Valuations, deformations, and toric geometry

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To David Rees, on the occasion of his eightieth Birthday,
and Pierre Samuel, on the occasion of his seventy-seventh Birthday

Summary

Given a nœtherian local integral domain R and a valuation ν of its field of fractions which is non negative on R, I study a geometric specialization of R to the graded ring $\mathrm{gr}_\nu R$ determined by the valuation. I show that if the residue field of R is algebraically closed, for 0-dimensional valuations this graded ring corresponds to an essentially toric variety, possibly of infinite embedding dimension, and consider some possible applications of this fact to local uniformization by deformation of a resolution of singularities of the toric variety.

This leads to the study of the ν-adic completion of a nœtherian local domain and of the structure and regularity of certain semigroup algebras and graded algebras associated to valuations. A result on the structure of ν-adically complete equicharacteristic nœtherian local rings and some phenomena of algebraic geometry in finite Krull dimension but countable embedding dimension play an important role.

Contents

1 Introduction

This text is an attempt to clarify the problems which one meets when trying to prove local uniformization of a valuation at a singular point of an algebraic or analytic variety along the lines of what is done in [G-T] for plane branches, by re-embedding the singularity in such a way that the valuation can be uniformized (in an *embedded* manner) by a *single* toric modification of the ambient space. The reader is assumed to be aware of the basics of valuation theory contained in [V], of the theory of toric varieties found in [Cox] and of the results of [G-T].

Given the local ring R of an algebraic variety, and assuming that it is an integral domain with fraction field K, a valuation of R is a valuation ν of K which is non-negative on R. It is also the datum of a valuation ring R_ν of K containing R. The problem of local uniformization is to find a *regular* local R-algebra R', essentially of finite type over R and contained in R_ν. Moreover we wish R' to be obtained by localizing an affine chart of a proper algebraic map $Z \to \mathrm{Spec} R$ which is described as precisely as possible, for example as a composition of blowing ups with non singular centers or as a proper and birational toric map with respect to some system of generators of the maximal ideal of R.

In agreement with the spirit that Herwig Hauser wished to give to the Working Week, this text is somewhat programmatic, even speculative, in nature since there are several statements which may not be in their final form, and some proofs are only sketched. The corresponding propositions are between asterisks.

In higher dimension, the first serious difficulty is that the graded ring $\mathrm{gr}_\nu R$ associated to the filtration of R determined by the valuation ν is not nœtherian in general; its structure depends on the semigroup of the values which the valuation takes on the ring, which is not always finitely generated.

The good news is that for rational valuations, that is for valuations of a local ring R such that the inclusion $R \subset R_\nu$ in the ring R_ν of the valuation induces a trivial residue field extension, the space $\mathrm{Specgr}_\nu R$ is, up to a very simple deformation, a toric variety, of finite Krull dimension but possibly of infinite embedding dimension (see Proposition 6.2); it is this toric variety which generalizes the monomial curve of [G-T]. The valuation ν induces a natural valuation ν_{gr} on $\mathrm{gr}_\nu R$, and the uniformization of ν_{gr} is a combinatorial problem (see §6).

The basic idea is to obtain the uniformisation of a valuation of an excellent integral local ring R as a *deformation* (the opposite of the specialization above, after suitable completions which have to be exorcized) of a *toric* uniformization of the valuation ν_{gr} of the graded ring $\mathrm{gr}_\nu R$. In the case where the residue field of R is algebraically closed, this uniformization of ν_{gr} should contain all the combinatorics associated to the uniformization of ν. Moreover, it is blind to the characteristic.

In the case of plane branches studied in [G-T], we chose to abandon the simplicity of embedding dimension two in exchange for the simplicity of dealing only with a monomial curve in affine $(g+1)$-space, where $g+1$ is the number of generators of the semigroup of the branch (in characteristic zero, g is the number of its Puiseux exponents), because the resolution of singularities of the monomial curve is a purely combinatorial problem. Here I have to accept an infinite embedding dimension because the semigroup of values of our valuation on R is not of finite type in general.

In order to relate the uniformizations of ν and ν_{gr}, the main tool is the *valuation algebra* associated to a valuation ν of a domain R; its spectrum is the total space of a (faithfully flat) specialization of the ring R to $\mathrm{gr}_\nu R$.

Because it is in the nature of generators of the graded algebra to lift to elements of R which are in some sense "coordinates", the new embedding mentioned at the beginning should be described by elements of R whose initial forms in $\mathrm{gr}_\nu R$ generate it as an algebra. However, these elements behave as coordinates adapted to the valuation only in the completion $\hat{R}^\nu$ of R for the topology defined by the valuation, and this is a reason to work at first in $\hat{R}^\nu$; we shall see an avatar of this reason below.

The cases of valuations which are not rational are essentially deformations of this case, as I show using the simple behavior of the valuation algebra under composition of valuations. The reason is that, if we assume that the residue field of R is algebraically closed, for any valuation ν_0 there are rational valuations ν composed with ν_0, of which ν_0 may be deemed to be a deformation.

In fact I have to consider suitably completed valuation algebras, and the key fact for this simple behavior even after completion (see 5.2) is that whenever a valuation ν of a nœtherian analytically irreducible local ring is composed with another valuation ν_1, the completions $\hat{R}^{\nu}$ and $\hat{R}^{\nu_1}$ of R with respect to the topologies defined by ν and ν_1 are isomorphic. Using this one can prove by reduction to the case of a valuation of height one that these completions are quotients of the $\mathbf{p}$-adic completion $\hat{R}^{\mathbf{p}}$ of R, where $\mathbf{p}$ is the center in R of the valuation of height one with which ν is composed. This implies that they are nœtherian, and even henselian (Proposition 5.4), which is crucial (see below).

This work was initially partly motivated by the desire to understand from the viewpoint of [T1] the inspiring attempt of Spivakovsky in [S2] to prove local uniformization and resolution of singularities; because of this change in approach, substantial differences soon appeared, which were outlined in [G-T] for the case of plane branches. The viewpoint developped here is therefore different from that of [S2]. Spivakovsky generalizes to arbitrary valuations (or at least to those of height one) the construction which for plane branches describes resolution as a composite of *plane* toric maps, each of these being in turn a composition of blowing ups of points in the plane (see sections 7 and 8 of [G-T]). The graded algebra $\mathrm{gr}_{\nu}R$ also plays an important, although quite different, role (see the introduction and §7 of [G-T]). Many difficulties in this approach stem from the fact whose avatar for curves is that after each plane toric map one must choose again local coordinates in which the next toric map will be described. I allow from the start a change of embedding, even to infinite dimensions, such that in this new embedding, a *single* toric map (or its appropriate generalization to infinite dimension) will uniformize, and I do not seek, in the first phase, to construct a sequence of blowing-ups with non singular centers.

From a more analytical viewpoint, an important difference with Spivakovsky's and Zariski's approaches is that I do not start from a set of coordinates and equations and try to improve them, then follow them through a sequence of blowing-ups, but I produce directly from the ring of the singularity, given a valuation, a candidate for a "natural" embedding in a (possibly infinite-dimensional) space equipped with a natural class of coordinates $(u_i)_{i\in I}$, liftings of the generators of the graded algebra, and candidates for a "natural" set of equations, which are as simple as one might wish (deformations of binomials) and in particular are "nondegenerate" in the natural coordinates (see Propositions 5.14 and 5.15). A good part of the work is then to show that they truly are coordinates and equations in a useful sense with respect to the given valuation.

This involves completions; it was already apparent in the case of branches studied in [G-T], where R is a one-dimensional complex analytic integral algebra, that one should just effect on the "natural" coordinates the toric modifications which resolve $\mathrm{gr}_{\nu}R$ (no changes of coordinates!) and then one needs to apply the implicit function theorem only once, to prove that smoothness and transversality to the exceptional divisor of the strict transform by the toric modification are preserved when one deforms from $\mathrm{gr}_\nu R$ to R. But apply it one must indeed, and so one must in the general case complete R with respect to the ν-adic topology in order to be able to apply the implicit function theorem or Hensel's lemma.

Finally the nœtherian and henselian properties of the completion $\hat{R}^\nu$ are used again to ensure that the lift to $\hat{R}^\nu$ of finitely many steps of the (possibly infinite) combinatorial process of resolution of $\mathrm{gr}_\nu R$ suffices to uniformize the extension $\hat{\nu}$ of ν to $\hat{R}^\nu$.

Here the main point is that all but finitely many of the equations defining $\mathrm{Spec}\hat{R}^\nu$ in the possibly infinite-dimensional affine space with coordinates $(u_i)_{i\in I}$ serve only to express the coordinates $u_i,\ i > N_0$ in $\hat{R}^\nu$ in terms of the finitely many coordinates $(u_1, \ldots, u_{N_0})$ which suffice to generate its maximal ideal and, by their valuations, to rationally generate the value group. If there were also infinitely many equations not serving this purpose, they would indefinitely add singularities, and the method could not work. As it is, it suffices finally, in spite of the fact that $\mathrm{gr}_\nu R$ may be of a smaller dimension than $\hat{R}^\nu$, that a toric process *partially* resolves $\mathrm{Specgr}_\nu R$ to uniformize $\hat{\nu}$, because of the "abyssal" effect generated by this property of the equations (see the third example of subsection 4.2, subsection 6.5 and subsection 7.2).

defined in R provided that it is excellent. This last step uses the important observation, due (in another language) to Spivakovsky, that if R is excellent, $\mathrm{Spec}\hat{R}^\nu$ is not contained in the singular locus of $\mathrm{Spec}\hat{R}^{\mathbf{P}}$ (see Proposition 5.7).

Another difficulty which appears is the necessity to approximate in a useful way the (possibly infinite-dimensional) toric map which resolves our toric variety by finite-dimensional ones. This is possible thanks to the fact that for any valuation ring R_ν with residue field k_ν, the graded algebra $\mathrm{gr}_\nu R_\nu$ is the direct limit of a filtering (in fact a nested) system of maps between polynomial k_ν-subalgebras in a fixed number of variables, and this system can be described rather explicitly; for archimedian valuations, it is essentially described by Perron's algorithm (see section 4).

Perron's algorithm is not the only fact appearing in different guises here and in Zariski's proof; the classification of the different valuations by height is another example. From the viewpoint exposed here this is due to the fact that Zariski computes in a deformation of the graded algebra which he follows through blowing-ups, while I first compute the "resolution" of the graded algebra and then deform it. This procedure may have other applications.

Yet another difference is that I must go directly to the "worst" valuations from the viewpoint of the value group, the rational ones, to take advantage of the toric nature of the corresponding graded algebra, while Zariski and Spivakovsky both begin by the case of height one (archimedian) valuations (see [Z1], top of page 858). That the center of gravity of the proof, as Zariski puts it, is the uniformization of rational valuations has the great advantage that it makes sense to assume that the residue field is algebraically closed, whereas in Zariski's approach, in order to deal with valuations of height > 1, one localizes, and so one cannot make such an assumption. We may therefore, in this first phase, confine our attention to the case of an algebraically closed residue field. The rational valuations are also the only ones which appear naturally in the complex-analytic case.

I believe that this strategy makes clearer the nature of the problem by separating as much as possible combinatorial problems from choices of coordinates and equations. Once the toric setup is clarified, finding centers of blowing-up in a given embedding of the singularity should be considerably easier.

In the redaction, some facts must be applied to valuation rings R_ν, which are not nœtherian in general, and so parts are written without that assumption. Basically, I try to do here what is needed for local uniformization of an arbitrary valuation on a local nœtherian equicharacteristic excellent analytically irreducible ring with an algebraically closed residue field. The assumption of analytic irreducibility is not difficult to get rid of, while the case where the residue field is not algebraically closed requires other techniques in this approach. In the first sections I assume the existence of a base field, since it is only after ν-adic completion that one may use the existence of a field of representatives (see Proposition 5.5).

It was during Heisuke Hironaka's wonderful Summer meeting at Ohnuma in 1993 that this train of thought started. I wish to thank Antonio Campillo, David Eisenbud, Franz-Viktor Kuhlmann, Mark Spivakovsky and Michel Vaquié for stimulating comments and suggestions. I am especially grateful to Mark Spivakovsky, whose work rekindled the ever burning coals of my interest for the role of the monomial curve and the Riemann-Zariski manifold, and to Franz-Viktor Kuhlmann, Michel Vaquié and the referees for commenting on a very preliminary version. I am also grateful to Nicolas Bourbaki for his incomparably careful and lucid exposition of completion problems.

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2 Valuative sorites

This section[1] introduces the specialization of a ring R to the graded ring associated to a valuation. It generalizes to filtrations indexed by a totally ordered group a construction found, for filtrations indexed by $\mathbf{Z}$, in the work of Gerstenhaber [G1], [G2], Rees [R], and in [T1]. One difference is that I do not use the valuation algebra

[1] Its title is that of an embryo version of this paper. The following was added at the request of a referee: the origin of the word *sorite* is an abbreviation (found e.g. in Galien) of the Greek which refers to a syllogistic type of reasoning based on the accumulation of premises; means "I accumulate". It is used by some mathematicians to designate such an accumulation, adding a slightly depreciatory nuance. According to the Concise Oxford Dictionary, it also evokes "a form of sophism leading by gradual steps from truth to absurdity".

as a formal algebraic construction, asking only whether it is finitely generated or not, but use it as a geometric object.

2.1 The valuation algebra

Recall that a valuation of a commutative field K is a map $\nu\colon K^* \to \Phi$ to a totally ordered commutative group Φ (the value group) such that $\nu(xy) = \nu(x) + \nu(y)$, $\nu(x+y) \geq \inf(\nu(x),\ \nu(y))$, with equality if $\nu(x) \neq \nu(y)$. The element $0 \in K$ has a valuation greater than any element of Φ. The set $\{x \in K \mid \nu(x) \geq 0\}$ is the ring R_ν of the valuation; it is local with maximal ideal $m_\nu = \{x \in K \mid \nu(x) > 0\}$. If the first condition is relaxed to $\nu(xy) \geq \nu(x) + \nu(y)$, one has a *pseudo-valuation*. The trivial valuation has value group $\{0\}$. Let R be an integral domain with field of fractions K and let ν be a valuation of K such that its valuation ring R_ν contains R. Except when there is explicit mention of the contrary, all valuations are assumed to be non trivial. Let us denote by Φ the totally ordered value group of the valuation ν and by k_ν its residue field R_ν/m_ν. Denote by Φ_+ (resp. Φ_-) the semigroup of positive (resp. negative) elements of Φ and set $\Gamma = \nu(R \setminus \{0\}) \subset \Phi_+ \cup \{0\}$; it is the *semigroup* of (R, ν).

For $\phi \in \Phi$, set:

$$\begin{aligned} \mathcal{P}_\phi(R) &= \{x \in R \mid \nu(x) \geq \phi\} \\ \mathcal{P}^+_\phi(R) &= \{x \in R \mid \nu(x) > \phi, \} \end{aligned}$$

where we agree that $0 \in \mathcal{P}_\phi$ for all ϕ, since its value is larger than any ϕ, so that by the properties of valuations the $\mathcal{P}_\phi$ are ideals of R. Note that the intersection $\bigcap_{\phi\in\Phi_+} \mathcal{P}_\phi = (0)$ and that if ϕ is in the negative part Φ_- of Φ, then $\mathcal{P}_\phi(R) = \mathcal{P}^+_\phi(R) = R$. Consider the graded algebra à la Gerstenhaber-Rees (see the papers quoted above), which we may call the *valuation algebra* of (R, ν):

$$\mathcal{A}_\nu(R) = \bigoplus_{\phi\in\Phi} \mathcal{P}_\phi(R) v^{-\phi} \subset R[v^\Phi]$$

where $R[v^\Phi]$ is the group algebra of Φ with coefficients in R, the element $x_\phi[\phi]$ being written $x_\phi v^{-\phi}$.

When there is no ambiguity we shall denote this algebra simply by $\mathcal{A}$. It is a domain, not nœtherian in general even if R is, as we shall see below. Note that when ν is the trivial valuation, with $\Phi = \{0\}$, we have $\mathcal{A}_\nu(R) = R$.

