b\cdot g(V_0^+)^\varepsilon$. So in order to prove the claim it is enough to find realizations $m,m'\in M$ of G_0^+ with $G=m\cdot V,H=m'\cdot V$ such that $g(V^+)<m$ iff $g(V^+)<m$. We may assume that $G\subseteq H$. As $G^+,H^+\not\sim_A V^+$ we get $g(V^+)<G^+$ or $H^+<g(V^+)$. If $g(V^+)<G^+$ we take $m,m'\in M$ with $g(V^+)<m,m'$ such that $G=mV$ and $H=m'V$. If $H^+<g(V^+)$, then by saturation there are $m\in G,m'\in H$ such that m,m' realize G_0^+ and $G=m\cdot V$, $H=m'\cdot V$. In any case m,m' are the elements we are looking for and the claim is proved.

Now the theorem in the case 2.1. follows since (M,V) is $|A|^+$-strong homogeneous: there is an A-automorphism $\sigma:(M,V)\longrightarrow(M,V)$ with $\sigma(m')=m$. Hence $\sigma(H)=G$ and $(M,G)\cong_A(M,H)$.
Case 2.2. $\operatorname{sign}^* G^+=\operatorname{sign}^* H^+=0$.
Let α, β be realizations of G^+, H^+ respectively. Let G_1,H_1 be the largest convex subgroups of $M\langle\alpha\rangle$, $M\langle\beta\rangle$ lying over G, H respectively. Since $\operatorname{sign}^* G^+=0$ we know that $\operatorname{sign}^* G_1^+=1$ and $Z^*(G_1^+)$ is the set of realizations of G^+ in $M\langle\alpha\rangle$. Since $\operatorname{sign}^* H^+=0$ we have that $\operatorname{sign}^* H_1^+=1$ and $Z^*(H_1^+)$ is the set of realizations of H^+ in $M\langle\beta\rangle$. Moreover the requirement on the map f remain valid. Item (c) with the same map g remains valid, since by (7.2)(ii) we have $V^+\not\sim G^+,H^+$. So we can apply case 2.1 and get $(M\langle\alpha\rangle,G_1)\equiv_A(M\langle\beta\rangle,H_1)$, hence $(M\langle\alpha\rangle,V(G_1),G_1)\equiv_A(M\langle\beta\rangle,V(H_1),H_1)$. Because

$\operatorname{sign}^* H^+ = 0$ we know that (M,V,H) is existentially closed in $(M\langle\beta\rangle, V(H_1), H_1)$ from (5.5). From these assertions we get $(V,G) \equiv\rangle_{\exists,A}(V,H)$. Since T^0 is model complete relative $\mathscr{L}$, it follows that $(M,V,G) \equiv_A (M,V,H)$, in particular $(M,G) \equiv_A (M,H)$. $\square$

(8.3) COROLLARY. *Let $A \prec M,N$ be models of T and let $G \subseteq M, H \subseteq N$ be proper convex subgroups. Suppose the following conditions hold:*
(a) $V(H) \cap A = V(G) \cap A =: V_0$, $Z^*(G) \cap A = Z^*(H) \cap A$, $H \cap A = G \cap A =: G_0$,
(b) $\operatorname{sign}^* G^+ = \operatorname{sign}^* H^+$,
(c) $V(G)$ and $V(H)$ are T-convex.
(d) $V(G_0)$ is T-convex.
Then

$$(M,G) \equiv_A (N,H).$$

PROOF. We only have to prove condition (c) of (8.2). We may again assume that $N = M$ and $V := V(G) = V(H)$ by (8.1). So assume that $V_0^+ \sim G_0^+$. Since V_0 and $V(G_0)$ are T-convex, theorem (5.2) implies that $V_0 = V(G_0)$ and the group G_0 is of the form $b\cdot V_0^\varepsilon$ for some $b \in A$ and some $\varepsilon \in \{\pm 1\}$. We claim that the map $g(x) := b\cdot x^\varepsilon$ fulfills $g(V^+) < G^+ \Leftrightarrow g(V^+) < H^+$ as required in (8.2)(c). If $g(V^+) = G^+$, then $Z^*(G) \cap A \neq \emptyset$ by (5.2), hence $H = G$ by items (a) and (b). So we may assume that $g(V^+) \neq G^+$ and similar $g(V^+) \neq H^+$. Suppose $G^+ < g(V^+) < H^+$, hence $\frac{1}{b}G^+ < (V^+)^\varepsilon < \frac{1}{b}H^+$. Since $V(G) = V(H) = V$ this is only possible if $\frac{1}{b}G^+ < 1 < \frac{1}{b}H^+$ in contradiction to $b \in A$ and $G \cap A = H \cap A$. $\square$

Corollary (8.3) does not give a quantifier elimination result as we get (8.7) from (8.6) below. The reason is that (8.3)(d) is not an elementary statement in the structure (M,G). Here is an example, which shows that we can not drop assumption (d) in (8.3). Let T be the theory of $\mathbb{R}_{\exp}$ and let $\omega > \mathbb{R}$. Let $A := \mathbb{R}\langle\omega\rangle$ and let V_0 be the convex hull of $\mathbb{R}$ in A. Let G_0 be the convex valuation ring of A with upper edge $\exp(\omega\cdot V_0^+)$. Let (M,V) be an elementary extension of (A,V_0) such that there are realizations $\alpha,\beta \in M$ of G_0^+ with $\alpha < \exp(\omega\cdot V^+) < \beta$. Finally let $G := \alpha\cdot V$ and $H := \beta\cdot V$. Then $(M,G) \not\equiv_A (M,H)$, items (a) and (b) of (8.3) holds and $V(H) = V(G) = V$ is T-convex. By (6.3) it is also possible to get a counter example from this situation in the case $\operatorname{sign}^* G^+ = \operatorname{sign}^* H^+ = 0$.

If each convex valuation ring of a model of T is T-convex, we don't need condition (c) and (d) of (8.3). These theories are precisely the polynomially bounded theories with archimedean prime model. We briefly state the definitions and known facts which we use.

(8.4) DEFINITION. An o-minimal expansion of a real closed field is called **polynomially bounded** if for each 0-definable map f there is some $n \in \mathbb{N}$ with $f(x) \leq x^n$ for sufficiently large x.

(8.5) PROPOSITION. *If T is an o-minimal expansion of ordered fields, then the following are equivalent:*
(i) T is polynomially bounded and has an archimedean prime model.
(ii) Each convex valuation ring of a model of T is T-convex.

PROOF. This follows easily from [vdD-Lew], (4.2). $\square$

For the rest of this paper we work with a polynomially bounded $\mathscr{L}$-theory T which has an archimedean prime model.