2.2 The valuation $\mu_{\mathcal{A}}$

We now define a valuation $\mu_{\mathcal{A}}$ on $\mathcal{A}$ with value group Φ by

$$\mu_{\mathcal{A}}(\sum x_\phi v^{-\phi}) = \min(\nu(x_\phi) - \phi).$$

By construction of $\mathcal{A}$ this valuation is non-negative on $\mathcal{A}$.
There are several rings connected with $\mathcal{A}_\nu(R)$: its quotient field is the quotient field $K(v^\Phi)$ of $K[v^\Phi]$, and it is not difficult to describe the valuation ring $\mathcal{R}_{\mu_\mathcal{A}}$ of $\mu_\mathcal{A}$; it is the set of quotients $\sum x_\phi v^{-\phi}(\sum y_\psi v^{-\psi})^{-1}$ such that $\min\bigl(\nu(x_\phi)-\phi\bigr)\geq \min\bigl(\nu(y_\psi)-\psi\bigr)$.

We use the notation $\mathcal{P}_\phi(R_\nu)=\{x\in R_\nu \mid \nu(x)\geq\phi\}$ as above, and similarly for $\mathcal{P}^+_\phi(R_\nu)$ and $\mathcal{P}_\phi(K)$, and remark that $\mathcal{P}_\phi(K)$, $\mathcal{P}^+_\phi(K)$ are R_ν-submodules of K, and that for $\phi\in\Phi_+$ we have $\mathcal{P}_\phi(K)=\mathcal{P}_\phi(R_\nu)$. With this notation, we have:

$$\mathcal{R}_{\mu_\mathcal{A}}\cap K[v^\Phi]=\mathcal{A}_\nu(K)=\bigoplus_{\phi\in\Phi_-}\mathcal{P}_\phi(K)v^{-\phi}\bigoplus R_\nu\bigoplus\bigoplus_{\phi\in\Phi_+}\mathcal{P}_\phi(R_\nu)v^{-\phi}$$

and

$$\mathcal{R}_{\mu_\mathcal{A}}\cap R_\nu[v^\Phi]=\mathcal{A}_\nu(R_\nu)=\bigoplus_{\phi\in\Phi}\mathcal{P}_\phi(R_\nu)v^{-\phi}\subset R_\nu[v^\Phi].$$

The maximal homogeneous ideal of $\mathcal{A}_\nu(R_\nu)$ containing m_ν is:

$$\mathcal{M}_\nu=\bigoplus_{\phi\in\Phi_-}R_\nu v^{-\phi}\bigoplus m_\nu\bigoplus\bigoplus_{\phi\in\Phi_+}\mathcal{P}_\phi(R_\nu)v^{-\phi},$$

so that the corresponding residue field is the residue field k_ν of the valuation, while the ideal of elements of positive value is

$$\tilde{\mathcal{Q}}^+_0=\bigoplus_{\phi\in\Phi_-}R_\nu v^{-\phi}\bigoplus m_\nu\bigoplus\bigoplus_{\phi\in\Phi_+}\mathcal{P}^+_\phi(R_\nu)v^{-\phi}.$$

The maximal homogeneous ideal of $\mathcal{A}_\nu(K)$ containing m_ν is

$$\bigoplus_{\phi\in\Phi}\mathcal{P}^+_\phi(K)v^{-\phi}$$

which induces $\tilde{\mathcal{Q}}^+_0$ on $\mathcal{A}_\nu(R_\nu)$. In any case, the ideal of elements of positive $\mu_\mathcal{A}$-value of $\mathcal{A}_\nu(R)$ is

$$\mathcal{Q}^+_0=\bigoplus_{\phi\in\Phi_-}Rv^{-\phi}\bigoplus(m_\nu\cap R)\bigoplus\bigoplus_{\phi\in\Phi_+}\mathcal{P}^+_\phi(R)v^{-\phi},$$

and we remark that it is exactly the ideal of $\mathcal{A}_\nu(R)$ generated by the family of elements (v^{Φ_+}). Indeed, it is generated by elements $x_\phi v^{-\phi}$ with $\nu(x_\phi)>\phi$; but then,

$$x_\phi v^{-\phi}=v^{\nu(x_\phi)-\phi}x_\phi v^{-\nu(x_\phi)}\in(v^{\Phi_+})\mathcal{A}_\nu(R).$$

The other inclusion is clear.

For any ring A, let us denote by $A[v^{\Phi_+}]$ the algebra of the monoid $\Phi_+\cup\{0\}$ with coefficients in A.

2.3 The specialization to the graded algebra

The graded algebra associated with the valuation ν was introduced in ([L-T], [T1]) for the very special case of a plane branch (see [G-T]), and in [S 1], [S 2] in full generality. It is

$$\mathrm{gr}_\nu R = \bigoplus_{\phi\in\Gamma} \mathcal{P}_\phi(R)/\mathcal{P}^+_\phi(R).$$

We remark that it is indexed by the *semigroup* of (R,ν).
For each element $x \in R$, there is a unique $\phi \in \Phi_+ \cup \{0\}$ such that $x \in \mathcal{P}_\phi \setminus \mathcal{P}^+_\phi$; the image of x in the quotient $(\mathrm{gr}_\nu R)_\phi = \mathcal{P}_\phi(R)/\mathcal{P}^+_\phi(R)$ is the *initial form* $\mathrm{in}_\nu(x)$ of x. When there is no ambiguity, I will often denote it by $\overline{x}$.

Remark. 1) Since R is an integral domain and ν is a valuation, the ring $\mathrm{gr}_\nu R$ is an integral domain.
2) If the value group Φ has finite rational rank, it is countable, since it is torsion free and hence the map $\Phi \to \mathbf{Q}\otimes_{\mathbf{Z}} \Phi = \mathbf{Q}^r$ is injective. Therefore the $R/(m_\nu \cap R)$-algebra $\mathrm{gr}_\nu R$ has a countable system of generators, and so does the k_ν-algebra $\mathrm{gr}_\nu R_\nu$.
3) If the ring R is nœtherian, each $\mathcal{P}_\phi(R)/\mathcal{P}^+_\phi(R)$ is an $R/(m_\nu \cap R)$-module of finite type.
4) If R is a local ring with maximal ideal m, by definition of domination, the valuation ring R_ν dominates R if $m_\nu \cap R = m$, and we will sometimes, but not always, be in that situation. Then $\mathrm{gr}_\nu R$ is a k_R-algebra, where k_R is the residue field of R. If we want to emphasize that R and R_ν have the same field of fractions, we say that R_ν *birationally dominates* R.

Let us denote by $(v^{\Phi_+})\mathcal{A}_\nu(R)$ the ideal generated in $\mathcal{A}_\nu(R)$ by the elements of (v^{Φ_+}).

Proposition 2.1. *a) The natural map*

$$\mathcal{A}_\nu(R) \longrightarrow \mathrm{gr}_\nu R$$

defined by

$$x_\phi v^{-\phi} \mapsto x_\phi \mod.\mathcal{P}^+_\phi(R)$$

induces an isomorphism of graded rings

$$\mathcal{A}_\nu(R)/\mathcal{Q}^+_0 = \mathcal{A}_\nu(R)/(v^{\Phi_+})\mathcal{A}_\nu(R) \longrightarrow \mathrm{gr}_\nu R.$$

b) The natural inclusion $R[v^{\Phi_+}] \longrightarrow \mathcal{A}$ obtained by considering only the part of negative degree (i.e., $\phi \in \Phi_-$) of $\mathcal{A}$ induces, after taking rings of fractions by the multiplicative subset (v^{Φ_+}), an isomorphism

$$(v^{\Phi_+})^{-1}R[v^{\Phi_+}] \longrightarrow (v^{\Phi_+})^{-1}\mathcal{A}_\nu(R);$$

the quotient $\mathcal{A}_\nu(R)/R[v^{\Phi_+}]$ is a torsion $\mathbf{Z}[v^{\Phi_+}]$-module.

c) Given a homomorphism $\phi \mapsto e(\phi)$ from Φ to the multiplicative group of units of R, it induces a surjection $\mathcal{A}_\nu(R) \to R$ defined by $\sum x_\phi v^{-\phi} \mapsto \sum x_\phi e(-\phi)$. The kernel of this surjection is the ideal $\big((v^\phi - e(\phi))_{\phi\in\Phi_+}\big)\mathcal{A}_\nu(R)$. If R contains a field k, this is true in particular for homomorphisms $\Phi \to k^*$, and in all cases it is enough to define the homomorphism on Φ_+.

Proof. the map in a) is clearly surjective, and its kernel is homogeneous and generated by elements $x_\phi v^{-\phi}$ with $\nu(x_\phi) > \phi$; but we saw in 2.2 that it is $(v^{\Phi_+})\mathcal{A}_\nu(R) = \mathcal{Q}_0^+$.
To prove b), consider an element $v^{-\psi}\sum_{\phi\in\Phi_-} x_\phi v^{-\phi}$ of the ring of fractions $(v^{\Phi_+})^{-1}R[v^{\Phi_+}]$, and rewrite it as $v^{-\psi}\sum_{\phi\in\Phi_-} v^{-\nu(x_\phi)} x_\phi v^{-\phi+\nu(x_\phi)}$. Since $-\phi \geq 0$ and $\nu(x_\phi) \geq 0$, this is clearly an element of $(v^{\Phi_+})^{-1}\mathcal{A}_\nu(R)$, and every element can be written in this way; if $\sum x_\phi v^{-\phi} \in \mathcal{A}_\nu(R)$, let ϕ_+ be the largest degree appearing in this sum. If it is negative, the element is in $R[v^{\Phi_+}]$. Otherwise, rewrite the sum as $v^{-\phi_+}\sum x_\phi v^{-(\phi-\phi_+)} \in (v^{\Phi_+})^{-1}R[v^{\Phi_+}]$. This proves both assertions of b). Assertion c) is easily verified. ○

To interpret this geometrically, let us assume in this section that R contains[2] a field k ; we have a composed map of algebras $k[v^{\Phi_+}] \to R[v^{\Phi_+}] \to \mathcal{A}_\nu(R)$. Then:

Proposition 2.2. *The $k[v^{\Phi_+}]$-algebra $\mathcal{A}_\nu(R)$ is faithfully flat.*

Proof. The criterion for faithful flatness of [B3], Chap. I. §3, No. 7, Proposition 13 is that every solution $(y_k,\ 1 \leq k \leq n)$, $y_k \in \mathcal{A}_\nu(R)$, of a system

$$(*) \qquad \sum_{k=1}^{n} c_{ki} y_k = d_i \quad (1 \leq i \leq m)$$

of linear equations with coefficients and right-hand sides in $k[v^{\Phi_+}]$ can be written in the form

$$y_k = x_k + \sum_{j=1}^{p} z_{jk} b_j \quad (1 \leq k \leq n)$$

where (x_k) is a solution of $(*)$ in $k[v^{\Phi_+}]$, the b_j belong to $\mathcal{A}_\nu(R)$ and the z_{jk} for each fixed j are solutions in $k[v^{\Phi_+}]$ of the homogeneous system associated to $(*)$. Writing $y_k = \sum y_{k\psi} v^{-\psi}$, we notice that the elements with $\nu(y_{k\psi}) > 0$ must disappear in the equation $(*)$. Our equation splits into two systems:
Let us denote by $y_k^{(0)}$ the sum $\sum_{\{(k,\psi)|\nu(y_{k\psi})=0\}} y_{k\psi} v^{-\psi}$ and by $y_k^{(1)}$ the difference $y_k - y_k^{(0)}$. Our system is equivalent to

$$\text{a)} \ \sum_k c_{ki} y_k^{(0)} = d_i \quad , \qquad \text{b)} \ \sum_k c_{ki} y_k^{(1)} = 0 \quad (1 \leq i \leq m).$$

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[2]In this paper, whenever we introduce a base field k, it will be tacitly assumed that the valuations studied are trivial on k; whenever we have inclusions $k \subset R \subset R_\nu$, the intersection $m_\nu \cap k$ is (0).

We may view each system as a system of linear equations with unknowns $y_{k\psi} \in R$ and coefficients and right-hand side in k, since they are the coefficients of the polynomials $c_{ki},\ d_i \in k[v^{\Phi_+}]$. Since R is a faithfully flat k-algebra (or by invoking linear algebra), we see that there are elements $x_k^{(0)},\ w_{kp\psi}^{(0)} \in k$ and $b_p^{(0)} \in R$ such that

$$y_{k\psi}^{(0)} = x_{k\psi}^{(0)} + \sum_p w_{kp\psi}^{(0)} b_p^{(0)},$$

and

$$\sum_k c_{ki}(\sum_\psi x_{k\psi}^{(0)} v^{-\psi}) = d_i,\ \sum_k c_{ki}(\sum_\psi w_{kp\psi}^{(0)} v^{-\psi}) = 0 \qquad \forall p.$$

Let us now consider the second system; here we have to deal with restrictions on the valuations of the elements; a solution of this system defines a finite dimensional k-vector subspace V of R, generated by the coefficients $y_{k\psi}^{(1)}$ of the solutions $y_k^{(1)}$. Let us choose a basis $(b_q)_{1\leq q\leq r}$ of this vector space which is compatible with the finite filtration of V by the vector subspaces $\mathcal{P}_\phi \cap V$. We can now write

$$y_{k\psi}^{(1)} = \sum_{q=1}^{r} w_{kq\psi}^{(1)} b_q \text{ with } \ w_{kq\psi}^{(1)} = 0 \text{ if } \nu(b_q) < \psi,$$

and the coefficients $w_{kq\psi}^{(1)}$ satisfy the system of linear equations

$$\sum c_{ki}(\sum_\psi w_{kq\psi}^{(1)} v^{-\psi}) = 0 \quad \forall q.$$

This means that we may write

$$y_k^{(1)} = \sum_q (\sum_\psi w_{kq\psi}^{(1)} v^{\nu(b_q)-\psi}) b_q v^{-\nu(b_q)},$$

and this is indeed a combination of elements of $\mathcal{A}_\nu(R)$ with coefficients in $k[v^{\Phi_+}]$ satisfying b). Adding up the decompositions: $y_k = y_k^{(0)} + y_k^{(1)}$ after remarking that the only ψ's which appear in $y_k^{(0)}$ have to be ≤ 0, we see that we have satisfied the criterion for faithful flatness. ○

Remark. Let $(\xi_i)_{i\in I}$ be elements of R such that their images (initial forms) in $\mathrm{gr}_\nu R$ generate it as a $R/(m_\nu \cap R)$-algebra. Let us denote by S the k-subalgebra of R generated by the elements $(\xi_i)_{i\in I}$ and let $\mathcal{B}_\nu(S)$ the sub $k[v^{\Phi_+}]$-algebra of $\mathcal{A}_\nu(S)$ generated by the elements $\xi_i v^{-\nu(\xi_i)}$. The proof we have just seen can be adapted to show that the $k[v^{\Phi_+}]$-algebra $\mathcal{B}_\nu(S)$ is faithfully flat.

The inclusion $k[v^{\Phi_+}] \to \mathcal{A}$ defines a map of schemes

$$\mathrm{Spec}\mathcal{A} \longrightarrow \mathrm{Spec}k[v^{\Phi_+}].$$

Part a) of Proposition 2.1 is equivalent to the fact that the special fiber of this map is $\mathrm{Spec}\,\mathrm{gr}_\nu R$ and parts b) and c) are ways of stating that its generic and general fibers are isomorphic to $\mathrm{Spec}R$. Since this map is faithfully flat by Proposition 2.2, we can say that we have made R appear as a deformation of its associated graded ring, in a family parametrized by$\mathrm{Spec}k[v^{\Phi_+}]$.

The center of the valuation $\mu_{\mathcal{A}}$ on $\mathcal{A}_\nu$ is the generic point of the subscheme $\mathrm{Spec}\,\mathrm{gr}_\nu R$ of $\mathrm{Spec}\mathcal{A}$; as we saw, this subscheme is integral since ν is a valuation and not a pseudo-valuation.

Let us now compute the associated graded ring of $\mathcal{A}$ with respect to the valuation $\mu_{\mathcal{A}}$. We set $\mathcal{P}_\phi = \mathcal{P}_\phi(R)$ and

$$\mathcal{Q}_\delta = \{a \in \mathcal{A} \mid \mu_{\mathcal{A}}(a) \geq \delta\};$$

it is a graded ideal generated by the $\{x_\phi v^{-\phi} \mid \nu(x_\phi) - \phi \geq \delta\}$, and similarly for $\mathcal{Q}_\delta^+$. So we have

$$\mathcal{Q}_\delta = \mathcal{A} \cap \bigoplus_{\phi\in\Phi} \mathcal{P}_{\phi+\delta} v^{-\phi},$$

$$\mathcal{Q}_\delta^+ = \mathcal{A} \cap \bigoplus_{\phi\in\Phi} \mathcal{P}_{\phi+\delta}^+ v^{-\phi}.$$

So $\mathcal{Q}_\delta = \mathcal{A}$ for $\delta \leq 0$ and we can write

$$\mathrm{gr}_{\mu_{\mathcal{A}}}\mathcal{A} = \bigoplus_{\delta\in\Phi_+\cup\{0\}} (\mathcal{Q}_\delta/\mathcal{Q}_\delta^+) = \bigoplus_{\delta\in\Phi_+} \Big(\bigoplus_{\{\phi \mid \phi+\delta\in\Phi_+\cup\{0\}\}} \mathcal{P}_{\phi+\delta}/\mathcal{P}_{\phi+\delta}^+ \Big).$$

Thus $\mathrm{gr}_{\mu_{\mathcal{A}}}\mathcal{A}$ has a $\Phi_+ \oplus \Phi_+$-graduation (one must also add $\{0\}$ to the semigroups) if we give degree $\phi+\delta$ to the elements of $\mathcal{P}_{\phi+\delta}/\mathcal{P}_{\phi+\delta}^+$ as suggested by the graduation of $\mathcal{A}$. Let us denote by $\overline{R}$ the quotient $R/(m_\nu \cap R)$, and remark that $\mathrm{gr}_\nu R$ is an $\overline{R}$-algebra.

We have a map of $\overline{R}$-algebras

$$\mathrm{gr}_{\mu_{\mathcal{A}}}\mathcal{A} \to (\mathrm{gr}_\nu R) \otimes_{\overline{R}} \overline{R}[v^{\Phi_+}]$$

induced by the map $\mathcal{Q}_\delta \to (\mathrm{gr}_\nu R) \otimes_{\overline{R}} \overline{R}[v^{\Phi_+}]$ defined by:

$$\sum_{\nu(x_\phi)-\phi\geq\delta} x_\phi v^{-\phi} \mapsto \sum_{\nu(x_\phi)-\phi=\delta} \overline{x_\phi} \otimes v^\delta,$$

where $\overline{x_\phi}$ is the initial form in $\mathcal{P}_{\phi+\delta}/\mathcal{P}_{\phi+\delta}^+$ of x_ϕ. This map is homogeneous with respect to the $\Phi_+ \oplus \Phi_+$ graduation of $\mathrm{gr}_{\mu_{\mathcal{A}}}\mathcal{A}$ described above if we give $(\mathrm{gr}_\nu R) \otimes_{\overline{R}} \overline{R}[v^{\Phi_+}]$ the tensor product graduation where elements of degree $\delta \oplus \psi \in \Phi_+ \oplus \Phi$ are tensor products with v^δ of elements of degree ψ of $\mathrm{gr}_\nu R$.

Proposition 2.3. *With the gradings just described, this map is a ($\Phi \oplus \Phi_+$)- graded isomorphism of $\overline{R}$-algebras*

$$\mathrm{gr}_{\mu_{\mathcal{A}}}\mathcal{A} \to (\mathrm{gr}_\nu R) \otimes_{\overline{R}} \overline{R}[v^{\Phi_+}].$$

Proof. The map is clearly surjective, and its injectivity is easily verified degree by degree. Note that upon taking $\phi + \delta = 0$, this map induces the identity of $\overline{R}[v^{\Phi_+}]$. Note also that the valuation $\mu_{\mathcal{A}}$ takes its positive values in Φ_+, and not in Γ. ○

Remark. The meaning of this Proposition is that the deformation of graded algebras corresponding to the deformation of rings encoded by $\mathcal{A}_\nu$ and to $\mu_{\mathcal{A}}$ is "trivial" in the sense that the total space $\mathrm{Specgr}_{\mu_{\mathcal{A}}}$ is the product by $\mathrm{Spec}\overline{R}[v^{\Phi_+}]$ of the special fiber $\mathrm{Specgr}_\nu R$.

2.4 The valuation $\nu_{\mathcal{A}}$

There is another natural valuation on the ring $\mathcal{A}_\nu(R)$: it is the valuation $\nu_{\mathcal{A}}$ defined by

$$\nu_{\mathcal{A}}(\sum x_\phi v^{-\phi}) = \min(\nu(x_\phi)).$$

Its center is the subscheme of $\mathrm{Spec}\mathcal{A}_\nu(R)$ defined by the ideal of $\mathcal{A}_\nu(R)$ generated by $(m_\nu \cap R, \bigoplus_{\phi \in \Phi_+} \mathcal{P}_\phi v^{-\phi})$. This subscheme is isomorphic to $\mathrm{Spec}\overline{R}[v^{\Phi_+}]$. In the special case where R is a local k-algebra dominated by R_ν and the extension $k \to R/m$ is trivial, this means that the center of $\nu_{\mathcal{A}}$ is the image of a section

$$\sigma\colon \mathrm{Spec}k[v^{\Phi_+}] \to \mathrm{Spec}\mathcal{A}_\nu(R).$$

Let us examine the valuation algebra of $\mathcal{A}_\nu(R)$ with respect to $\nu_{\mathcal{A}}$. Set

$$\mathcal{S}_\delta = \{\sum x_\phi v^{-\phi} \in \mathcal{A}_\nu(R) \mid \min\nu(x_\phi) \geq \delta\} = \bigoplus_{\phi \in \Phi} \mathcal{P}_{\max(\delta,\phi)}(R)v^{-\phi},$$

$$\mathcal{S}_\delta^+ = \{\sum x_\phi v^{-\phi} \in \mathcal{A}_\nu(R) \mid \min\nu(x_\phi) > \delta\} = \bigoplus_{\phi \in \Phi}(\mathcal{P}_\delta^+(R) \cap \mathcal{P}_\phi(R))v^{-\phi};$$

we have the equalities

$$\mathrm{gr}_{\nu_{\mathcal{A}}}\mathcal{A}_\nu(R) = \bigoplus_{\delta \in \Phi} \mathcal{S}_\delta/\mathcal{S}_\delta^+ = \bigoplus_{\phi \in \Phi}(\bigoplus_{\delta \geq \phi}(\mathcal{P}_\delta(R)/\mathcal{P}_\delta^+(R))).$$

Let us define the "degree algebra" of a graded algebra $H = \bigoplus_{\delta \in \Phi_+ \cup \{0\}} H_\delta$ in a way similar to the definition of the valuation algebra, replacing valuation by degree, and setting $H_\delta = 0$ for $\delta \in \Phi_-$

$$\mathcal{A}_{\deg}(H) = \bigoplus_{\phi \in \Phi}(\bigoplus_{\delta \geq \phi} H_\delta)v^{-\phi} \subset H[v^\Phi].$$

This algebra is "trivial" in view of the

Lemma 2.4. *The map*

$$\mathcal{A}_{\deg} \to H \otimes_{H_0} H_0[v^{\Phi_+}]$$

determined by

$$\sum_{\phi}(\sum_{\delta \geq \phi} x_\delta) v^{-\phi} \mapsto \sum_{\phi,\delta \in \Phi,\ \delta \geq \phi} x_\delta \otimes_{H_0} v^{\delta-\phi}$$

is a graded isomorphism of H_0-algebras when one gives the right hand side the twisted $\Phi \oplus \Phi_+$ graduation for which a homogeneous element of degree $\delta \oplus \psi$ is of the form $x_\delta \otimes v^{\delta-\psi}$ where x_δ is of degree δ.

Proof. Indeed, the inverse map is given by $x_\delta \otimes v^\psi \mapsto x_\delta v^{-(\delta-\psi)}$. ○

In order to show that the valuation algebra $\mathrm{gr}_{\nu_\mathcal{A}} \mathcal{A}_\nu(R)$ is "trivial", as in the case of the valuation $\mu_\mathcal{A}$, it only remains to check the obvious fact that

Remark. If we view $\mathrm{gr}_\nu(R)$ as a graded algebra indexed by $\Phi_+ \cup \{0\}$ with a 0 component for $\phi \notin \Gamma$, we have the equality

$$\mathrm{gr}_{\nu_\mathcal{A}} \mathcal{A}_\nu(R) = \mathcal{A}_{\deg}(\mathrm{gr}_\nu(R)).$$

Finally, we have proved

Proposition 2.5. *With the twisted grading on $(\mathrm{gr}_\nu R) \otimes_{\overline{R}} \overline{R}[v^{\Phi_+}]$ described above, the map of Lemma 2.4 is a graded isomorphism*

$$\mathrm{gr}_{\nu_\mathcal{A}} \mathcal{A}_\nu(R) \to (\mathrm{gr}_\nu R) \otimes_{\overline{R}} \overline{R}[v^{\Phi_+}],$$

determined by the fact that the $\nu_\mathcal{A}$-initial form of $\xi v^{-\nu(\xi)}$ is mapped to $\overline{\xi} \otimes 1$, where $\overline{\xi}$ is the ν-initial form of ξ.
Note that

$$\mathcal{A}_\nu(R)/\mathcal{S}_0 = \overline{R}[v^{\Phi_+}] = \overline{R} \otimes_{\overline{R}} \overline{R}[v^{\Phi_+}].$$

Remark. 1) The isomorphisms of the two graded algebras of $\mathcal{A}_\nu(R)$ with respect to $\mu_\mathcal{A}$ and $\nu_\mathcal{A}$ with $(\mathrm{gr}_\nu R) \otimes_{\overline{R}} \overline{R}[v^{\Phi_+}]$ have quite different geometric significations, as one can see by considering the analogous constructions for the Rees algebra $\mathcal{R}(I) = \bigoplus_{n \in \mathbf{Z}} I^n v^{-n} \subset R[v, v^{-1}]$ of an ideal I of a ring R, where $I^n = R$ for $n \leq 0$. The valuation μ corresponds to the v-adic filtration; it is the (pseudo)valuation associated to this filtration, and the valuation ν corresponds to the (pseudo)valuation induced by the I-adic filtration of $R[v, v^{-1}]$. The triviality of the $\mathrm{gr}_{\mu_\mathcal{A}} \mathcal{A}_\nu(R)$ generalizes the fact that since the special fiber $\mathrm{Specgr}_I R$ in this case is a divisor, its normal cone in $\mathrm{Spec}\mathcal{R}(I)$ is $\mathrm{Spec}(\mathrm{gr}_I R)[v]$. The triviality for $\nu_\mathcal{A}$ generalizes the fact, more specific to the Rees algebra, that the graded ring with respect to the I-adic filtration is trivial.

2) We shall see below in subsection 3.2 that there is a valuation $\tilde{\nu}_{\mathcal{A}}$ on $\mathcal{A}_\nu(R)$ which is composed with the natural valuation ν_{gr} on the graded ring $\mathrm{gr}_\nu R$, defined by

$$\nu_{\mathrm{gr}}(\sum \overline{x_\phi}) = \min(\nu(x_\phi)),$$

where $x_\phi \in R$ is a representative of $\overline{x_\phi}$, and the valuation does not depend on its choice. Moreover, this valuation $\tilde{\nu}_{\mathcal{A}}$ induces in a natural way the valuation ν on R. This will gives a precise meaning to the intuition that ν_{gr} is the specialization of ν to $\mathrm{gr}_\nu R$.

3) The graded ring $\mathrm{gr}_{\nu_{\mathrm{gr}}}\mathrm{gr}_\nu R$ is isomorphic to $\mathrm{gr}_\nu R$.

An example Let R be a discrete valuation ring, and π a generator of its maximal ideal. The value group of the valuation of R is $\mathbf{Z}$ and we have $\mathcal{P}_n(R) = \pi^n R$ for $n \geq 0$, $\mathcal{P}_n(R) = R$ for $n \leq 0$, so that

$$\mathcal{A}_\nu(R) = R[v, \pi v^{-1}] \subset R[v, v^{-1}].$$

In this case, $\mathrm{gr}_\nu R = (R/\pi R)[t]$, where t is the initial form of π. I leave it as an exercise to determine the maps $\mathcal{A}_\nu(R) \to R$ and $\mathcal{A}_\nu(R) \to \mathrm{gr}_\nu R$ in this case (hint: map πv^{-1} to t). This applies in particular when $R = \mathbf{C}\{t\}$, as we shall see in subsection 5.4, or to $k[[t]]$ where k is any field. It also applies to the non-equicharacteristic case, for example to $R = \hat{\mathbf{Z}}_p$, the ring of p-adic integers, and more generally when $R = W(k)$ is the ring of Witt vectors associated to a perfect field k of characteristic p, with $\pi = p$ (see [B3], Chap. IX, §§1,2, [Ei], §7). We shall see below in subsection 5.2 that the Spec of the p-adic completion of $\mathcal{A}_\nu(W(k))$ is the total space of a deformation of $\mathrm{Spec}k[[t]]$ to $\mathrm{Spec}W(k)$. This is the very first step in the extension of the method presented here to the non-equicharacteristic case.

3 Valuation algebras and composition of valuations

In this section I examine the behavior of valuation algebras when one changes a valuation for an other one with which it is composed, or for the corresponding residual valuation.

3.1 The composition of valuations

In this paragraph, we study the relation of composition of valuations; if ν is composed with ν_1, both non-negative on our local ring R, it means that we have inclusions of local rings

$$R \subset R_\nu \subset R_{\nu_1},$$

and of ideals

$$m_{\nu_1} \subset m_\nu \subset R_\nu,$$

$$m_{\nu_1} \cap R \subseteq m_\nu \cap R \subseteq m.$$

We assume known the theory of composition of valuations, referring to ([V], section 4, [Z-S], Vol 2, Ch. VI, [A1], [B2]), and fix notations.

Let us recall that the rank $\mathrm{r}(\Phi)$ of the $\mathbf{Q}$-vector space $\mathbf{Q} \otimes_{\mathbf{Z}} \Phi$ is called the *rational rank* of Φ; if Φ is the group of a valuation ν, it is also denoted by $\mathrm{r}(\nu)$. Recall also that in the case where R_ν dominates R, the transcendence degree $\mathrm{t}_{k_R}(\nu)$ of the extension $(k_\nu : k_R)$ is called the residual transcendence degree, and that the *height* $\mathrm{h}(\nu)$ (often also called the *rank*) of the valuation ν is the number (in general the ordinal type) of elements of a maximal chain

$$R_\nu \subset R_{\nu_1} \subset \cdots \subset R_{\nu_{h-1}}$$

of valuation rings of K containing R_ν; it is also the Krull dimension of the ring R_ν. Each inclusion $R_\nu \subset R_{\nu_i}$ corresponds to an monotone (non-decreasing) map of totally ordered groups $\lambda_i \colon \Phi \to \Phi_i$ such that $\nu_i = \lambda_i \circ \nu$. A monotone non-decreasing map means here that if $\phi \leq \psi$, then $\lambda(\phi) \leq \lambda(\psi)$. The kernel Ψ_i of such a map is an isolated (or *convex*) subgroup of Φ, so that $\mathrm{h}(\nu)$ is also the ordinal type of the chain of isolated subgoups of Φ; it depends only on Φ and we will speak of the height of Φ.

A totally ordered group of height one is archimedian and can be embedded as an ordered subgroup of $\mathbf{R}$. Following [A1], we will say that a group of height one is *discrete* if it is well ordered, and we will say that a group Φ of finite height is discrete if the quotient of two successive elements of the sequence

$$(0) \subset \Psi_{h-1} \subset \cdots \subset \Psi_1 \subset \Phi$$

of the isolated subgroups of Φ is discrete.