Hence from now on every convex valuation ring of a model of T is T-convex. This fact and (9.3) below gives us enough information to determine the theory of all (M, p^L) in the language $\mathscr{L}(\mathscr{D})$ over some parameter set A, where p runs through the cuts of M.

(8.6) COROLLARY. *Let $A \prec M, N$ be models of T and let $G \subseteq M, H \subseteq N$ be proper convex subgroups. Then*

$$(M, G) \equiv_A (N, H)$$

if and only if
(a) $V(H) \cap A = V(G) \cap A =: V_0$, $Z^*(G) \cap A = Z^*(H) \cap A$, $H \cap A = G \cap A =: G_0$ and
(b) $\operatorname{sign}^* G^+ = \operatorname{sign}^* H^+$.

PROOF. By (8.3). □

It is not difficult to see that convex subgroups of $(M, +, <)$ are in 1-1 correspondence with paires (V, ξ), where V is a convex valuation ring of M and ξ is a cut of Γ_V with $G(\xi) \in \{0, \Gamma_V\}$. If G corresponds to (V, ξ) and $v : M \longrightarrow \Gamma$ denotes the valuation corresponding to V then Corollary (8.6) says that the theory of (M, G) with parameters from A is determined by the theory of (Γ, ξ^L) with parameters in $v(A^*)$. For the theory of cuts coming from residue fields of convex valuation rings see (9.7) below. We do not make use of this point of view.

(8.7) COROLLARY. *Let $\mathcal{O}$, $\mathscr{Z}^*$ and $\mathscr{G}$ be new unary predicates. Let $\mathscr{L}^{group} := \mathscr{L}(\mathcal{O}, \mathscr{Z}^*, \mathscr{G})$. For $\varepsilon \in \{-1, 0, +1\}$ let T^ε be the $\mathscr{L}^{group}$-theory which extends T and says the following things about a model (M, V, Z^*, G) of T^ε:*
(i) G is a convex subgroup of M with $\operatorname{sign}^* G^+ = \varepsilon$.
(ii) $V = V(G)$ and $Z^* = Z^*(G)$.

Suppose T has quantifier elimination and a universal system of axioms. Then T^ε has quantifier elimination.

PROOF. By (8.6). □

(8.8) COROLLARY. *Let $G \subseteq M$ be a convex subgroup and let $A \prec M$. If p is a cut of M, such that p^L is definable in (M, G) with parameters from A and $\alpha \in M$, then the 1-type $t^{(M,G)}(\alpha/A)$ is uniquely determined by the following datas:*
(a) The cut $t(\alpha/A)$.
(b) $\alpha \in f(r)^L$, where $f : M \longrightarrow M$ is A-definable, $r \in \{V(G)^+, z^*(G^+), G^+\}$ and $f(r \restriction A) = t(\alpha/A)$.
(c) $\alpha \in f(r)^R$, where $f : M \longrightarrow M$ is A-definable, $r \in \{V(G)^+, z^*(G^+), G^+\}$ and $f(r \restriction A) = t(\alpha/A)$.

PROOF. By (8.7). □

9. The elementary theory of Cuts. Again T denotes a polynomially bounded o-minimal expansion of ordered fields. First we repeat some results of [Mi2], which are also explained in [vdD3], §3.

Let $\mathscr{P}$ denote the prime model of T. Since $\mathscr{P}$ is assumed to be archimedean we may assume that the underlying field of $\mathscr{P}$ is an ordered subfield of $\mathbb{R}$. Hence the field $\mathbb{R}$ can be extended to an elementary extension $\mathscr{R}$ of $\mathscr{P}$.

A **power function** is a definable endomorphism f of the multiplicative group $\mathscr{P}^{>0}$. The extension of f to $\mathscr{R}$ is of the form x^λ for some exponent $\lambda \in \mathbb{R}$ and f is uniquely determined by λ. The set K of all these exponents λ is an ordered subfield of $\mathbb{R}$. If V is a convex valuation ring of a model M of T, then the value group Γ of V is a K-vectorspace with multiplication $\lambda \cdot v(m) := v(m^\lambda)$ for $m \in M,\ m > 0$.

(9.1) THEOREM. *(Valuation Property)*
Let $M \prec N$ be models of T with $\dim N/M = 1$. Let $V \subseteq N$ be T-convex and let $v : N \longrightarrow \Gamma \cup \{+\infty\}$ be the valuation according to V. Suppose $v(M) \neq v(N)$. Then
(a) If $\alpha \in N \setminus M$ then there is some $a \in M$ such that $v(\alpha - a) \notin v(M)$.
(b) If $\gamma \in v(N) \setminus v(M)$ then $v(N^*) = v(M^*) \oplus \gamma \cdot K$.

PROOF. (a) is [vdD-S],9.2 and (b) is [vdD-3],5.4. □

We make use of another formulation of (9.1):

(9.2) COROLLARY. *Let $G \subseteq M \models T$ be a convex subgroup and let $f : M \longrightarrow M$ be M-definable and non constant in each neighborhood of G^+. Then there is a unique exponent λ with the following property: there are $a, b \in M$ such that for all $N \succ M$ and all convex subgroups H of N with $H \cap M = G$ we have*

$$f(H^+) = a + b \cdot (H^+)^\lambda.$$

We have $\lambda \neq 0$ and in the case $\operatorname{sign}^* G^+ = 0$, λ is the unique exponent with the property $f(G^+) = c + d \cdot (G^+)^\lambda$ for some $c, d \in M$.

PROOF. First we prove that there are $a, b \in M$ and an exponent $\lambda \neq 0$ as claimed. Let α be a realization of G^+ and let V be the convex hull of $\mathbb{Q}$ in $M\langle\alpha\rangle$. If $a \in M, a > 0$ and $v(\alpha) = v(a)$, then there are $n, m \in \mathbb{N}$ such that $a < n\alpha$ and $\alpha < ma$. Since $\alpha \models G^+$ and G is a convex subgroup, this is not possible.

Therefore $v(\alpha) \notin v(M)$ and we can apply (9.1). Since $f(\alpha) \notin M$ there is some $a \in M$ such that $v(f(\alpha) - a) \notin \Gamma_{V \cap M}$. Hence there are $b \in M$ and an exponent $\lambda \neq 0$, such that $v(f(\alpha) - a) = v(b) + \lambda v(\alpha)$. We claim that a, b and λ are the elements we are looking for.

To see this take $N \succ M$ and a convex subgroup H of N with $H \cap M = G$. Let β be a realization of H^+ and let W be the convex hull of $\mathbb{Q}$ in $N\langle\beta\rangle$. Furthermore let $V := W \cap M\langle\beta\rangle$. By what we have shown before we have $v(f(\beta) - a) = v(b) + \lambda v(\beta)$. Hence $w(f(\beta) - a) = w(b) + \lambda w(\beta)$ as well. Since $\lambda \neq 0$, we know that $f(\beta) - a$ realizes the same cut of N as $b \cdot \beta^\lambda$. Hence $t(f(\beta)/N) = t(a + b \cdot \beta^\lambda / N)$ and a, b and λ have the required properties.

Now we prove the second uniqueness statement. So assume $\operatorname{sign}^* G^+ = 0$, $c, d \in M$ and μ is an exponent such that $f(G^+) = c + d\cdot(G^+)^\mu$. Hence $b\cdot(G^+)^\lambda = c - a + d\cdot(G^+)^\mu$, thus $G^+ = (\frac{d}{b})^{\frac{1}{\lambda}}\cdot(G^+)^{\frac{\mu}{\lambda}}$. We write $e := (\frac{d}{b})^{\frac{1}{\lambda}}$ and $\eta := \frac{\mu}{\lambda}$. Let α be a realization of G^+ and let V be the convex hull of $V(G)$ in $M\langle\alpha\rangle$ with corresponding valuation v. Since $\operatorname{sign}^* G^+ = 0$ there is some $m \in G$ with $\frac{\alpha}{e\alpha^\eta} < m$. Hence $v(\alpha) = v(e\alpha^\eta) = v(e) + \eta\cdot v(\alpha)$. Now $\eta \neq 1$ implies $v(\alpha) = \frac{1}{1-\eta}v(a) \in v(M)$ in contradiction to $\operatorname{sign}^* G^+ = 0$. So $\eta = 1$, i.e. $\mu = \lambda$, which proves the second uniqueness statement.

The first uniqueness statement concerning λ follows from the second one, since by (6.6) there is some $N \succ M$ and a convex subgroup H of N, lying over G with $\operatorname{sign}^* H^+ = 0$. □

Lemma (9.2) applied to $N = M$ gives a strengthening of the signature alternative ((7.2)(i)) for polynomially bounded theories with archimedean prime model:

(9.3) COROLLARY. *Let G be a convex subgroup of a model M of T. If p is a cut of M with $p \sim G^+$, then* $\operatorname{sign} p \neq 0$. □

The multiplicative signature alternative ((7.2)(ii)) can not be strengthened in this way. To see an example, let R be a real closed field, let t be infinitesimal, positive over R and let S be the real closure of $R(t)$. Let v be the valuation of S according to the convex hull of R in S and let $G := \{a \in S \mid v(a) > \sqrt{2}\}$. Let $p := G^+$ and $q := 1 + p$. Since $1 + G$ is a convex subgroup of $(S^{>0}, \cdot, <)$ we have $\operatorname{sign}^* q = 1$. But $\operatorname{sign}^* p = 0$ and $p \sim q$.

If T is an expansion of $Th(\mathbb{R}, \exp)$ and R is a model of T, then we can apply the logarithm of R to the example above. This shows that (9.3) fails without the assumption that T is polynomially bounded.

We come to our main result:

(9.4) THEOREM. *Let T be polynomially bounded with archimedean prime model. Let M, N be models of T, $A \prec M, N$ and let p, q be cuts of M, N respectively. Then*

$$(M, p^L) \equiv_A (N, q^L)$$

if and only if the following four conditions are fulfilled:

(i) $p \restriction A = q \restriction A =: p_0$, $Z(p) \cap A = Z(q) \cap A$, $G(p) \cap A = G(q) \cap A =: G_0$, $Z^*(\hat{p}) \cap A = Z^*(\hat{q}) \cap A$ *and* $V(p) \cap A = V(q) \cap A =: V_0$.

(ii) p is dense if and only if q is dense, p is definable if and only if q is definable, $\operatorname{sign} p = \operatorname{sign} q$ *and* $\operatorname{sign}^* \hat{p} = \operatorname{sign}^* \hat{q}$.

(iii) If $p_0 \sim V_0^+$ then there are $a, b \in A$ and $\varepsilon \in \{\pm 1\}$ with $a + b(V_0^+)^\varepsilon = p_0$ such that

(a) $a + b(V(p)^+)^\varepsilon < p \Leftrightarrow a + b(V(q)^+)^\varepsilon < q$ *and*

(b) $a + b(V(p)^+)^\varepsilon < z(p) \Leftrightarrow a + b(V(q)^+)^\varepsilon < z(q)$.

(iv) If $p_0 \sim G_0^+$ then there is some $a \in A$ with $|p_0 - a| = \hat{p}_0$ such that for all $b \in A$ and each exponent λ with $a + b(G_0^+)^\lambda = p_0$ we have

(a) $a + b\cdot(\hat{p})^\lambda < p \Leftrightarrow a + b\cdot(\hat{q})^\lambda < q$,

(b) $a + b\cdot(\hat{p})^\lambda < z(p) \Leftrightarrow a + b\cdot(\hat{q})^\lambda < z(q)$,

(c) $a + b\cdot z^*(\hat{p})^\lambda < p \Leftrightarrow a + b\cdot z^*(\hat{q})^\lambda < q$ *and*

(d) $a + b\cdot z^*(\hat{p})^\lambda < z(p) \Leftrightarrow a + b\cdot z^*(\hat{q})^\lambda < z(q)$.

PROOF. Suppose first that $(M,p^L)\equiv_A(N,q^L)$. Clearly (i) and (ii) holds. Item (iv) holds, since $G_0^+\sim p_0$ implies that $\operatorname{sign}p_0\neq 0$ ((9.3)). Item (iii) holds since $V_0^+\sim p_0$ implies that there are $a,b\in A$ and $\varepsilon\in\{+1,-1\}$ with $a+b(V_0^+)^\varepsilon=p_0$ ((5.2)).

Conversely suppose that (i)-(iv) holds. If p and q are definable, then $(M,p^L)\equiv_A(N,q^L)$, since $p\restriction A=q\restriction A$, $\operatorname{sign}p=\operatorname{sign}q$ and $Z(p)\cap A=Z(q)\cap A$. If p and q are dense, then $(M,p^L)\equiv_A(N,q^L)$ follows from $p\restriction A=q\restriction A$ and (3.3). So we may assume that p and q are neither dense nor definable. By (8.6) conditions (i) and (ii) imply $(M,G(p))\equiv_A(N,G(q))$. Since $z(q)$ and $z^*(\hat{q})$ are 0-definable in (M,q^L) we may assume that $M=N$, $G(p)=G(q)=:G$ and $V(p)=V(q)=:V$ by (8.1).
<u>Case 1.</u> $\operatorname{sign}p=\operatorname{sign}q\neq 0$. We do the case $\operatorname{sign}p=\operatorname{sign}q=1$, the case $\operatorname{sign}p=\operatorname{sign}q=-1$ is similar. By (i) and (ii) we have $z(p)=z(q)$ and $z^*(\hat{p})=z^*(\hat{q})$.
<u>Case 1.1.</u> $z(p)\restriction A=z(q)\restriction A\neq p_0$. Hence there are $a,b\in A$ with $p=a+G^+$ and $q=b+G^+$. Then $b-a\in G$, so $p=q$ and we are done.
<u>Case 1.2.</u> $z(p)\restriction A=z(q)\restriction A=p_0$, hence there are no $a\in A$ and no convex subgroup H of $(M,+,<)$ such that $p=a+H^+$ or $q=a+H^+$.