We have the inequality

$$\mathrm{h}(\nu) \leq \mathrm{r}(\nu)$$

and for a valuation ring R_ν birationally dominating a nœtherian local ring R, we have *Abhyankar's inequality* (see [V], section 9, Th. 9.2)

$$\mathrm{r}(\nu) + \mathrm{t}_{k_R}(\nu) \leq \dim R.$$

Abhyankar's inequality has a nice interpretation in terms of the graded algebra

Proposition 3.1. (Piltant, [P]) *If R is a nœtherian local domain with uncountable residue field k_R, birationally dominated by the valuation ring R_ν, the Krull dimension of* $\mathrm{gr}_\nu R$ *is equal to* $\mathrm{r}(\nu) + \mathrm{t}_{k_R}(\nu)$. *If* $\mathrm{t}_{k_R}(\nu) = 0$, *the assumption on the cardinality is unnecessary.*

In this case Abyankar's inequality becomes

$$\dim \mathrm{gr}_\nu R \leq \dim R.$$

Note that if we have $R \subset R_\nu$, the local ring R_ν dominates $R_{m_\nu \cap R}$.

Spivakovsky had remarked that the transcendence degree over R/m of $\mathrm{gr}_\nu R$ is equal to $\mathrm{r}(\nu)+\mathrm{t}_{k_R}(\nu)$ (see [S 3], Remark 3.7).

Remark. I will use Piltant's result only in the case where $\mathrm{t}_{k_R}(\nu)=0$. For the convenience of the reader, I sketch the proof of Piltant's result in this case. In the beginning, I suppose only that $m_\nu\cap R=m$, i.e. that the valuation ring R_ν dominates R. Let S be the multiplicative set of homogeneous elements of positive degree in $\mathrm{gr}_\nu R$, and let $[F]_0$ denote the part of degree zero of an homogenous ring of fractions F of a graded algebra by a multiplicative set of homogeneous elements.

Lemma 3.2. *The map*

$$[S^{-1}\mathrm{gr}_\nu R]_0\to k_\nu,\quad \frac{\overline{x}_\phi}{\overline{y}_\phi}\mapsto\frac{x_\phi}{y_\phi}\mathrm{mod}.m_\nu$$

is well defined and is an isomorphism.

Proof. This follows from a direct verification. ○

Let now $(u_1,\dots,u_r)$ be elements of R whose valuations $(\delta_1,\dots,\delta_r)$ rationally generate the group Φ, where r is the rational rank $\mathrm{r}(\nu)$ of Φ; this means that $(\delta_1,\dots,\delta_r)$ form a basis of the $\mathbf{Q}$-vector space $\Phi\otimes_{\mathbf{Z}}\mathbf{Q}$. Let $\Gamma^{(u)}=\mathbf{Z}\delta_1\oplus\cdots\oplus\mathbf{Z}\delta_r\subseteq\Phi$ be the subgroup generated by the δ_i; set $\Gamma^{(u)}_+=\Gamma^{(u)}\cap\Phi_+$ and define the subalgebra

$$\mathrm{gr}^{(u)}_\nu R=\bigoplus_{\phi\in\Gamma^{(u)}_+}\mathcal{P}_\phi(R)/\mathcal{P}^+_\phi(R).$$

Let us denote by $S^{(u)}$ the intersection $S\cap\mathrm{gr}^{(u)}_\nu R$ and by $k^{(u)}$ the induced field $[(S^{(u)})^{-1}\mathrm{gr}^{(u)}_\nu R]_0$. Remember that an injection $A\to B$ of commutative rings is said to be radicial if for each $y\in B$ there exists $n\in\mathbf{N}$ such that $y^n\in A$; then the map $\mathrm{Spec}B\to\mathrm{Spec}A$ is an homeomorphism; in particular A and B have the same Krull dimension.

Lemma 3.3. *The extensions* $\mathrm{gr}^{(u)}_\nu R\to\mathrm{gr}_\nu R$ *and* $k^{(u)}\to k_\nu$ *are radicial.*

Proof. It suffices to check that any homogeneous element $x_\phi v^{-\phi}\in\mathrm{gr}_\nu R$ has a power which is $\mathrm{gr}^{(u)}_\nu R$. Since $\Gamma^{(u)}$ rationally generates Φ, there is an integer n such that $n\phi\in\Gamma^{(u)}$, so that $x^n v^{-n\phi}\in\mathrm{gr}^{(u)}_\nu R$. ○

Given an element $\phi\in\Gamma^{(u)}$, it can be written uniquely as

$$\phi=a_1\delta_1+\cdots+a_r\delta_r\quad\text{with } a_i\in\mathbf{Z}.$$

Lemma 3.4. *With the notation just introduced, the map*

$$\iota_u:\mathrm{gr}^{(u)}_\nu R\to k^{(u)}[t^{\Gamma^{(u)}_+}],\quad \overline{x}_\phi\mapsto\frac{\overline{x}_\phi}{\overline{u}_1^{a_1}\dots\overline{u}_r^{a_r}}t^\phi$$

induces an isomorphism

$$(S^{(u)})^{-1}\mathrm{gr}_\nu^{(u)}R \to k^{(u)}[x_1^{\pm 1},\ldots,x_r^{\pm 1}].$$

If k_ν is algebraic over k_R, the map ι_u is integral.

Proof. For the first part, since $\Gamma^{(u)}$ is a free abelian group of rank r, it suffices to show that $(S^{(u)})^{-1}\mathrm{gr}_\nu^{(u)}R \to k^{(u)}[t^{\Gamma^{(u)}}]$ is an isomorphism. Remarking that $\iota_u(S^{(u)}) \subseteq \{\lambda t^\gamma,\ \gamma \in \Gamma_+^{(u)},\ \lambda \in k^{(u)*}\}$, we see that $(S^{(u)})^{-1}\mathrm{gr}_\nu^{(u)}R \subset k^{(u)}[t^{\Gamma^{(u)}}]$. The surjectivity is easy to check.
If the extension $k_R \to k_\nu$ is algebraic, so is $k_R \to k^{(u)}$. To prove that ι_u is integral, it suffices to show that any element λt^γ with $\lambda \in k^{(u)},\ \gamma \in \Gamma^{(u)}$ is integral. This follows from the fact that λ is algebraic over k_R. ○

From these lemmas we deduce

Proposition 3.5. (Spivakovsky, Piltant) *The transcendence degree over k_R of $\mathrm{gr}_\nu R$ is equal to $\mathrm{r}(\nu) + \mathrm{t}_{k_R}(\nu)$.*

Proof. Because the extension is radicial, this transcendence degree is equal to that of $\mathrm{gr}_\nu^{(u)}R$, hence to that of $(S^{(u)})^{-1}\mathrm{gr}_\nu^{(u)}R$, which is that of $k^{(u)}[t^{\Gamma^{(u)}}]$ and also of $k_\nu[t^{\Gamma^{(u)}}]$ since the extension $k^{(u)} \to k_\nu$ is radicial. ○

Now Piltant uses the following corollary to Cohen's dimension inequality

Proposition 3.6. *Let B be an integral domain containing a nœtherian ring A. Then*

$$\dim B \le \dim A + \mathrm{t}_A(B),$$

where $\mathrm{t}_A(B)$ is the transcendence degree of B over A.

Proof. The proof is like the classical one (see [Ei], Chap. 13) except that we do not suppose that B is finitely generated over A. But if $(0) = \mathbf{p}_0 \subset \mathbf{p}_1 \subset \mathbf{p}_2 \subset \cdots \subset \mathbf{p}_d$ is a chain of distinct prime ideals of B, choosing for each $i,\ 1 \le i \le d$ an element $y_i \in \mathbf{p}_i \setminus \mathbf{p}_{i-1}$, we see that the $\mathbf{p}_i \cap A[y_1,\ldots,y_d]$ form a chain of distinct prime ideals, so that $\dim A[y_1,\ldots,y_d] \ge d$, and we are reduced to the case of a finitely generated algebra. ○

Applying this result with $A = k_R$ and $B = \mathrm{gr}_\nu R$, remarking that we have used the nœtherianity of R only to ensure that $\mathrm{r}(\nu) + \mathrm{t}_{k_R}(\nu) < \infty$ and using Proposition 3.5 gives the first part of the

Proposition 3.7. (Piltant, [P])*) Whenever the valuation ring R_ν birationally dominates the local ring R and $\mathrm{r}(\nu) + \mathrm{t}_{k_R}(\nu) < \infty$, the following inequality holds*

$$\dim \mathrm{gr}_\nu R \le \mathrm{r}(\nu) + \mathrm{t}_{k_R}(\nu).$$

If $\mathrm{t}_{k_R}(\nu) = 0$,

$$\dim \mathrm{gr}_\nu R = \mathrm{r}(\nu).$$

Proof. To prove the second part, we use the lemmas above: the map

$$(S^{(u)})^{-1}\mathrm{gr}_\nu^{(u)}R \to k^{(u)}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$$

is an isomorphism, so these two rings have the same dimension $\mathrm{r}(\nu)$. Since dimension cannot increase by localization, this gives $\dim \mathrm{gr}_\nu^{(u)}R \geq \mathrm{r}(\nu)$ and finally, because the extension $\mathrm{gr}_\nu^{(u)}R \to \mathrm{gr}_\nu R$ is radicial, $\dim \mathrm{gr}_\nu R \geq \mathrm{r}(\nu)$. But we have the opposite inequality by the first part of the proposition. $\circlearrowleft$

Remark. Piltant's result implies that if $\mathrm{t}_{k_R}(\nu) = 0$ and $\mathrm{r}(\nu) < \infty$ we have

$$\dim \mathrm{gr}_\nu R_\nu = \mathrm{r}(\nu)$$

and

$$\dim \mathrm{gr}_\nu R = \mathrm{t}_{k_R}(\mathrm{gr}_\nu R),$$

the transcendence degree of $\mathrm{gr}_\nu R$ over k_R. We will use this last equality in several examples. The first one should be contrasted with the fact that $\dim R_\nu = \mathrm{h}(\nu) \leq \mathrm{r}(\nu)$.

Question. Does the proof in the case where $\mathrm{t}_{k_R}(\nu) = 0$ also show that $\mathrm{gr}_\nu R$ is catenary?
The proof of the fact that equality in the inequality of Proposition 3.7 continues to hold without the assumption that $\mathrm{t}_{k_R}(\nu) = 0$ provided that k_R is uncountable is more delicate. Piltant has given in [P] an example of a valuation ν on the ring $R = \overline{\mathbf{Q}}[s,t,x,y]_{(s,t,x,y)}$ with value group $\mathbf{Z}^2_{lex}$ and residue field $\overline{\mathbf{Q}}(\frac{s}{t})$ such that $\mathrm{gr}_\nu R$ has Krull dimension $3 = \mathrm{r}(\nu) + \mathrm{t}_{k_R}(\nu)$ but is not catenary; it possesses a saturated chain of prime ideals of length two. It would be interesting to decide whether the completion $\widehat{\mathrm{gr}_\nu R}$ with respect to ν_{gr} which we shall meet in subsection 5.2 is catenary.
Strict inequality in Abhyankar's inequality may happen, but this does not contradict the semi-continuity theorem for the dimensions of the fibers of the map $\mathrm{Spec} A_\nu(R) \to \mathrm{Spec} k[v^{\Phi_+}]$ since semicontinuity is proved only for morphisms which are essentially of finite type. Under the assumptions of Piltant's theorem, in order that $\mathcal{A}_\nu(R)$ may be a $k[v^{\Phi_+}]$-algebra of finite type, it is necessary that Abhyankar's equality holds.

Since R is assumed to be nœtherian we will in this text remain in the situation where $\mathrm{r}(\Phi), \mathrm{t}_{k_R}(\nu)$, etc.. are finite.

Recall that Φ is said to be rationally complete ([A], II, 8, p.44), or *divisible*, if for every $\phi \in \Phi$ and every non zero $n \in \mathbf{Z}$ there is $\phi' \in \Phi$ such that $n\phi' = \phi$. In *loc.cit*, Proposition 2.10, it is shown that if Φ is divisible and of height h, there are h ordered subgroups Φ_i of $\mathbf{R}$ (of height one) such that Φ is order isomorphic to the lexicographically ordered sum $\Phi_1 \oplus \cdots \oplus \Phi_h$. We will say that a valuation is rationally complete if its group is. This notion is used to show that the group of a valuation is always an ordered subgroup of $\mathbf{Q}^{\mathrm{r}(\nu)}$ ordered lexicographically.

Moreover, (see [B3], Chap. VI, §10, No. 2), if Φ is finitely generated and if $\mathrm{r}(\nu) = \mathrm{h}(\nu)$, then Φ is order isomorphic to $\mathbf{Z}^{\mathrm{h}(\nu)}$ with the lexicographic order.

We will often make use of the following fact

Proposition 3.8. (Zariski; see ([Z-S], Appendix 3, [S2], remark 2.7, [V], Prop. 9.1.) *If R is nœtherian, the value semigroup $\Gamma = \nu(R\setminus\{0\}) \subset \Phi_+$ is well ordered.*

Proof. Zariski's proof is in terms of valuation ideals of R, which are the distinct $\mathcal{P}_\phi(R)$; a decreasing sequence of elements of Γ gives an increasing sequence of ideals $\mathcal{P}_\phi(R)$, which has to be stationary. ◌

A surjective monotone non-decreasing map of ordered groups $\lambda\colon \Phi \to \Phi_1$ corresponds to a composite valuation $\nu_1 = \lambda\circ\nu$ and another valuation ring $R_{\nu_1} \supset R_\nu$. However, the classical terminology is that ν is *composed* with ν_1, and we will respect it. The map λ extends to a map

$$\tilde\lambda\colon k[v^{\Phi_+}] \to k[w^{\Phi_{1+}}].$$

We have a map $\mathcal{A}_\nu(R) \to \mathcal{A}_{\nu_1}(R)$ and in fact a commutative diagram

$$\begin{array}{lccc} \tilde\lambda_{\mathcal{A}}(R)\colon & \mathcal{A}_\nu(R) & \longrightarrow & \mathcal{A}_{\nu_1}(R) \\ & \uparrow & & \uparrow \\ \tilde\lambda\colon & k[v^{\Phi_+}] & \longrightarrow & k[w^{\Phi_{1+}}] \end{array}$$

where the top arrow is induced by the inclusions $\mathcal{P}_\phi(R) \subset \mathcal{P}_{\lambda(\phi)}(R)$, i.e., $x_\phi v^{-\phi} \mapsto x_\phi w^{-\lambda(\phi)}$, and the bottom arrow is $\tilde\lambda$. The map $\tilde\lambda$ is surjective since $\lambda\colon \Phi \to \Phi_1$ is order preserving, and its kernel is the ideal generated by the $(v^\psi - 1)_{\psi\in\Psi_+}$ where Ψ is the kernel of the map $\Phi \to \Phi_1$. Let us show that $\tilde\lambda_{\mathcal{A}}(R)$ is also surjective: given $xw^{-\mu}$, where $\mu = \lambda(\phi)$ and $\nu_1(x) \geq \mu$, either $\nu(x) \geq \phi$ and $xw^{-\mu}$ is the image of $xv^{-\phi}$, or we have not chosen ϕ well, but since λ is order preserving, if $\nu(x) < \phi$, we have $\nu_1(x) = \lambda(\nu(x)) \leq \lambda(\phi) = \mu$ so that $\lambda(\nu(x)) = \mu$ and $\phi - \nu(x) \in \Psi$. Finally, we see that $xw^{-\mu}$ is the image of $xv^{-\nu(x)}$ and we have shown the surjectivity. The kernel is the ideal of $\mathcal{A}_\nu$ generated by $(v^\psi - 1)_{\psi\in\Psi_+}$.

Finally, we have shown that the formation of $k[v^{\Phi_+}] \to \mathcal{A}_\nu$ commutes with the composition of valuations; we have

Proposition 3.9. *The natural morphism*

$$\begin{array}{cc} \mathcal{A}_\nu(R) \otimes_{k[v^{\Phi_+}]} k[w^{\Phi_{1+}}] & \to \mathcal{A}_{\nu_1}(R) \\ x_\phi v^{-\phi} \otimes w^{\phi_1} & \mapsto x_\phi w^{\phi_1 - \lambda(\phi)} \end{array}$$

is an isomorphism of graded $k[v^{\Phi_+}]$-algebras.

Remark. The fact that

$$\mathcal{A}_{\nu_1}(R) = \mathcal{A}_\nu(R)/\big((v^\psi - 1)_{\psi\in\Psi_+}\big)$$

is true without assuming that R contains a field.