Let $\alpha,\beta\in M$ such that $p=\alpha+G^+$ and $q=\beta+G^+$. We prove that $t^{(M,G)}(\alpha/A)=t^{(M,G)}(\beta/A)$. For this we have to go through conditions (a)-(c) of (8.8). First we have $t(\alpha/A)=t(\beta/A)=p_0$ by our assumption of case 1.2. Let $r\in\{V^+,z^*(G^+),G^+\}$ and let $f:M\longrightarrow M$ be A-definable with $f(r\restriction A)=p_0$. In order to check conditions (b) and (c) of (8.8) we have to show that $f(r)<\alpha$ if and only if $f(r)<\beta$. Since r is the edge of a convex subgroup we can use (9.2) to find $a',b\in A$ and an exponent $\lambda\neq 0$ such that $f(r)=a'+b{\cdot}r^\lambda$. Hence we may assume that $f(x)=a'+b{\cdot}x^\lambda$. Suppose $\alpha<f(r)<\beta$. We may assume that $p<q$, thus $p<\beta$ and $p<\beta-G^+=z(q)$.
<u>Case 1.2.1.</u> $r=G^+$.

Take $a\in M$ such that $|p_0-a|=\hat{p}_0$ and such that (iv)(a)-(d) holds. Since $\alpha-a$ and $\beta-a$ realize the same edge of the subgroup $G(p_0)$ of A we have that $(f(G^+)-a)\restriction A$ is this edge. Now $(f(G^+)-a)\restriction A=a'-a+b{\cdot}(G_0^+)^\lambda$, so $a'-a\in b{\cdot}G_0^\lambda$ and $a'-a\in b{\cdot}G^\lambda$. It follows $a+b(G^+)^\lambda=f(r)<\beta<q$, hence $z(p)<\alpha<a+b(G^+)^\lambda<p<z(q)$, by the choice of a and (iv)(a). This contradicts (iv)(b).
<u>Case 1.2.2.</u> $r=z^*(G^+)$.

If $z^*(G^+)\restriction A\neq G_0^+$, then $z^*(G^+)\sim_A G^+$ and we can use case 1.2.1. If $z^*(G^+)\restriction A=G_0^+$ then the same proof as in case 1.2.1. leads to a contradiction (where we use (iv)(c) and (d) instead of (a), (b) now).
<u>Case 1.2.3.</u> $r=V^+$.

Take $a_0,b_0\in A$ and $\varepsilon\in\{\pm 1\}$ as in (iii). Since $r=V^+$ and $p_0=a+b\cdot(V^+)^\lambda$ we may assume that $a_0=a'$ as in the proof of case 1.2.1. So $f(r)=a_0+b\cdot V^\lambda$ and $b{\cdot}(V_0^+)^\lambda=G(p_0)$. Since $a_0+b_0(V_0^+)^\varepsilon=p_0$ it follows that $b{\cdot}(V_0^+)^\lambda=b_0\cdot(V_0^+)^\varepsilon$, hence $b{\cdot}(V^+)^\lambda=b_0\cdot(V^+)^\varepsilon$, too. This shows that we can replace b by b_0 and λ by ε, hence $f(r)=a_0+b_0\cdot(V^+)^\varepsilon$. It follows $a_0+b_0\cdot(V^+)^\varepsilon<\beta<q$, hence $a_0+b_0\cdot(V^+)^\varepsilon<p$ by (iii)(a). So $z(p)<\alpha<a_0+b_0{\cdot}(V^+)^\varepsilon<p<z(q)$ which contradicts (iii)(b).

This proves that $t^{(M,G)}(\alpha/A)=t^{(M,G)}(\beta/A)$. Since (M,G) is strong $|A|^+$-homogeneous there is an A-automorphism σ of (M,G) which maps β to α. Hence the cut q is mapped to the cut p under this automorphism, which shows $(M,p^L)\equiv_A(N,q^L)$.

Case 2. $\operatorname{sign} p = \operatorname{sign} q = 0$.

Then $z(p) = p$ and $z(q) = q$. Let α, β be realizations of p, q respectively and let p_1, q_1 be the largest extensions of p,q on $M\langle\alpha\rangle$, $M\langle\beta\rangle$ respectively.

Claim A. $(M\langle\alpha\rangle, p_1^L) \equiv_A (M\langle\beta\rangle, q_1^L)$. To see claim A we may use case 1, since $\operatorname{sign} p_1 = \operatorname{sign} q_1 = 1$. Hence it is enough to check conditions (i)-(iv) for p_1 and q_1. Condition (ii) clearly holds for p_1 and q_1, since $\hat{p}_1$, $\hat{q}_1$ are the unique extensions of $\hat{p}$, $\hat{q}$ respectively, so the signature does not change.

Since $\operatorname{sign} p = 0$ the cut $z^*(\hat{p}_1)$ is the unique extension of $z^*(\hat{p})$ on $M\langle\alpha\rangle$. Since $p_1 = \alpha + \hat{p}_1$ we have $z(p_1) = \alpha - \hat{p}_1$ is the least extension of p on $M\langle\alpha\rangle$. Moreover $V(p_1)$ is the convex hull of V on $M\langle\alpha\rangle$. From these datas it follows that conditions (i),(iii) and (iv) are fulfilled for p and p_1. Similarly they are fulfilled for q and q_1. Hence (i)-(iv) holds for p_1 and q_1.

Claim B. $(V^+, G^+, z^*(G^+), p) \equiv\rangle_{\exists,A} (V^+, G^+, z^*(G^+), q)$.

This follows from claim A if we know that $(V_{q_1}^+, \hat{q}_1, z^*(\hat{q}_1), q_1) \equiv\rangle_{\exists,A} (V^+, G^+, z^*(G^+), q)$. As $\operatorname{sign} q = 0$, the cuts G^+ and V^+ are omitted in $M\langle\beta\rangle$, in particular $\operatorname{sign}^* \hat{q}_1 = \operatorname{sign}^* \hat{q}$ and $Z^*(\hat{q}_1) \cap M = Z^*(\hat{q})$. So by (7.3)(ii) we even know

$$(V_{q_1}^+, \hat{q}_1, z^*(\hat{q}_1), q_1) \equiv\rangle_{\exists,M} (V^+, G^+, z^*(G^+), q).$$

In order to prove the theorem we use (7.4). Let $Z_M^* := \{a \in M^* \mid a \cdot V(p) = G(p) \text{ or } a \cdot \mathfrak{m}(p) = G(p)\}$ and $Z_N^* := \{a \in N^* \mid a \cdot V(r) = G(r) \text{ or } a \cdot \mathfrak{m}(r) = G(r)\}$. Let $\mathfrak{M}$ be the $\mathscr{L}^*$-structure (M, V, G, Z_M^*, p^L) and let $\mathfrak{N}$ be the $\mathscr{L}^*$-structure (M, V, G, Z_N^*, q^L). Let $\varepsilon := \operatorname{sign}^* \hat{p}$. We know that $\mathfrak{N}, \mathfrak{M} \models T_0^\varepsilon$. Claim B says that for each existential $\mathscr{L}^*(A)$-sentence φ we have $\mathfrak{M} \models \varphi \Rightarrow \mathfrak{N} \models \varphi$. Since T_0^ε is model complete we get $\mathfrak{M} \equiv_A \mathfrak{N}$, hence $(M, p^L) \equiv_A (M, q^L)$. □

In section 10 we construct examples, which shows that (i),(ii) and (iv) of (9.4) do not imply (iii) and that (i),(ii),(iii) and (iv)(a) in the case $\operatorname{sign} p = 0$ do not imply (iv)(b).

(9.5) COROLLARY. *There is some $a \in A$ such that*

$$(M, p^L) \equiv_A (N, q^L) \Leftrightarrow (M, p^L) \equiv_{\{a,b\}} (N, q^L) \text{ for each } b \in A.$$

PROOF. With the notation as in (9.4) we can take $a = 0$ if $\operatorname{sign} p_0 = 0$ and $a \in A$ with $|p_0 - a| = \hat{p}_0$ otherwise. □

By (10.1) below it is possible to have $(M, p^L) \not\equiv_A (N, q^L)$ and $(M, p^L) \equiv_{\{b\}} (N, q^L)$ for each $b \in A$.

Special cases of (9.4) are formulated in the next corollaries:

(9.6) COROLLARY. *Let M, N be models of T, $A \prec M, N$ and let p, q be cuts of M, N respectively. Suppose*

(i) $p \restriction A = q \restriction A =: p_0$, $Z(p) \cap A = Z(q) \cap A$, $G(p) \cap A = G(q) \cap A =: G_0$, $Z^*(\hat{p}) \cap A = Z^*(\hat{q}) \cap A \neq \emptyset$ *and* $V(p) \cap A = V(q) \cap A =: V_0$.

(ii) p *and* q *are neither dense nor definable with* $\operatorname{sign} p = \operatorname{sign} q$.

(iii) If $p_0 \sim V_0^+$, *then there are* $a, b \in A$ *and* $\varepsilon \in \{\pm 1\}$ *with* $a + b(V_0^+)^\varepsilon = p_0$ *such that*

(a) $p < a + bV(p)^\varepsilon \Leftrightarrow q < a + bV(q)^+$ *and*

(b) $z(p) < a + bV(p)^\varepsilon \Leftrightarrow z(q) < a + bV(q)^+$.
Then

$$(M, p^L) \equiv_A (N, q^L).$$

PROOF. Since $\mathrm{sign}^* \hat{p} = \mathrm{sign}^* \hat{q}$ is implied by (i) and (ii), we only have to check condition (iv) of (9.4). By (8.6) we know $(M, G(p)) \equiv_A (N, G(q))$ and we may again assume that $M = N$, $V := V(p) = V(q)$ and $G := G(p) = G(q)$. Moreover we assume that $\mathrm{sign}^* G^+ = 1$, the case $\mathrm{sign}^* G^+ = -1$ is similar. Let $c \in Z^*(\hat{p}) \cap A = Z^*(\hat{q}) \cap A$, so $G = c \cdot V$ and $z^*(G^+) = c \cdot \mathfrak{m}^+$.

Now let $p_0 \sim G_0^+$ as in (9.4)(iv) assumed. Since $G_0 = c \cdot V_0$ it follows $p_0 \sim V_0^+$, hence by (iii) there are $a, b \in A$ and some $\varepsilon \in \{\pm 1\}$ with $a + b(V_0^+)^\varepsilon = p_0$ and $p < a + b(V^+)^\varepsilon \Leftrightarrow q < a + bV^+$. We prove (9.4)(iv)(a), the other cases are similar. We have to show that for all $b_1 \in A$ and each exponent λ the cut $a + b_1(G^+)^\lambda$ does not lie between p and q. Since p and q extends p_0, we may assume that $a + b_1(G^+)^\lambda$ extends $p_0 = a + b \cdot (V^+)^\varepsilon$, hence $b_1(cV_0^+)^\lambda = b \cdot (V_0^+)^\varepsilon$, in other words the A-definable map $f(x) := (\frac{b_1(cx)^\lambda}{b})^{\frac{1}{\varepsilon}}$ fixes V_0^+. Hence f fixes V^+, too. Thus $b_1(cV^+)^\lambda = b \cdot (V^+)^\varepsilon$, which gives the claim by the choice of a and b. □

A cut p of M is of the form $P^{-1}(\xi^L)^+$ where $P : M \longrightarrow M_0 \cup \{+\infty\}$ is a real place and ξ is a cut of M_0 with definable invariance group if and only if the valuation ring of P is $V(p)$, $G(p) = \mathfrak{m}(p)$ and $|p| \le V(p)^+$. For these cuts the theory of (M, p^L) with parameters in $A \prec M$ is determined by the theory of (M_0, ξ^L) with parameters in $P(A)$:

(9.7) COROLLARY. *Suppose p,q are neither dense nor definable such that* $\mathrm{sign}\, p = \mathrm{sign}\, q$, *$Z(p) \cap A = Z(q) \cap A$, $p \restriction A = q \restriction A$, $V(p) \cap A = V(q) \cap A$, $G(p) = \mathfrak{m}(p)$, $G(q) = \mathfrak{m}(q)$, $|p| \le V(p)^+$ and $|q| \le V(q)^+$. Then $(M, p^L) \equiv_A (N, q^L)$.*