Corollary. *The rings $\mathrm{gr}_\nu R$ and $\mathrm{gr}_{\nu_1} R$ are both specializations of the same ring over the polynomial ring $k[(t_\psi)_{\psi\in\Psi_+}]$.*

Proof. The ring $\mathrm{gr}_{\nu_1} R$ is isomorphic to the quotient ring of $\mathcal{A}_\nu(R)$ by $\big((v^\psi - 1)_{\psi\in\Psi_+}, (v^\phi)_{\phi\in\Phi_+\setminus\Psi_+})\big)$, and $\mathrm{gr}_\nu R = \mathcal{A}_\nu(R)/\big((v^\phi)_{\phi\in\Phi_+}\big)$. These rings are specializations of

$$\mathcal{A}_\nu(R)\otimes_k k[(t_\psi)_{\psi\in\Psi_+}]/\big((v^\psi - t_\psi)_{\psi\in\Psi_+}, (v^\phi)_{\phi\in\Phi_+\setminus\Psi_+})\big).$$

$\circlearrowleft$

Remark. If we have a sequence of valuation rings as above, with $R\subset R_\nu$,

$$R_\nu\subset R_{\nu_1}\subset\cdots\subset R_{\nu_{h-1}},$$

where h is the height of the valuation ν, all the graded algebras $\mathrm{gr}_{\nu_s} R$ are specializations of the same algebra of the form

$$\mathcal{A}_\nu(R)\otimes_k k[t_\omega]/\big((v^\omega - t_\omega)_{\omega\in\Omega_1}, (v^\omega)_{\omega\in\Omega_2}\big),$$

and we can in a number of cases use a system of generators of $\mathrm{gr}_\nu R$ to produce a system of "formal" generators for each $\mathrm{gr}_{\nu_s} R$ (see below).

3.2 The valuation $\tilde\nu_{\mathcal{A}}$

Let us denote by $\tilde\nu_{\mathcal{A}}$ the valuation on $\mathcal{A}_\nu(R)$ with values in the group $\Phi\oplus\Phi$ ordered lexicographically, and defined by

$$\tilde\nu_{\mathcal{A}}(\Sigma x_\phi v^{-\phi}) = \big(\mu_{\mathcal{A}}(\Sigma x_\phi v^{-\phi}), \nu_{\mathcal{A}}(\Sigma x_\phi v^{-\phi})\big).$$

According to what we saw in §1, the center of this valuation is defined by the ideal $\big((v^{\Phi_+})\mathcal{A}_\nu(R), \oplus_{\phi\in\Phi_+}\mathcal{P}_\phi v^{-\phi}\big)$ of $\mathcal{A}_\nu(R)$. Geometrically, this center is $\mathrm{Spec}\overline{R}$.

This valuation is composed with $\mu_{\mathcal{A}}$ in the manner corresponding to the first projection $\Phi\oplus\Phi\to\Phi$, and the residual valuation of $\tilde\nu$ on the center $\mathrm{gr}_\nu R$ of $\mu_{\mathcal{A}}$ is ν_{gr}.

This valuation is of interest because it is a "specialization" of the valuation ν on R to the valuation ν_{gr}. Indeed, the valuation $\nu_{\mathcal{A}}$ extends to a valuation of $(v^{\Phi_+})^{-1}A_\nu(R) = (v^{\Phi_+})^{-1}R[v^{\Phi_+}]$ (Proposition 2.1, b)) which is the algebra of the "generic fiber" of the faithfully flat map defined in §2.3, and as we saw it induces the natural valuation on the special fiber. Moreover, heuristically, we may think of the inclusion $\mathcal{A}_\nu(R)\to\mathcal{A}_\nu(R_\nu)$ as a uniformization of the valuation $\tilde\nu$ (see below, §4), forgetting for a moment the fact that it is not an extension of finite type. The center of $\tilde\nu$ in $\mathcal{A}_\nu(R_\nu)$ is the origin, and, still heuristically, $\mathcal{A}_\nu(R_\nu)$ is "regular". Then the corresponding map of schemes is a simultaneous uniformization for ν and ν_{gr} since the induced map $\mathrm{gr}_\nu R\to\mathrm{gr}_\nu R_\nu$ is also (in the same heuristic sense) a uniformization of ν_{gr}.

3.3 The valuation algebra of the residual valuation

Let us now come back to the inclusion $R_\nu \subset R_{\nu_1}$ associated to an monotone non-decreasing surjection $\lambda\colon \Phi \to \Phi_1$ of ordered groups, with kernel Ψ; it is a classical fact of valuation theory (see [A], pp. 56-57 and [V]) that $m_{\nu_1} \subset R_\nu$ and the image $\overline{R}_\nu = R_\nu/m_{\nu_1} \subset R_{\nu_1}/m_{\nu_1}$ is a valuation ring of the field R_{ν_1}/m_{ν_1}, corresponding to a valuation $\overline{\nu}$ with value group Ψ.

If we endeavor to find the valuation algebra of $\overline{R}_1 = R/(m_{\nu_1} \cap R)$ with respect to $\overline{\nu}$, we see that if we denote by $\mathcal{A}_\Psi(R)$ the subalgebra $\bigoplus_{\phi\in\Psi} \mathcal{P}_\phi(R)v^{-\phi}$ of $\mathcal{A}_\nu(R)$, then $\mathcal{A}_{\overline{\nu}}(\overline{R}_1)$ is nothing else than the image of $\mathcal{A}_\Psi(R)$ in $\overline{R}_1[v^\Phi]$ by the canonical map $R[v^\Phi] \to \overline{R}_1[v^\Phi]$. It is important to note that $\overline{R}_1$ may have a field of fractions smaller than R_{ν_1}/m_{ν_1}, and that, setting as usual $\Gamma = \nu(R\setminus\{0\}) \cup \{0\}$, the group generated by $\Gamma \cap \Psi$ may be smaller than Ψ. This happens in particular if $m_{\nu_1} \cap R = m_\nu \cap R$ since in that case the valuation $\overline{\nu}$ is trivial on $\overline{R}_1$; this is the same as saying that we have $\Gamma \cap \Psi = \{0\}$.

A typical example is the monomial valuation on $R = k[u,v]_{(u,v)}$ with values in $\mathbf{Z}^2$ ordered lexicographically determined by $\nu(u) = (1,0)$, $\nu(v) = (1,1)$; it is composed with the (u)-adic valuation, Γ is the semigroup generated by $(1,0)$, $(1,1)$ and Ψ is the subgroup $\{(i,j) |\ i+j=0\}$. Note that the two valuations have the same center in R and that for a given $\phi \in \Phi_+$ there are only finitely many ideals $\mathcal{P}_\psi$ between $\mathcal{P}_{\lambda(\phi)}$ and $\mathcal{P}^+_{\lambda(\phi)}$.

3.4 Comparison of graded algebras for composed valuations

The result of Proposition 3.9 suggests that we seek a closer relation between $\mathrm{gr}_\nu R$ and $\mathrm{gr}_{\nu_1} R$ when ν is composed with ν_1. First, let us remark that with the usual notations, we have for $\phi \in \Phi_+ \cup \{0\}$ the following inclusions[3]

$$\mathcal{P}^+_{\lambda(\phi)} \subset \mathcal{P}^+_\phi \subset \mathcal{P}_\phi \subset \mathcal{P}_{\lambda(\phi)},$$

so that we get a filtration of each

$$(\mathrm{gr}_{\nu_1} R)_{\phi_1} = \frac{\mathcal{P}_{\phi_1}}{\mathcal{P}^+_{\phi_1}}$$

by the $(\mathcal{P}_\phi/\mathcal{P}^+_{\lambda(\phi)})_{\lambda(\phi)=\phi_1}$. In the case where $\phi_1 = 0$, it is nothing else than the filtration of $\overline{R} = R/(m_{\nu_1} \cap R)$ by the images of the $(\mathcal{P}_\phi)_{\phi\in\Psi}$ where $\Psi = \mathrm{Ker}\lambda$, i.e., the filtration corresponding to the ideals $(\overline{\mathcal{P}}_\psi)_{\psi\in\Psi}$ associated to the residual valuation $\overline{\nu}$ on $\overline{R}$. Let us denote by $\overline{\mathcal{P}}(\phi_1)$ this filtration, for each $\phi_1 \in \Phi_{1+}$. We have immediately

[3] The fact that $\lambda(\phi) \in \Phi_1$ is deemed sufficient, here and in the sequel, to indicate that the first and last ideals correspond to the valuation ν_1.

Lemma 3.10. *The natural map*

$$\mathcal{P}_\phi \to \frac{\mathcal{P}_\phi}{\mathcal{P}^+_{\lambda(\phi)}}$$

induces an isomorphism

$$\mathrm{gr}_\nu R \to \bigoplus_{\phi_1 \in \Phi_{1+} \cup \{0\}} \mathrm{gr}_{\overline{\mathcal{P}}(\phi_1)}(\mathrm{gr}_{\nu_1} R)_{\phi_1}.$$

It is graded in the sense that an element of degree ϕ is mapped to an element of bidegree $(\phi, \lambda(\phi))$. Moreover, for $x \in R$, we have

$$\mathrm{in}_\nu x = \mathrm{in}_{\overline{\mathcal{P}}(\nu_1(x))}(\mathrm{in}_{\nu_1} x).$$

In particular, taking $\lambda(\phi) = 0$, we find that it induces a graded isomorphism

$$\bigoplus_{\psi \in \Psi_+ \cup \{0\}} (\mathrm{gr}_\nu R)_\psi \to \mathrm{gr}_{\overline{\nu}} \overline{R},$$

which can also be deduced from 3.3. Note that when R is nœtherian, and so Γ is well ordered by Proposition 3.8, the topology on $R/(m_{\nu_1} \cap R)$ corresponding to the $\overline{\nu}$-adic filtration is finer than the image by the natural surjection $R \to R/(m_{\nu_1} \cap R)$ of the ν-adic topology on R. This follows from the inclusions $\mathcal{P}_\psi \subset \mathcal{P}_\phi + m_{\nu_1} \cap R$ for $\psi \in \Psi,\ \psi \geq \min_{\{x | \nu_1(x)=0, \nu(x) \geq \phi\}} \nu(x)$. Thus, the $\overline{\nu}$-adic topology on $R/(m_{\nu_1} \cap R)$ is the same as the image of the ν-adic topology of R. Note that they may be the discrete topology; more generally, if ν and ν_1 have the same center in R, the filtrations $\overline{\mathcal{P}}(\phi_1)$ are finite ([Z-S], Vol. II, Appendix 3, Corollary).

Remark. In fact, whenever we have a specialization $R \subset R_\nu \subset R_{\nu_1}$ of valuations of R, the valuation ν induces, as described above, a filtration on the ring $\mathrm{gr}_{\nu_1} R$, to which we can associate as in subsection 2.1 a "filtration algebra" which determines a specialization of $\mathrm{gr}_{\nu_1} R$ to $\mathrm{gr}_\nu R$. This specialization is the same as that coming from the embedding of $\mathrm{Spec}\mathcal{A}_{\nu_1}(R)$ in $\mathrm{Spec}\mathcal{A}_\nu(R)$.

Example. Let R be a local integral domain, and $f \in R \setminus \{0\}$ be such that $R/(f)$ is again integral. Let us choose a valuation $\overline{\nu}$ of $R/(f)$ with semigroup of values $\overline{\Gamma}$ generating a group Ψ. Since each element x of R can be written in a unique way $x = f^n y$ with $y \notin (f)$, we may define a valuation ν on R with group $\mathbf{Z} \oplus \Psi$ ordered lexicographically by

$$\nu(f^n y) = (n, \overline{\nu}(\overline{y})) \ \text{ where } \overline{y} = y \bmod.(f).$$

The map $f^n y \mapsto n \in \mathbf{N}$ defines a valuation ν_1 on R, with value group $\mathbf{Z}$. and with which ν is composed. With the notations introduced above, we have $\overline{R} = R/(f)$, and

$$\mathrm{gr}_{\nu_1} R = \overline{R}[\xi] \quad \text{with } \xi = \mathrm{in}_{\nu_1}(f),$$

and from the considerations above we see that because it is a polynomial ring in one variable, we have:

$$\mathrm{gr}_\nu R = (\mathrm{gr}_{\overline{\nu}} \overline{R})[\xi].$$

3.5 Comparison of residual valuations

In this subsection I recall some direct consequences of the basic facts on the composition of valuations found in ([A], pp. 56-57, [V], section 4). Keeping the notations of the beginning of this section, let us consider a sequence

$$R_\nu \subset R_{\nu_1} \subset R_{\nu_2}$$

of valuation rings; it corresponds to a sequence

$$\Phi \to \Phi_1 \to \Phi_2$$

of value groups; let $\Psi_2 \subset \Psi_1 \subset \Phi$ be the kernels of the maps $\Phi \to \Phi_1$ and $\Phi \to \Phi_2$ respectively. To this situation is associated an inclusion

$$R_\nu/(m_{\nu_2} \cap R_\nu) \subset R_{\nu_1}/(m_{\nu_2} \cap R_{\nu_1})$$

of valuation rings of the residue field k_{ν_2} of R_{ν_2}. Note that in fact $m_{\nu_2} \cap R_\nu = m_{\nu_2}$ since $m_{\nu_2} \subset R_\nu$ by the general properties of composed valuations (see [V]), and similarly for R_{ν_1}; recall also that R_{ν_i} is the localization of R_ν at the prime m_{ν_i}. The value group of the first valuation ring is Ψ_1, and the value group of the second is Ψ_1/Ψ_2. If ν_1 and ν_2 are consecutive valuation rings, in the sense that there is no other valuation ring between them, the group Ψ_1/Ψ_2 is of height one.

Assuming that this is the case, given a subring $R \subset R_\nu$, we see that ν_1 induces a valuation of height at most one, with valuation ring $R_{\nu_1}/(m_{\nu_2} \cap R_{\nu_1})$, on the quotient $R/(m_{\nu_2} \cap R)$; the center of this valuation is the ideal $(m_{\nu_1} \cap R)/(m_{\nu_2} \cap R)$, which is also the kernel of the natural morphism

$$R/(m_{\nu_2} \cap R) \to R/(m_{\nu_1} \cap R).$$

More precisely, it may happen that $m_{\nu_1} \cap R = m_{\nu_2} \cap R$, and then the residual valuation is trivial on $R/(m_{\nu_2} \cap R)$; in general the quotient $R/(m_{\nu_2} \cap R)$ need not have the same field of fractions as $R_{\nu_1}/(m_{\nu_2} \cap R_{\nu_1})$, as we saw in subsection 3.3.

3.6 Specialization of valuations and rational valuations

Let k be a field, and (R, m) a local k-algebra which is an integral domain. Let R_{ν_0} a valuation ring in the fraction field of R, containing R, with residue field k_{ν_0}, and assume that the transcendence degree over k of k_{ν_0} is greater than zero. Then, if $R/(m_{\nu_0} \cap R) \neq k_{\nu_0}$, there exists a k-valuation ring V of k_{ν_0} containing $R/(m_{\nu_0} \cap R)$ and different from k_{ν_0}. If we denote by $\pi\colon R_{\nu_0} \to k_{\nu_0}$ the canonical projection, we see that $\pi^{-1}(V)$ is a valuation ring R_{ν_1} containing R and contained in R_{ν_0}. The residue field of R_{ν_1} is equal to the residue field of V, and therefore its transcendence degree over k is less than that of k_{ν_0}. We may iterate this construction as long as the residue field of R_{ν_1} is transcendental over k and $R/(m_{\nu_1} \cap R) \neq k_{\nu_1}$. When we stop, we have built a sequence

$$R \subset R_{\nu_t} \subset R_{\nu_{t-1}} \subset \cdots \subset R_{\nu_0}$$

such that $t \leq \mathrm{t}_k(\nu_0)$ and either R_{ν_t} dominates R and the residual extension is trivial, or the residue field of R_{ν_t} is algebraic over k. In this case, the ring $R/(m_{\nu_t} \cap R)$ contains k and is contained in k_{ν_t}, so it is a field. This means that $m_{\nu_t} \cap R = m$, so R_{ν_t} dominates R. So at worse we end up the specialization process with a valuation ring dominating R and an algebraic residual extension.

This is a special case of the construction found in [Z-S], Vol. 2, Chap VI, §16.