PROOF. Say $p, q > 0$. We use (9.6) and we only have to check condition (iii) there. We do statement (a), statement (b) is similar. Since $0 < p \le V^+$ and $G(p) = \mathfrak{m}$ we have $\mathfrak{m}_0^+ \le p_0 \le V_0^+$, so if $p_0 = V_0^+$ we can take $a = 0$, $b = 1$ and $\varepsilon = 1$. So we assume that $\mathfrak{m}_0 \le p_0 < V_0^+$. We assume again that $M = N$ and $V = V(p) = V(q)$. Suppose $p_0 = a + b(V_0^+)^\varepsilon$ and $p \le a + b(V^+)^\varepsilon \le q$. If $p = a + b(V^+)^\varepsilon$ or $q = a + b(V^+)^\varepsilon$, then $p = q$ since $Z(p) \cap A = Z(q) \cap A$ and we are done. So we assume that $p < a + b(V^+)^\varepsilon < q$. Since $\mathfrak{m}_0^+ \le b(V_0^+)^\varepsilon < V_0^+$ we must have $\mathfrak{m}_0 = b(V_0^+)^\varepsilon$, hence $\varepsilon = -1$ and $b \in V_0^*$. Since $V_0^* \subseteq V^*$ we may assume that $|b| = 1$, Hence $p < a \pm \mathfrak{m}^+ < q$, which contradicts $G(p) = G(q) = \mathfrak{m}$ and $p \restriction A = q \restriction A$. □

(9.8) COROLLARY. *Let T be polynomially bounded with archimedean prime model. Let M, N be models of T, $A \prec M, N$ and let p, q be cuts of M, N respectively. Suppose*
(i) $p \restriction A = q \restriction A$ has signature 0, $G(p) \cap A = G(q) \cap A$, $Z^*(\hat{p}) \cap A = Z^*(\hat{q}) \cap A$ and $V(p) \cap A = V(q) \cap A =: V_0$.
(ii) p is dense if and only if q is dense, p is definable if and only if q is definable, $\mathrm{sign}\, p = \mathrm{sign}\, q$ *and* $\mathrm{sign}^* \hat{p} = \mathrm{sign}^* \hat{q}$.
Then

$$(M, p^L) \equiv_A (N, q^L)$$

PROOF. Since $p_0 := p \restriction A$ has signature 0 we have $Z(p) \cap A = \emptyset = Z(q) \cap A$. Moreover $p_0 \not\sim V_0^+$ and $p_0 \not\sim G_0^+$, where $V_0 = V(p) \cap A$ and $G_0 = G(p) \cap A$. Hence (i)-(iv) of (9.4) are fulfilled and the Corollary follows. $\square$

Theorem (9.4) implies a quantifier elimination result for the theories T_δ^ε in an extended language:

(9.9) COROLLARY. *Suppose T has quantifier elimination and a universal system of axioms. Let $\varepsilon, \delta \in \{-1, 0, +1\}$. We use the notation of (7.4); since T is polynomially bounded with archimedean prime model the language $\mathscr{L}^*$ is $\mathscr{L}(\mathscr{O}, \mathscr{G}, \mathscr{Z}, \mathscr{Z}^*, \mathscr{D})$ (no constants are needed) and does not depend on a given 0-definable map as in (7.4). Let $\mathscr{L}^{cut}$ be the language $\mathscr{L}^*$ together with binary predicates $R_\eta^1, R_\eta^2, S_\lambda^1, ..., S_\lambda^4$ for each exponent $\lambda \neq 0$ and $\eta \in \{\pm 1\}$. Let $\overline{T_\delta^\varepsilon}$ be the $\mathscr{L}^{cut}$-theory, which extends T_δ^ε and which says in addition the following things about a model $(M, p^L, G, V, ...)$: for all $a, b \in M$ we have*
(a) $R_\eta^1(a, b) \leftrightarrow a + b\cdot(V^+)^\eta < p$
(b) $R_\eta^2(a, b) \leftrightarrow a + b\cdot(V^+)^\eta < z(p)$
(c) $S_\lambda^1(a, b) \leftrightarrow a + b\cdot(G^+)^\lambda < p$
(d) $S_\lambda^2(a, b) \leftrightarrow a + b\cdot(G^+)^\lambda < z(p)$
(e) $S_\lambda^3(a, b) \leftrightarrow a + b\cdot z^*(G^+)^\lambda < p$ and
(f) $S_\lambda^1(a, b) \leftrightarrow a + b\cdot z^*(\mathscr{G}^+)^\lambda < z(p)$,

Then $\overline{T_\delta^\varepsilon}$ has quantifier elimination.

PROOF. By (9.4), two models of $\overline{T_\delta^\varepsilon}$ which induce the same $\overline{\mathscr{L}}$-structure on a common substructure $\mathscr{A}$ are elementary equivalent over $\mathscr{A}$ (since T has a universal system of axiom, $\mathscr{A}$ is an expansion of a common elementary restriction A of the underlying T-models). This is a reformulation of quantifier elimination. $\square$

10. Counter examples. We give here some examples which shows in which way one can not improve (9.4). T is again polynomially bounded with archimedean prime model.

(10.1) EXAMPLE. There are models $A \prec M, N$ of T and cuts p, q of M, N respectively with $(M, p^L) \equiv_b (N, q^L)$ for all $b \in A$ such that $(M, p^L) \not\equiv_A (N, q^L)$. In other words we can have $(M, p^L) \not\equiv_A (N, q^L)$, although for every $\mathscr{L}(\mathscr{D})$-formula $\varphi(x)$ in one variable, the sets $X := \{m \in M \mid (M, p^L) \models \varphi(m)\}$ and $Y := \{n \in N \mid (N, q^L) \models \varphi(n)\}$ fulfill $X \cap A = Y \cap A$.

PROOF. We choose A such that there are $a, b \in A$, $a, b > 0$ with $v_0(a) < n\cdot v_0(b) < 0$ for all $n \in \mathbb{N}$, where v_0 is a convex valuation on A with valuation ring V_0. Let $p_0 := a + bV_0^+$ and let $M_1 \succ A$ such that there are a convex valuation ring V_1 lying over V_0 and realizations $\alpha, \beta \in M$ of V_0^+ with $\alpha \in V_1 < \beta$. Let $p_1 := a + b\alpha + V^+$ and let $q_1 := a + b\beta + V^+$. Certainly p and q extend p_0. By (6.3) there is some $M \succ M_1$ and cuts p, q of M extending p_1, q_1 respectively, such that $G(p) = G(q) =$ the convex hull V of V_1 in M and such that $\operatorname{sign} p = \operatorname{sign} q = 0$. So $Z(p) = Z(q) = \emptyset$, $G(p) = G(q) = V$ and $Z^*(\hat{p}) = Z^*(\hat{q}) =$

$(V^*)^{>0}$. In particular conditions (i) and (ii) of (9.6) are fulfilled for any $A_0 \prec A$. We have $(M, p^L) \not\equiv_A (M, q^L)$, since $p = a + b\alpha + V^+ < a + bV^+ < a + b\beta + V^+ = q$.