In the case where R_ν dominates R and the residual extension is trivial, one says that the valuation ν on R is *rational*. So we have:

Proposition 3.11. *If the residue field of R is algebraically closed, we can specialize any valuation ν_0 of R to a rational valuation.*

Remark. We may, at each step in the preceding construction, choose a discrete divisorial valuation; then the transcendence degree of the residual valuation drops exactly by one at each step, while the height of the value group increases exactly by one since the value group of V is the kernel of the map $\Phi_{\nu_1} \to \Phi_{\nu_0}$. Finally we have:

$$\mathrm{h}(\nu_t) = \mathrm{h}(\nu_0) + \mathrm{t}_{k_R}(\nu_0),$$

and since $\mathrm{h}(\nu_t) \leq \mathrm{r}(\nu_t)$, in order to prove Abhyankar's inequality in this case it suffices to check that $\mathrm{r}(\nu) \leq \dim R$ when R_ν dominates R, but this is readily done, at least in the algebraic or formal case, if we present the fraction field K of R as an algebraic extension of a field of rational functions of transcendence degree $\dim R$ over k_R, since extending valuations to algebraic extensions does not affect the rational rank. This suggests a proof of Abhyankar's inequality, not very different from that described in [V].

3.7 Monomial valuations

This section explains in what sense the liftings $(\xi_i)_{i\in I}$ of generators $(\overline{\xi}_i)_{i\in I}$ of the graded $R/(m_\nu \cap R)$-algebra $\mathrm{gr}_\nu R$ form a system of coordinates "adapted" to the valuation. The idea is that, at least in the case of trivial residual extension, valuations which are monomial in a certain system of coordinates are exactly those which can be uniformized by a single toric modification of the ambiant space in which this system of coordinates embeds our singularity.

Given a valuation ν of a local or graded ring R, we generalize the definition given in [S 2] for a regular local ring as follows

Definition. The valuation ν of the local (or graded) ring R is monomial with respect to some system of generators $(\xi_i)_{i\in I}$ of the center $m_\nu \cap R$ of ν in R if $(\xi_i)_{i\in I}$ is a generating sequence for ν in the sense (see [S 2]) that any element $x \in R$ can be written as a finite sum

$$x = \sum_{\nu(\xi^\gamma) \geq \nu(x)} c_\gamma \xi^\gamma$$

with $\gamma \in I^b$, for some $b \in \mathbf{N}$, $c_\gamma \in R$. In particular, $\nu(x) = \min_\gamma(\nu(\xi^\gamma))$.

This is equivalent to asking that for each $\phi \in \Phi_+$ the ideal $\mathcal{P}_\phi(R)$ is generated by monomials in the $(\xi_i)_{i\in I}$, and also to the fact that the initial forms $(\overline{\xi}_i)_{i\in I}$ in $\mathrm{gr}_\nu R$ of the elements $(\xi_i)_{i\in I}$ generate it as $R/(m_\nu \cap R)$-algebra. In particular, the valuation ν is monomial with respect to a *finite* set of generators of $m_\nu \cap R$ if and only if the $R/(m_\nu \cap R)$-algebra $\mathrm{gr}_\nu R$ is finitely generated.

Example. 1) Let (S, m) be a regular local ring and ν be a valuation of S having the property of the above definition with respect to a system of regular parameters $(\xi_0, \ldots, \xi_n)$ of S. If we denote by $\mathcal{R}_\phi$ the ideal of R generated by the monomials in $(\xi_0, \ldots, \xi_n)$ of valuation $\geq \phi$, we have $x \in \mathcal{R}_{\nu(x)}$ and therefore ν is monomial in the sense of [S 2].
2) If R is the algebra of a plane analytic branch C, and ν its unique valuation, then ν is monomial with respect to two well chosen generators of the maximal ideal of R if and only if C has only one characteristic pair (see [G-T], [T1]).

Proposition 3.12. *Let R be an integral domain and ν a valuation on R, i.e., $R \subset R_\nu$. Set $\mathbf{p} = m_\nu \cap R$. The valuation ν_{gr} on $\mathrm{gr}_\nu R$ is monomial with respect to any system of homogeneous generators $(\xi_i)_{i\in I}$ of the graded $R/\mathbf{p}$-algebra $\mathrm{gr}_\nu R$.*

Proof. by the definition of ν_{gr}, it suffices to check the definition for homogeneous elements. Every homogeneous element $\overline{x}_\phi$ can be written as a finite sum of monomials of the same degree $\overline{x}_\phi = \sum_\gamma c_\gamma \xi^\gamma$ with $c_\gamma \in R/\mathbf{p}$ and $\nu_{\mathrm{gr}}(c_\gamma \xi^\gamma) = \phi$, as required. ◯

4 The graded algebras of valuation rings

In this section I show that the graded ring $\mathrm{gr}_\nu R_\nu$ is the direct limit of graded algebras (or subalgebras) smooth over the residue field k_ν of the valuation. This means that it is "regular", and hence can be deemed to be a resolution of singularities of the algebra $\mathrm{gr}_\nu R$, if we forget for a moment that $\mathrm{gr}_\nu R_\nu$ is not an extension of finite type of $\mathrm{gr}_\nu R$, and remember only that they have the same field of fractions.

It turns out that the result for $\mathrm{gr}_\nu R_\nu$ is equivalent to its counterpart for $k[t^{\Phi_+}]$ where k is any field. This should not surprise us since we saw that $\mathrm{gr}_\nu R_\nu$ should be "equisingular" with $k_\nu[t^{\Phi_+}]$.

The result for $k[t^{\Phi_+}]$ in turn follows from the stronger statement that the semigroup Φ_+ is a direct limit of a system of linear maps between finitely generated free monoids, which will give us toric maps between the corresponding polynomial algebras.

We are going to see that this is true, even with finitely generated free *submonoids*, whenever Φ is of finite height.

4.1 $\mathrm{gr}_\nu R_\nu$ as a limit of polynomial algebras

We begin with an example: the case when $\Phi = \mathbf{Z}^d$ with the lexicographic order.

Indeed, consider for each $(d-1)$-uple $(r_1, r_2, \ldots, r_{d-1})$ of positive integers the map

$$e_{r_1, r_2, \ldots, r_{d-1}} : \mathbf{N}^d \to \mathbf{Z}^d$$

defined by

$$e_{r_1, r_2, \ldots, r_{d-1}}(i_1, \ldots, i_d) = (i_1, i_2 - r_1 i_1, \ldots, i_d - r_{d-1} i_{d-1});$$

its image is in the semigroup $\mathbf{Z}^d_+$ of positive elements, and it is injective. It defines an injective map

$$e_{r_1, r_2, \ldots, r_{d-1}} : k[z_1, \ldots, z_d] \to k[t^{\mathbf{Z}^d_+}],$$

and clearly this last algebra is the union of these subalgebras as $r_1, r_2, \ldots, r_{d-1}$ vary. Note that whenever $(r_1, r_2, \ldots, r_{d-1}) \leq (s_1, s_2, \ldots, s_{d-1})$ in the sense that $r_j \leq s_j$, $1 \leq j \leq d-1$, there is a map of algebras $k[z_1, \ldots, z_d] \to k[w_1, \ldots, w_d]$ which is toric, i.e., given by monomials, and that $k[t^{\mathbf{Z}^d_+}]$ is the direct limit of this system of toric maps.

When Φ is a subgroup of $\mathbf{R}$, consisting of numbers of the form of the form $\Sigma_{i=1}^m a_i \tau_i$ with given $\tau_i > 0$ and $a_i \in \mathbf{Z}$, it is also true that $k[t^{\Phi_+}]$ is a direct limit of polynomial subalgebras over k. This is again because the half space in $\mathbf{R}^m$ defined by $\Sigma_{i=1}^m r_i \tau_i \geq 0$ can be approximated by regular rational simplicial cones (generated by integral vectors which form a basis of the integral lattice).

More precisely, let ν be a valuation of height one, i.e., with archimedian value group $\Phi \subset \mathbf{R}$ (see [A], [Z-S], Vol. II). Assume first that Φ is generated by m rationally independent positive real numbers $\tau_1, \ldots, \tau_m$. We use the Perron algorithm as expounded in ([Z 1], B. I, p. 861), but with a somewhat different interpretation. The algorithm consists in writing

$$\tau_1 = \tau_m^{(1)}, \tau_2 = \tau_1^{(1)} + a_2^{(0)} \tau_m^{(1)}, \ldots, \tau_m = \tau_{m-1}^{(1)} + a_m^{(0)} \tau_m^{(1)},$$

where

$$a_j^{(0)} = [\tau_j / \tau_1], \quad j = 2, \ldots, m,$$

and repeating this operation after replacing $(\tau_1, \ldots, \tau_m)$ by $(\tau_1^{(1)}, \ldots, \tau_m^{(1)})$, and so on. After h steps, one has written

$$\tau_i = A_i^{(h)} \tau_1^{(h)} + \cdots + A_i^{(h+m-1)} \tau_m^{(h)}$$

or, if we denote by w the (weight) vector $(\tau_1, \ldots, \tau_m) \in \mathbf{R}^m$ and by $A^{(h)}$ the vector $(A_1^{(h)}, \ldots, A_m^{(h)})$,

$$w = \tau_1^{(h)} A^{(h)} + \tau_2^{(h)} A^{(h+1)} + \cdots + \tau_m^{(h)} A^{(h+m-1)}$$

where the $\tau_j^{(h)}$ are positive, the coefficients $A_i^{(j)}$ are non negative integers, and the matrix of the vectors

$$A^{(h)}, A^{(h+1)}, \ldots, A^{(h+m-1)}$$

has determinant $(-1)^{h(m-1)}$. Moreover, the directions in $\mathbf{P}^{m-1}(\mathbf{R})$ of the vectors $A^{(h)}$ tend to the direction of w. So we have a sequence of vectors $A^{(h)}$ with positive integral coordinates (becoming longer and longer) whose directions in $\mathbf{P}^{m-1}(\mathbf{R})$ spiral to the direction of w and such that any consecutive m of them as above form a basis of the integral lattice such that w is contained in the convex cone $\sigma^{(h)} = \langle A^{(h)}, A^{(h+1)}, \dots, A^{(h+m-1)}\rangle$ which they generate. The convex dual $\check{\sigma}^{(h)}$ of $\sigma^{(h)}$ (see [Cox], §2, [E], V, 2, p. 149) is contained in the half space $\sum_{j=1}^m a_j\tau_j \geq 0$, the integral points of which form the semigroup Φ_+. The algebra of the semigroup $\check{\sigma}^{(h)} \cap \mathbf{Z}^m$ is a polynomial algebra $k[x_1^{(h)}, \dots, x_m^{(h)}]$ (*loc.cit.*, VI,2) contained in $k[t^{\Phi_+}]$, and since by assumption there are no integral points on the hyperplane $\sum_{j=1}^m a_i\tau_i = 0$ except the origin, the semigroup Φ_+ is the union of the $\check{\sigma}^{(h)} \cap \mathbf{Z}^m$ as $h \to \infty$. This proves that $k[t^{\Phi_+}]$ is the union, or direct limit, of these polynomial subalgebras.

Note that by construction we have for each $h \geq 1$ the equality

$$A^{(h)} - A^{(h+m)} + a_2^{(h)}A^{(h+1)} + \cdots + a_m^{(h)}A^{(h+m-1)} = 0,$$

which shows, since the $a_j^{(h)} = [\tau_j^{(h)}/\tau_1^{(h)}]$ are non-negative, that we have

$$A^{(h+m)} \in \sigma^{(h)} = \langle A^{(h)}, \dots, A^{(h+m-1)}\rangle$$

and therefore $\sigma^{(h+1)} \subset \sigma^{(h)}$, that is $\check{\sigma}^{(h)} \subset \check{\sigma}^{(h+1)}$ and

$$k[x_1^{(h)}, \dots, x_m^{(h)}] \subset k[x_1^{(h+1)}, \dots, x_m^{(h+1)}],$$

so that our direct system is in fact a nested sequence of polynomial subalgebras.

If we now consider a group with one more generator $\tau_{m+1} > 0$ which is rationally dependent on $\tau_1, \dots, \tau_m$, Zariski shows in ([Z 1], B. I, p. 862) that the new weight vector $w = (\tau_1, \dots, \tau_m, \tau_{m+1}) \in \mathbf{R}^{m+1}$ is contained in a rational simplicial cone $\sigma \subset \mathbf{R}^{m+1}$ generated by m integral vectors of the first quadrant forming part of a basis of the integral lattice. Indeed w is contained in a unique rational hyperplane. The dual cone $\check{\sigma} \subset \check{\mathbf{R}}^{m+1}$ is the product of an m dimensional strictly convex cone by a 1-dimensional vector space (see [E], V, 2) generated by an integral vector which is the dual of the rational hyperplane containing w. Note that the map $f : \mathbf{Z}^{m+1} \to \mathbf{R}$ defined by $(a_1, \dots, a_{m+1}) \mapsto \sum_{i=1}^{m+1} a_i\tau_i$ is no longer injective; the primitive vector corresponding to the variable x_{m+1} is in the kernel. Let us set $\widetilde{\Phi_+} = f^{-1}(\Phi_+ \cup \{0\})$. By refining as above by the Perron algorithm for w inside the linear span $< \sigma >$ of σ, starting with the barycentric coordinates of w in σ, we find a sequence of regular simplicial cones $\sigma^{(h)} \subset \sigma$ whose duals $\check{\sigma}^{(h)} \subset \widetilde{\Phi_+}$ correspond ([E], VI, Th. 2.12) to algebras of the form $k[x_1^{(h)}, \dots, x_m^{(h)}, x_{m+1}^{\pm 1}] \subset k[t^{\widetilde{\Phi_+}}]$. The free semigroups $\check{\sigma}^{(h)} \cap \mathbf{Z}^{m+1}$ fill up $\widetilde{\Phi}_+$ as $h \to \infty$ since the only rational points of the hyperplane $\sum_{j=1}^{m+1} a_j\tau_j = 0$ are on the dual of the hyperplane containing w, which is contained in all the $\check{\sigma}^{(h)}$. So the direct limit of the images of the maps $k[t^f]\colon k[x_1^{(h)}, \dots, x_m^{(h)}, x_{m+1}^{\pm 1}] \to k[t^{\Phi_+}]$ is

$k[t^{\Phi_+}]$. But these images are isomorphic to $k[x_1^{(h)},\ldots,x_m^{(h)},x_{m+1}^{\pm 1}]/(x_{m+1}-1)$ so that they are again polynomial rings $k[x_1^{(h)},\ldots,x_m^{(h)}]$. If we have more generators rationally dependent on $\tau_1,\ldots,\tau_m$, we can repeat the argument after taking as new generators the coordinates of the weight vector with respect to the m primitive vectors of σ. In both examples, the fact that k is a field plays no role, so we have proved:

Lemma 4.1. *Let Φ be a totally ordered finitely generated group of height one (i.e., archimedian), or $\mathbf{Z}^d$ with the lexicographic order. For any ring A the semigroup algebra $A[t^{\Phi_+}]$ of Φ_+ with coefficients in A is the direct limit of a direct system of graded subalgebras which are polynomial algebras $A[x_1,\ldots,x_m]$ over A with $m=\mathrm{r}(\Phi)$.*

Moreover, the maps between these algebras are toric maps, i.e., each variable of one is sent to a monomial in the variables of the other, and there is a cofinal subsystem which is a chain of nested subalgebras.

Remark. 1) Here and in the sequel, for simplicity's sake I write $A[t^{\Phi_+}]$ for what should properly be written $A[t^{\Phi_+\cup\{0\}}]$.
2) In both cases, the smooth subalgebras are produced by an algorithm.
3) The result, especially in view of its algorithmic nature, is much more useful than its consequence that $\mathrm{Spec}A[t^{\Phi_+}]$ is a pro-object in the category of smooth affine toric schemes over $\mathrm{Spec}A$ with toric maps.
4) The proof also shows that the semigroup algebra of the semigroup

$$\widetilde{\Phi_+}=\{(a_1,\ldots,a_m,a_{m+1},\ldots,a_{m+r})\in\mathbf{Z}^{m+r}\mid\sum_{i=1}^{m+r}a_i\tau_i\geq 0\},$$

where $\tau_1,\ldots,\tau_m$ are rationally independent and the others are rationally dependent upon them, is a toric direct limit of toric subalgebras of the form

$$A[x_1,\ldots,x_m,x_{m+1}^{\pm 1},\ldots,x_{m+r}^{\pm 1}].$$

5) The result for subgroups of $\mathbf{R}$ holds without the finiteness assumption if we allow m to vary, since any abelian group is a direct limit of its subgroups of finite type.

Let now Φ be a totally ordered group of finite height $h>1$. We have a surjective monotone non-decreasing map $\lambda\colon\Phi\to\Phi_1$ where Φ_1 is of height $h-1$, and the kernel Ψ of λ is of height 1. By induction on the height we may assume that Φ_{1+} is the union of sub-semigroups isomorphic to $\mathbf{N}^m$, and we know from the lemma above that the same is true for Ψ_+.

Let us denote the semigroups that fill Φ_{1+} by F_i, and let $\tilde{F}_i\subset\Phi_+$ be the subsemigroup generated by elements $e_1,\ldots,e_{r_i}$ which lift to Φ_+ the generators of F_i. Similarly let us denote by $G_j\subset\Psi_+$ free semigroups which fill Ψ_+, generated say by $f_1,\ldots,f_{s_j}$. Note that for $\phi\in\Phi_+,\ \psi\in\Psi,\ \phi+\psi\in\Phi_+\cup\{0\}$, and consider for r_is_j-tuples $n=(n_{s,t}, 1\leq s\leq r_i,\ 1\leq t\leq s_j)$ of non negative integers, the free semigroups $\tilde{F}_i(n)\subset \Phi_+\cup\{0\}$ generated by $e_1-\sum_t n_{1t}f_t,\dots,e_{r_i}-\sum_t n_{r_it}f_t$. Let us check that the direct sums of free semigroups $\tilde{F}_i(n)\oplus G_j$ fill up Φ_+; the proof is similar to the proof in the case of the lexicographic $\mathbf{Z}^d$. Given $\phi\in\Phi_+$, there exists an index i and a $\phi_1\in\tilde{F}_i$ such that $\phi-\phi_1\in\Psi$. If we have $\phi-\phi_1\in\Psi_+$, there exists an index j such that $\phi-\phi_1\in G_j$ so that indeed we have $\phi\in\tilde{F}_i\oplus G_j$. This happens in particular if $\phi\in\Psi$. If $\phi-\phi_1\in\Psi_-$, there exists an index j such that we can write

$$\phi=\sum_{i=1}^{r_i}k_ie_i-\sum_{j=1}^{s_j}\ell_jf_j \text{ with non negative integers } k_i,\ \ell_j.$$

Since $\phi\notin\Psi$, at least one of the k_i is not zero, say k_1. Choosing positive integers $\tilde{\ell}_j$ such that $k_1\tilde{\ell}_j>\ell_j$, we may rewrite ϕ as follows

$$\phi=k_1(e_1-\sum_{j=1}^{s_j}\tilde{\ell}_jf_j)+\sum_{i=2}^{r_i}k_ie_i+\sum_{j=1}^{s_j}(k_1\tilde{\ell}_j-\ell_j)f_j.$$

This shows that ϕ is indeed in $\tilde{F}_i(n)\oplus G_j$ with $n=\tilde{\ell}$.

Now if we assume, as we may by induction, that the F_i and G_j are nested sequences, and choose $n(i)$ given by $(n_{s,t}=i, 1\leq s\leq r_i,\ 1\leq t\leq s_j)$, we see that the corresponding groups $\tilde{F}_i(n(i))\oplus G_i$ form a nested sequence which is cofinal in the direct system. So we have:

Proposition 4.2. *For any totally ordered group Φ of finite height, and any commutative ring A, the semigroup algebra $A[t^{\Phi_+}]$ is a direct limit of a direct system of graded polynomial subalgebras over A with monomial maps, and there are cofinal nested subsystems of such polynomial subalgebras. If Φ is of finite rational rank $\mathrm{r}(\Phi)$, all the polynomial subalgebras may be chosen isomorphic to $A[x_1,\dots,x_{\mathrm{r}(\Phi)}]$.*

Proof. There remains only to prove the last sentence. This is done by induction on the height; we have shown above that the result is true for valuations of height one. Assume that the result is true for Ψ and Φ_1, and so the rank of the free monoid $\tilde{F}_i(n)\oplus G_j$ is the sum of the rational ranks of Ψ and Φ_1. But since the rational rank is additive in an exact sequence because $\mathbf{Q}$ is a flat $\mathbf{Z}$-module, this is the rational rank of Φ. ◯

Here and in the sequel, the notation V^m stands for the monomial $V_1^{m_1}\dots V_s^{m_s}$ with $m_i\in\mathbf{N}\cup\{0\}$.

Corollary. *If the rational rank of Φ is finite, the semigroup algebra $A[t^{\Phi_+}]$ is a quotient of a polynomial ring over A in countably many indeterminates $A[(V_j)_{j\in J}]$ by a binomial ideal of the form*

$$(V_j-V^{m(j)})_{j\in J'}$$

for some $J'\subset J$, with $|m(j)|\geq 2$.

Proof. If we pick a system of homogeneous generators of the A-algebra $A[t^{\Phi_+}]$, each of them must appear in one of the polynomial subalgebras of which $A[t^{\Phi_+}]$ is the direct limit, and we may assume that it appears as one of the generators of that algebra. The only relations between these generators are those corresponding to the toric inclusions $A[x_1, \dots, x_r] \to A[y_1, \dots, y_r]$, and they are of the announced type after we discard the trivial relations $x_i = y_j$ by removing some generators. ○

Remark. We can apply this to the subalgebra $k[v^{\Phi_+}]$ of $\mathcal{A}_\nu(R)$, and view the v^ϕ as coordinates linked by the binomial relations described in this Corollary. I will take the liberty to treat the v^ϕ as "coordinates" for $\mathrm{Spec} k[v^{\Phi_+}]$.

Proposition 4.3. *Given the ring R_ν of a valuation of finite rational rank $\mathrm{r}(\nu)$:*
a) the graded k_ν-algebra $\mathrm{gr}_\nu R_\nu$ is a quotient of a polynomial ring $k_\nu[(V_j)_{j\in J}]$ in countably many indeterminates over the residue field k_ν by a binomial ideal of the form

$$(V_j - \lambda_j V^{m(j)})_{j\in J'} \ , \ \lambda_j \in k_\nu^*$$

for some $J' \subset J$, with $|m(j)| \geq 2$.
b) The graded k_ν-algebra $\mathrm{gr}_\nu R_\nu$ is the union of a nested sequence of graded polynomial subalgebras $k_\nu[x_1^{(h)}, \dots, x_{r(\nu)}^{(h)}]$, where the inclusions are given by maps sending each variable $x_i^{(h)}$ to a constant times a monomial in the $x_j^{(h+1)}$, $1 \leq j \leq \mathrm{r}(\nu)$.

Proof. See Proposition 4.2 and the previous Corollary. We will see in Proposition 6.2 that $\mathrm{gr}_\nu R_\nu$ is isomorphic to a quotient of a polynomial algebra $k[(V_i)_{i\in I}]$ by a binomial ideal whose generators are of the form $V^m - \lambda_{mn} V^n$. Setting all the constants λ_{mn} equal to one gives the semigroup algebra $k[t^{\Phi_+}]$, hence assertion a). Assertion b) follows from this and the correspondence between the direct system of polynomial subalgebras and the binomial equations exhibited in the proof of the Corollary to Proposition 4.2. ○

It is a way to check that $\mathrm{gr}_\nu R_\nu$ is "regular" over k_ν; in some sense, all the equations have independent linear terms. This is not strictly true since the same V_j may appear linearly in infinitely many equations of the form $V_j - \lambda_j V'^{m'(j)} = 0$. However, the proof shows that all these equations eventually become the same after substituting monomials in V'' for some of the variables V', so that they may be thought of as one equation.

Remark. It is important to note that the direct system built in this way is rather precisely determined by Φ; we shall see on the next examples that at least in dimension two the structure of the binomial equations, or equivalently that of the direct system of polynomial subalgebras converging to $\mathrm{gr}_\nu R_\nu$, reflects the structure of the system of points in birational modifications of $\mathrm{Spec} R$ corresponding to the valuation ν, as well as the height and rational rank of ν.

Problem It is an interesting problem to make these correspondences precise, especially in dimensions ≥ 3. A very useful reference for dimension two is [A2].

Corollary. *The dimension of* $\mathrm{gr}_\nu R_\nu$ *is at most* $\mathrm{r}(\nu)$.

Proof. Let $(0)=\mathbf{p}_0\subset\mathbf{p}_1\subset\cdots\subset\mathbf{p}_d$ be a chain of distinct prime ideals of $\mathrm{gr}_\nu R_\nu$. Choose elements $y_i\in\mathbf{p}_i\setminus\mathbf{p}_{i-1}$. There is an integer h such that the subalgebra $k_\nu[x_1^{(h)},\ldots,x_r^{(h)}]$ contains all the y_j. The intersections $\mathbf{p}_i\cap k_\nu[x_1^{(h)},\ldots,x_r^{(h)}]$ are then distinct, so we must have $d\le r$. ○

Remark. We have seen in the remark following Proposition 3.7 that in fact equality holds. It would be nice to have a simple proof of the inequality $\dim \mathrm{gr}_\nu R_\nu\ge r(\nu)$.

4.2 A selection of examples

Example. 1) We give a slightly different proof of the fact, which we saw at the beginning of this section, that the algebra of the semigroup $\mathbf{Z}^d_+$ for the lexicographic order is a toric union of polynomial algebras; the relation between the two proofs is the idea for the induction on the height shown above. Let Φ be $\mathbf{Z}^d$ with the lexicographic order; the semigroup Φ_+ is generated by the following elements $e_j^{(1)}=(1,-j,0,\ldots,0), e_j^{(2)}=(0,1,-j,\ldots,0),\ldots,e_j^{(d-1)}=(0,0,\ldots,1,-j)$, $e_0^{(d)}=(0,0,\ldots,0,1)$ for $j=0,1,2,\ldots$. To check this, we proceed by induction on d; the case $d=1$ is obvious, so let $d\ge 2$ and $\phi=(a_1,\ldots,a_d)\in\mathbf{Z}^d_+$. If we have $a_1=0$ we are reduced to $\mathbf{Z}^{d-1}$, so we may assume $a_1>0$. If $a_2\ge 0$, we may write $\phi=a_1e_0^{(1)}+(0,a_2,\ldots,a_d)$ and we are again reduced to the case of $\mathbf{Z}^{d-1}$. If $a_2<0$, let ℓ be the smallest positive integer such that $\ell a_1+a_2\ge 0$; we can write $\phi=a_1e_\ell^{(1)}+(0,\ell a_1+a_2,a_3,\ldots,a_d)$ and again we may conclude by induction. It is easy to check, again by induction, that the relations are generated by $e_j^{(i)}=e_{j+1}^{(i)}+e_0^{(i+1)}$ for $1\le i\le d-1,\ j\ge 0$. The semigroup algebra $A[t^{\mathbf{Z}^d_+}]$ is isomorphic to

$$A[(V_{j_1}^{(1)})_{j_1\in\mathbf{N}_0},\ldots,(V_{j_{d-1}}^{(d-1)})_{j_{d-1}\in\mathbf{N}_0},V_0^{(d)}]/\big((V_j^{(i)}-V_{j+1}^{(i)}V_0^{(i+1)})_{j\in\mathbf{N}_0,1\le i\le d-1}\big),$$

where $\mathbf{N}_0=\mathbf{N}\cup\{0\}$. Note that from these equations we can read the fact that the valuation is of height d; they show that for each $(i,j),\ 1\le i\le d-1,\ j\ge 1$ and each integer n we have $V_j^{(i)}=V_{j+n}^{(i)}(V_0^{(i+1)})^n$, so that the value in Φ_+ of $V_j^{(i)}$ is greater than n times the value of $V_0^{(i+1)}$ for all $n\ge 1$. This implies the existence of $d-1$ non trivial convex subgroups of Φ; since we know that $\mathrm{h}(\Phi)\le\mathrm{r}(\Phi)$, we have $\mathrm{h}(\Phi)=d$.
In particular, for $d=2$, we see that

$$A[t^{\mathbf{Z}^2_+}]=A[W,V_0,V_1,\ldots,V_j,\ldots]/\big((V_j-WV_{j+1})_{j\in\mathbf{N}\cup\{0\}}\big).$$

Upon taking $A=k$, a field, we see that it corresponds to the sequence of infinitely near points in the plane $\mathbf{A}^2(k)$ obtained by endlessly repeating the following process: blow up the origin of an ordered set of coordinates (u_1,u_2), choose the chart of the blow up where the ideal (u_1,u_2) is generated by u_1, and in this chart, with coordinates $(u_1, u_2/u_1)$, blow up the origin. We see that the direct limit of the monomial maps $k[u_1, u_2] \to k[u_1, u_2/u_1]$ obtained by repeating this operation is $k[t^{\mathbf{Z}^2_+}]$ presented as above, with $V_j = u_2 u_1^{-j}$, $W = u_1$.

By the correspondence between sequences of points in blowing ups and valuations ([V]), this sequence of points corresponds to a valuation ν of the local ring $R = k[u, v]_{(u,v)}$, the monomial valuation in the coordinates (u, v) with value group $\mathbf{Z}^2$, lex determined by $\nu(u) = (1, 0)$, $\nu(v) = (0, 1)$. The graded algebra $\mathrm{gr}_\nu R_\nu$ is equal to $k[t^{\mathbf{Z}^2_+}]$. The valuation ν is monomial with respect to the coordinates (u, v), so that we have $\mathrm{gr}_\nu R = k[U, V]$, where U, V are the initial forms of u, v. It can be deemed to mean that the vanishing of a polynomial $P \in k[u, v]$ at the infinitely near points determined by the valuation depends only on the smallest of the valuations of the monomials which it contains. Things are quite similar for $d > 2$.

2) Given a positive real number τ, let us denote by

$$\tau = [s_1, s_2, \ldots, s_i, s_{i+1}, \ldots] = s_1 + \frac{1}{s_2 + \cdots}$$

its continued fraction expansion. Let $p_i/q_i = [s_1, \ldots, s_i]$ be the i-th approximant. Using the inductive relations

$$p_{i+1} = p_{i-1} + s_{i+1} p_i, \quad q_{i+1} = q_{i-1} + s_{i+1} q_i \quad \text{with } p_0 = 1,\ q_0 = 0,\ p_1 = s_1,\ q_1 = 1,$$

the reader will see that Perron's algorithm produces the vectors $A^{(h)} = (p_h, q_h) \in \mathbf{R}^2$ for $h \geq 1$. If Φ denotes the subgroup of $\mathbf{R}$ generated by $(1, \tau)$, the algorithm gives the following presentation, for any commutative ring A

$$A[t^{\Phi_+}] = A[(V_i)_{i \in \mathbf{N}}] / \big((V_i - V_{i+1}^{s_i} V_{i+2})_{i \in \mathbf{N}}\big).$$

It is equivalent to say that it presents $A[t^{\Phi_+}]$ as the direct limit of the maps

$$A[V_i, V_{i+1}] \to A[V_{i+1}, V_{i+2}] \text{ given by } V_i \mapsto V_{i+1}^{s_i} V_{i+2}, \quad V_{i+1} \mapsto V_{i+1}.$$

We shall see below in example 3) the interpretation of these maps in terms of blowing ups, the corresponding valuation of $k[u, v]_{(u,v)}$ and the graded algebra of its valuation ring.