But for each $a_0 \in A$ we have $(M, p^L) \equiv_{a_0} (M, q^L)$. To see this we have to show conditions (i)-(iii) of (9.6) for $A_0 := \mathrm{dcl}(a_0)$. We have already seen that conditions (i) and (ii) hold. We prove (iii) of (9.6) for A_0. Suppose there are $c, d \in A_0$ and $\varepsilon \in \{\pm 1\}$ such that $p \le c + d(V^+)^\varepsilon \le q$. Since p and q extend p_0 we have $c + d(V_0^+)^\varepsilon = a + bV_0^+$. Hence $\varepsilon = 1$ and $c + dV_0 = a + bV_0$. This means $v_0(a - c) \ge v_0(d) = v(b) < 0$. Because the definable closure of the emptyset is archimedean by assumption we know that d is not 0-definable in M. As $\dim A_0 = 1$ we have $c \in \mathrm{dcl}(d)$. Since T is polynomially bounded there is some $n \in \mathbb{N}$ with $n \cdot v_0(d) \le v_0(c)$. Hence $v_0(a) < n \cdot v_0(b) = n \cdot v_0(d) < v_0(c)$ and $v_0(a) = v_0(a - c) \ge v_0(d) = v_0(b)$, a contradiction. □

(10.2) Example. The following example shows that we need each exponent in (9.4). In particular we do not have quantifier elimination of $Th(\mathfrak{M})$ if $\mathfrak{M}$ is an expansion by definitions of (M, p^L) with finitely many new symbols.

Proof. Let $M_1 \succ M_0$ such that there is a convex subgroup G_1 of M_1 with $\mathrm{sign}^* G_1^+ = 0$ and $M_0 \subseteq V(G_1) \subseteq G_1$. Let $\lambda \ne 0$ and let $M_2 \succ M_1$ such that there are $b \in M_2$ with $b > M_1$ and a convex subgroup G_2 of M_2 such that $\mathrm{sign}^* G_2^+ = 0$, $G_2 \cap M_1 = G_1$ and $V(G_2) \cap M_1 = V(G_1)$. In particular $G_2^+ < b \cdot (G_2^+)^\lambda$. By compactness there is some $M_3 \succ M_2$ together with a convex subgroup G_3, $\mathrm{sign}^* G_3^+ = 0$, $G_3 \cap M_2 = G_2$, $V(G_3) \cap M_2 = V(G_2)$ together with realizations $\alpha, \beta \in M_3$ of $b \cdot (G_2^+)^\lambda$ such that $\alpha < b \cdot (G_3^+)^\lambda < \beta$. As $G_3^+ < b \cdot (G_3^+)^\lambda$ we have $\alpha + G_3^+ < b \cdot (G_3^+)^\lambda < \beta + G_3^+$ and $\alpha + G_3^+, \beta + G_3^+$ are extensions of $b \cdot (G_2^+)^\lambda$. Finally let $M \succ M_3$ such that there are extensions p, q of $\alpha + G_3^+, \beta + G_3^+$ on M respectively, with $\mathrm{sign}\, p = \mathrm{sign}\, q = 0$, $G(p) = G(q)$ lies over G_3, $V(p) = V(q)$ lies over $V(G_3)$ and $\mathrm{sign}^* G(p)^+ = 0$. Then we see:

(a) Conditions (i),(ii) and (iii) of (9.4) hold for $A = M_2$.

(b) Condition (iv) of (9.4) holds for $A = M_2$ and each exponent $\mu \ne \lambda$.

(c) $(M, p^L) \not\equiv_{M_2} (M, q^L)$ since condition (iv) of (9.4) does not hold for $A = M_2$ and λ.

(d) $(M, p^L) \equiv_{M_0} (M, q^L)$ by (a) and (b) and since condition (iv) of (9.4) holds for $A = M_0$ and λ.

□

(10.3) Example. The following example shows that conditions (i)-(iii) of (9.4) can hold such that $(p, V(p)^+)$ is not an heir of $(q, V(q)^+)$ over A and vice versa. Let $A \models T$ and let V_0 be a proper convex valuation ring of A. Let $\alpha \models V_0^+$, let W_1 be the convex hull of V_0 in $A\langle\alpha\rangle$ and let V_1 be the largest convex valuation ring of $A\langle\alpha\rangle$ lying over V_0. Let G_1 be a convex subgroup of $(A\langle\alpha\rangle, +, <)$ such that $V_1 \subseteq V(G_1)$ and such that $G_1 \subseteq W_1$ (for example we could take G_1 =the maximal ideal of V_1; if necessary we also can choose G_1 such that $\mathrm{sign}^* G_1^+ = 0$). Furthermore let p_1 be a cut of $A\langle\alpha\rangle$ such that $W_1 < p_1 < V_1$ and $G(p_1) = G_1$ (as $G_1 \subseteq W_1$, $\alpha + G_1^+$ is such a cut; if necessary we also can choose p_1 such that $\mathrm{sign}\, p_1 = 0$). By (6.5) there is an elementary extension $M \succ A\langle\alpha\rangle$ and a cut p of M, such that $p_1 \subseteq p$, $G(p)$ lies over G_1 and $V(p)$ lies over V_1 with $\mathrm{sign}\, p = \mathrm{sign}^* \hat{p} = 0$. Let W be a convex valuation ring of M, lying over W_1. Then $W \subseteq V(p)$, hence again by (6.5), there is an elementary extension $N \succ M$ and a cut q of N such that $p \subseteq q$, $G(q)$

lies over $G(p)$ and $V(q)$ lies over W with $\operatorname{sign} q = \operatorname{sign}^* \hat{q} = 0$. So we have constructed the following situation (where a line indicates extensions of cuts):

$$\begin{tikzcd}
\hat{q} \arrow[d, dash] & V(q)^+ \arrow[d, dash] & q \arrow[d, dash] & \\
\hat{p} \arrow[d, dash] & W^+ \arrow[d, dash] & p \arrow[d, dash] & V(p)^+ \arrow[d, dash] \\
G_1^+ & W_1^+ \arrow[dr, dash] & p_1 \arrow[d, dash] & V_1^+ \arrow[dl, dash] \\
 & & V_0^+ &
\end{tikzcd}$$

Hence q is even an heir of p. Moreover conditions (i)(ii) and (iv) of (9.4) are fulfilled (in particular $(M, G(p)) \equiv_A (N, G(q))$ by (8.6)). But $(M, p^L, V(p)) \models \exists x \; p < x \in V(p)$, $(N, q^L, V(q)) \models \neg\exists x \; q < x \in V(q)$ and $(M, p^L, V(p)) \models \neg\exists x \; V(p) < x < p$, $(N, q^L, V(q)) \models \exists x \; V(q) < x < q$. Hence $(p, V(p))$ is not an heir of $(q, V(q))$ over A and $(q, V(q))$ is not an heir of $(p, V(p))$ over A. $\square$

(10.4) EXAMPLE. Suppose $\operatorname{sign} p = \operatorname{sign} q = 0$, $Z^*(\hat{p}) \cap A = Z^*(\hat{q}) \cap A = \emptyset$, condition (i), (ii), (iii) and (iv)(a) of (9.4) holds. Then in general $(M, p^L) \not\equiv_A (N, q^L)$.

PROOF. We construct the cuts of the following diagram from bottom to top; again a line indicates extensions of cuts:

$$\begin{tikzcd}
W \arrow[d, dash] & & & & & \beta\cdot\mathfrak{m}(W) \arrow[d, dash] & \beta\cdot W \arrow[d, dash] & q \arrow[d, dash] \\
W_1 & {} < \frac{\beta}{\alpha} \in V_2 \arrow[d, dash] & & & & \beta\cdot\mathfrak{m}(W_1) \arrow[dr, dash] & \beta\cdot W_1 \arrow[d, dash] & p' \arrow[d, dash] \\
 & V \arrow[d, dash] & \frac{\alpha}{b}\cdot V \arrow[d, dash] & r \arrow[dl, dash] & \alpha\cdot\mathfrak{m}(V) \arrow[d, dash] & \alpha + r \arrow[d, dash] & \alpha\cdot V \arrow[d, dash] & b\cdot(\alpha + r) =: p \arrow[dd, dash] \\
 & V_1 \arrow[d, dash] & \frac{\alpha}{b}\cdot V_1 & & \alpha\cdot\mathfrak{m}(V_1) \arrow[dr, dash] & \alpha + \frac{\alpha}{b}\cdot V_1 \arrow[d, dash] & \alpha\cdot V_1 \arrow[dl, dash] & \\
 & V_0 & b & & & G_0 & & b\cdot G_0
\end{tikzcd}$$

Let V_0 be a proper convex valuation ring of A and let G_0 be a convex subgroup of A, with $V_0 \subseteq G_0$, $\operatorname{sign}^* G_0^+ = 0$ and $V(G_0) = V_0$. Furthermore let $V_0 < b \in G_0$. Let $\alpha \models G_0^+$ and let V_1 be the convex hull of V_0 in $A\langle\alpha\rangle$. Since $\operatorname{sign}^* G_0^+ = 0$, we have that V_1^+ is the unique cut of $A\langle\alpha\rangle$ extending V_0^+, $\alpha\cdot\mathfrak{m}(V_1)^+$ is the least and $\alpha\cdot V_1{}^+$ is the largest extension of G_0^+ on $A\langle\alpha\rangle$. Moreover $V_1^+ < \frac{\alpha}{b}\cdot V_1^+ < \alpha\cdot\mathfrak{m}(V_1)$.

By (6.2) there is some $M \succ A\langle\alpha\rangle$ and a cut r of M with $\operatorname{sign} r = 0$, such that r extends $\frac{\alpha}{b}\cdot V_1^+$ and such that $G(r)$ is the convex hull of $\frac{\alpha}{b}\cdot V_1$ in M. This means that $G(r) = \frac{\alpha}{b}\cdot V$, where V is the convex hull of V_1 in M. Since r extends $\frac{\alpha}{b}\cdot V_1^+$, the cut $\alpha + r$ extends the cut $\alpha + \frac{\alpha}{b}\cdot V_1^+$. The latter cut is $\alpha\cdot(1 + \frac{1}{b}\cdot V_1^+)$, so as $b > V_1$ we have $\alpha + \frac{\alpha}{b}\cdot V_1^+ < \alpha\cdot V_1^+$. Hence $\alpha{\cdot}\mathfrak{m}(V) < \alpha + r < \alpha{\cdot}V^+$. Let $p := b{\cdot}(\alpha + r)$. Then $G(p) = b{\cdot}G(\alpha + r) = b{\cdot}G(r) = \alpha{\cdot}V$ and $\operatorname{sign} p = 0$.

Let β be a realization of $\alpha\cdot V^+$. Since $\operatorname{sign} p = 0$, the unique cut p' of $M\langle\beta\rangle$ extending p has again signature 0 and $G(p')$ is the largest convex subgroup of $M\langle\beta\rangle$ lying over $G(p) = \alpha\cdot V$. Hence $G(p') = \beta\cdot V_2$, where V_2 is the largest convex valuation ring of $M\langle\beta\rangle$ lying over V. let $W_1 \subsetneqq V_2$ be a convex valuation ring of $M\langle\beta\rangle$ (for example we can take W_1 to be the convex hull of V in $M\langle\beta\rangle$). In particular $\beta\cdot W_1 \subseteq G(p')$.

Finally let $N \succ M$ and let q be a cut of N with $\operatorname{sign} q = 0$ such that $G(q)$ is the convex hull of $\beta{\cdot}W_1$ in N (by (6.2)). Hence $G(q) = \beta{\cdot}W$, where W is the convex hull of W_1 in N.

<u>Claim.</u> $\beta\cdot\mathfrak{m}(W_1)^+$ extends $\alpha\cdot V^+$. It is enough to prove the claim if W_1 is the convex hull of V. Since $W_1 < \varepsilon := \frac{\beta}{\alpha} \in V_2$, we get $\frac{1}{\varepsilon}\cdot\beta\cdot W_1^+ = \alpha\cdot W_1^+$ extends $\alpha\cdot V^+$ and $\frac{1}{\varepsilon}\cdot\beta{\cdot}W_1^+ < \beta{\cdot}\mathfrak{m}(W_1)^+$.
The claim implies that $\beta\cdot\mathfrak{m}(W)^+$ extends $\alpha\cdot V^+$. In particular $\frac{1}{b}\cdot q < \beta\cdot\mathfrak{m}(W)^+$ and $\frac{1}{b}\cdot q\cdot V(q)^+ < \hat{q}$. On the other hand $\frac{1}{b}\cdot p\cdot V(p)^+ = \hat{p}$, which proves

$$(N, q^L) \not\equiv_A (M, p^L)$$

But $(N, G(q)) \equiv_A (M, G(p))$ ($\operatorname{sign}^*_A \alpha V = \operatorname{sign}^*_A \beta W = 0$), q is an heir of p and for all A-definable maps f we have $f(p) < V(p)^+$ iff $f(q) < V(q)^+$ ($\operatorname{sign}^* G_0^+ = 0$). Note also that we can choose W_1 to be the convex hull of V, hence $V(q) = W$ lies over $V(p) = V$, $\hat{q}$ lies over $\hat{p}$ and q is an heir of p.

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