3) Recall the examples given in [Z-S]. Vol.2, Chap. VI, §15, p.102: let k be a field, $\{s_1, s_2, \ldots\}$ a sequence of positive integers such that the products $s_1 s_2 \cdots s_i$ tend to infinity with i, and $\{c_1, c_2, \ldots\}$ a sequence of elements of k^*. We define an infinite sequence of elements v_i in $k(u, v)$ by

$$v_1 = u, v_2 = v, \ldots, v_{i+2} = \frac{v_i - c_i v_{i+1}^{s_i}}{v_{i+1}^{s_i}}, \ldots.$$

Setting $R_i = k[v_{i-1}, v_i]$, $m_i = (v_{i-1}, v_i)R_i$, Zariski shows that the ring

$$R_\nu = \bigcup_{i=2}^{\infty} (R_i)_{m_i}$$

is the valuation ring of a valuation ν of $k(u,v)$ with value group

$$\Phi = \{\frac{n}{s_1 \cdots s_k}\ ,\ n \in \mathbf{Z},\ k = 1, 2, \ldots\},$$

that is, all rational numbers whose denominator is a product of s_i's. As $\nu(v_{i+2}) > 0$, we have $\nu(v_i - c_i v_{i+1}^{s_i}) > \nu(v_{i+1}^{s_i})$, so that it is not difficult to see that the graded algebra of R_ν is given by

$$\mathrm{gr}_\nu R_\nu = k[V_1, V_2, \ldots, V_i, \ldots]/\big((V_i - c_i V_{i+1}^{s_i})_{i \geq 1}\big).$$

One checks that the elements $\big((v_i)_{i\geq 1}\big)$ generate the maximal ideal of R_ν, and their initial forms generate the k-algebra $\mathrm{gr}_\nu R_\nu$. Note that this valuation is not monomial for $R = k[u,v]_{(u,v)}$ with respect to the coordinates (u,v).

In fact it is an interesting exercise to begin to compute $\mathrm{gr}_\nu R$ as a subalgebra of $\mathrm{gr}_\nu R_\nu$ in this case.

Set $R = k[u,v]_{(u,v)}$. Let us set $u_1 = u,\ u_2 = v$; from the equations defining the v_i, we see that $\nu(u_1) = 1,\ \nu(u_2) = \frac{1}{s_1}$ and $u_2^{s_1} - c_1^{-1}u_1 = c_1^{-1}u_2^{s_1}v_3 \in R$, We set $u_3 = c_1^{-1}u_2^{s_1}v_3$, so we have the following equation defining u_3

$$u_2^{s_1} - c_1^{-1}u_1 = u_3,$$

and we see similarly that $u_3^{s_2} - c_2^{-1}u_2^{s_1 s_2 + 1} = -c_2^{-1}u_3^{s_2}v_4 \in R$. We set $u_4 = -c_2^{-1}u_3^{s_2}v_4$ and so we have the equation

$$u_3^{s_2} - c_2^{-1}u_2^{s_1 s_2 + 1} = u_4$$

defining u_4, and so on. The minimal system of generators of the semigroup Γ corresponding to the values of ν on R begins with

$$\Gamma = \langle \frac{1}{s_1}, 1, 1 + \frac{1}{s_1 s_2}, s_2 + \frac{1}{s_1} + \frac{1}{s_1 s_2 s_3}, \ldots \rangle.$$

Here we must note the following important fact: the equations defining inductively the u_j are of the form

$$u_j^{s_{j-1}} - d_j u_1^{\ell_1^{(j)}} \ldots u_{j-1}^{\ell_{j-1}^{(j)}} = u_{j+1}$$

with $d_j \in k^*$, and the term on the right of the equation has greater valuation than the monomials appearing in the left hand side. When we compute $\mathrm{gr}_\nu R$ from these equations, using the remark following proposition 3.7, all the right-hand sides disappear because they have greater valuation, and in particular the first equation gives a relation $U_2^{s_1} - c_1^{-1}U_1 = 0$ between the initial forms of u_1 and of u_2, which decreases the Krull dimension of $\mathrm{gr}_\nu R$ to one in agreement with Piltant's theorem. In the ring R the first equation is captured, via its right-hand side u_3, in an abyssal[4] sequence of relations, and never gets a chance to decrease the dimension of R by one. If we stopped after a finite number of steps, setting $u_{j+1} = 0$, or $u_{j+1} = P(u_1, \dots, u_j)$ where P is any polynomial, the dimension would decrease to one.

This example displays for us a basic phenomenon: the decrease of dimension of the special fiber in the specialization of R to $\mathrm{gr}_\nu R$ in the case where Abhyankar's inequality is strict. Moreover, in both rings, all variables of index ≥ 3 are algebraically dependent upon u_1, u_2, but in $\mathrm{gr}_\nu R$ these two variables are dependent, while in R they are independent and all others are polynomials in them, which shows "from the equations" that R is regular of dimension two.

Let us now come back to $\mathrm{gr}_\nu R_\nu$. Since the rational rank of Φ is one, we expect the k-algebra $\mathrm{gr}_\nu R_\nu$ to be the direct limit of polynomial algebras in one variable, and indeed one sees easily that it is the direct limit of the system of monomial maps $k[V_i] \to k[V_{i+1}]$ given by $V_i \mapsto c_i V_{i+1}^{s_i}$.

While doing this, we have also presented the semigroup algebra of the semigroup Φ_+ as

$$A[t^{\Phi_+}] = A[(V_i)_{i\in\mathbf{N}}]/\big((V_i - V_{i+1}^{s_i})_{i\geq 1}\big).$$

We can describe the valuation in terms of blowing-ups as follows: in k^2 with ordered coordinates (v_1, v_2), we perform the following sequence of steps.

Step 1: reverse the order of coordinates.

Step 2: blow-up the origin and localize in the chart where v_2, now the first coordinate, generates the ideal (v_1, v_2), taking as coordinates $(v_2, v_1 v_2^{-1})$.

Step 3: repeat this second step $s_1 - 1$ times.

Step 4: change the second coordinate so that the point with coordinate c_1 on the axis of the second coordinate (the exceptional divisor just created) is the origin of coordinates, and localize at this point.

Repeat from step 1, replacing (s_1, c_1) by (s_2, c_2), and so on.

The valuation corresponds to the sequence of infinitely near points obtained in this way.

According to ([Z-S], Vol. 2, p. 104), any subgroup Φ of the additive group $\mathbf{Q}$ of rational numbers can be obtained in this manner.

This applies in particular to $\Phi = \mathbf{Q}$, obtained by taking $s_i = i \;\; \forall\, i \geq 1$.

Another construction of valuations whose value groups are given subgroups of $\mathbf{Q}$ is given by Zariski in [Z3], p.648, which leads to the distinction between those prime numbers which divide only a finite number of the s_i's and those which divide an infinite number of them. If we worked over a field whose characteristic belongs to the second group and tried to effectively build inductively elements like the v_i by

[4] From the Greek meaning *without bottom.*

solving equations of the type $c_i v_{i+1}^{s_i} - v_i + \cdots = 0$, we would run into substantial difficulties. However, this is not at all what we do here; we only use the direct system as a means to approximate $\mathrm{gr}_\nu R_\nu$ by polynomial subalgebras.
One may ask what happens if, in the sequence of steps just described, one makes no translation; it amounts to setting all the c_i equal to zero. Since the equations describing the formation of the v_i are

$$v_i = v_{i+1}^{s_i}(c_i + v_{i+2}),$$

we see that the graded algebra of our new valuation ν_0 becomes

$$\mathrm{gr}_{\nu_0} R_{\nu_0} = k[V_1, V_2, \ldots, V_i, \ldots]/\big((V_i - V_{i+1}^{s_i} V_{i+2})_{i \in \mathbf{N}}\big).$$

It follows from what we saw in example 2) that our valuation is the monomial valuation of height one and rational rank two defined by $\nu_0(u) = 1, \nu_0(v) = \tau$, where τ is the number with continued fraction expansion

$$\tau = [s_1, s_2, \ldots, s_i, s_{i+1}, \ldots].$$

Its value group is

$$\Phi_\tau = \mathbf{Z} + \mathbf{Z}\tau \ \subset \mathbf{R}$$

with the order induced by that of $\mathbf{R}$.
In particular, the valuations with value group $\mathbf{Q}$ constructed above become, upon setting all the c_i equal to zero, the rank two monomial valuation corresponding to $\nu(u) = 1,\ \nu(v) = \rho$, where

$$\rho = [1, 2, 3, \ldots, i, i+1, \ldots].$$

Problem. It would be interesting if, given any rational valuation ν of a nœtherian local integral domain R, one could specialize it (in the sense used just above, which differs from the usual one) to a valuation of R of rational rank equal to the dimension of R. Heuristically this means we can always "suppress the translations" continuously. As we shall see in Section 5 it is reasonable to try first for a valuation ν_0 of rational rank equal to the dimension of the ν_0-adic completion of R.
Can one use the results of [A2] to verify this statement in dimension two?

In [Z3], Zariski gives yet another construction, also found in [McL-S], of valuations on $k[u, v]_{(u,v)}$ with a value group $\Phi \subset \mathbf{Q}$ given in advance. Let $k[[u^{\mathbf{Q}_+}]]$ be the ring of power series with exponents forming a well ordered subset of $\mathbf{Q}_+$, and with the same s_i as above, given a sequence of integers m_i and a sequence of elements $d_i \in k^*$, consider the series

$$v(u) = d_1 u^{\frac{m_1}{s_1}} + d_2 u^{\frac{m_2}{s_1 s_2}} + \cdots + d_j u^{\frac{m_j}{s_1 \cdots s_j}} + \cdots ;$$

The map $u \mapsto u,\ v \mapsto v(u)$ induces an injection

$$k[u, v]_{(u,v)} \subset k[[u^{\mathbf{Q}_+}]]$$

and the u-adic valuation induces on $k(u, v)$ a valuation having as group of values the group of rational numbers whose denominator is a product of s_i's.

5 Completion problems

In this section I show that the ν-adic completion of a nœtherian analytically unramified local ring R has many of the nice properties of m-adic completions; it is nœtherian, henselian, and in the equicharacteristic case admits a field of representatives. Although completions of valued fields are a classical topic, the algebraic structure of the ν-adic completions of nœtherian rings do not seem to have been much studied. Perhaps the reason is that the nœtherian hypothesis is not natural from the viewpoint of the theory of valued fields, while completions other than those with respect to an ideal are not commonplace in algebraic geometry.
The method of proof is to reduce to the case of valuations of height one by considering the residual valuations in a sequence of composed valuations. I also show how to specialize the ν-adic completion of R to the ν_{gr}-adic completion of $\mathrm{gr}_\nu R$ by a suitable completion of the valuation algebra, and how to find a coordinate system and equations for the ring of this specialization.

5.1 The ν-adic completion of R

In general, even if the local ring R has a field of representatives, it is not possible to lift generators of the graded algebra $\mathrm{gr}_\nu R$ to generators of R; we need a hypothesis of completeness (see [B3], Chap. III). The arguments using the fact that a deformation preserves smoothness and transversality also use some form of the implicit function theorem, or Hensel's Lemma, the validity of which is another attribute of complete rings. This leads us to the study of the completion of a ring R with respect to the $(\mathcal{P}_\phi)_{\phi\in\Phi_+}$-filtration. We note that the quotients $R/\mathcal{P}_\phi$ form a countable projective system since the semigroup Φ_+ is countable and totally ordered. One can then define the ν-adic completion of R as

$$\hat{R}^\nu = \varprojlim_{\phi\in\Phi_+} R/\mathcal{P}_\phi.$$

There is a natural map $R \to \hat{R}^\nu$, which is injective since the filtration is separated. The valuation ν has a canonical extension $\hat{\nu}$ to a valuation of $\hat{R}^\nu$ with values in Φ. To see this it is enough to apply directly the explicit form of the definition of the ν-adic completion as a subset of the product of the quotients $R/\mathcal{P}_\phi$. We note that $\hat{R}^\nu$ is the closure of R in the completion $\hat{K}$ of its field of fractions with respect to the topology $\mathcal{T}_\nu$ defined by ν (see [B3], Chap VI, §5, No. 3). In particular, we have

$$R_{\hat{\nu}} = \hat{R}^\nu_\nu.$$

One sees also immediately that if we set

$$\hat{\mathcal{P}}_\phi = \{x \in \hat{R}^\nu \mid \hat{\nu}(x) \geq \phi\},\quad \hat{\mathcal{P}}^+_\phi = \{x \in \hat{R}^\nu \mid \hat{\nu}(x) > \phi\},$$

they are respectively the closure of $\mathcal{P}_\phi$, $\mathcal{P}^+_\phi$ in $\hat{R}^\nu$ ([B3], Chap. VI, §5 No. 2) and the natural inclusion $\mathcal{P}_\phi/\mathcal{P}^+_\phi \subset \hat{\mathcal{P}}_\phi/\hat{\mathcal{P}}^+_\phi$ is an equality, so that the inclusion $R \subset \hat{R}^\nu$

induces an isomorphism

$$\mathrm{gr}_\nu R = \mathrm{gr}_{\hat\nu} \hat{R}^\nu.$$

Moreover, the same description shows that if $\mathbf{p}$ is a prime ideal of R, its closure in $\hat{R}^\nu$ is a prime ideal, and clearly if $\hat{\mathbf{p}}$ is a prime ideal of $\hat{R}^\nu$, its intersection $\hat{\mathbf{p}} \cap R$ is a prime ideal of R.

We note that the semigroup of values $\hat\nu(\hat{R}^\nu \setminus \{0\}) \subset \Phi_+$ is the same as that of ν on R, and so is well ordered when R is nœtherian (Proposition 3.8).

Let us assume that R is a nœtherian local ring with maximal ideal m and ν is a valuation of K of height one such that its valuation ring R_ν contains R. Set $\mathbf{p} = m_\nu \cap R$; since R is nœtherian, we have $\nu(\mathbf{p}) > 0$, say $\nu(\mathbf{p}) = \phi_0$, and since ν is of height one, its group Φ is archimedian, so that for any $\phi \in \Phi$ there exists an integer $N(\phi)$ such that $N(\phi)\phi_0 \geq \phi$, that is $\mathbf{p}^{N(\phi)} \subset \mathcal{P}_\phi$. Then we have a projective system of surjective maps

$$R/\mathbf{p}^{N(\phi)} \to R/\mathcal{P}_\phi,$$

which induces a continuous morphism of completions $\hat{R}^{\mathbf{p}} \to \hat{R}^\nu$, where $\hat{R}^{\mathbf{p}}$ is the completion of R for the $\mathbf{p}$-adic topology. Since $\hat{R}^\nu$ is an integral domain, the kernel of this map is a prime ideal $H \subset \hat{R}^{\mathbf{p}}$. By the definition of completions, this kernel is $\bigcap_{\phi\in\Phi_+} \mathcal{P}_\phi \hat{R}^{\mathbf{p}}$, so that in the case where ν is of height one and R_ν dominates R, i.e., $\mathbf{p} = m$, it coincides with the ideal called the implicit ideal of $\hat{R}$ by Spivakovsky ([S2]). In any case, we have the following:

Lemma 5.1. *For a valuation ν of height one on an analytically irreducible nœtherian local ring R, with the notations introduced above, the natural injection*

$$\hat{R}^{\mathbf{p}}/H \to \hat{R}^\nu$$

is an isomorphism of topological rings. If the center of ν in R is the maximal ideal, we may drop the assumption that R is analytically irreducible.

Proof. By ([B3], Chap. III, §3, No.4, Prop.8), the ring $\hat{R}^{\mathbf{p}}/H$ is local (with maximal ideal $\hat{m} = m\hat{R}^{\mathbf{p}}/H$) and nœtherian, as a quotient of a ring having these properties. Moreover, the injection

$$\hat{R}^{\mathbf{p}}/H \subset \hat{R}^\nu$$

implies that the valuation ν extends to $\hat{R}^{\mathbf{p}}/H$ as a valuation $\hat\nu$ of height one.
The valuation ν is still ≥ 0 on the localization $R_{\mathbf{p}}$, and so we have a morphism of completions $\hat{R}_{\mathbf{p}}^{\mathbf{p}R_{\mathbf{p}}}/H_{\mathbf{p}} \subset \hat{R}^\nu_{\mathbf{p}}$. Let us replace R by $R_{\mathbf{p}}$ for a moment, so that $\mathbf{p}$ is the maximal ideal m. By [Z-S], Vol. 2, Appendix 3, Lemma 3 p. 343, the distinct valuation ideals (i.e., our $\mathcal{P}_\phi(\hat{R}^m/H)$ written without repetition) form a simple infinite descending chain of ideals which are primary for the maximal ideal $\hat{m}$ of $\hat{R}^m/H$ and have intersection zero. We denote them by $\hat{\mathcal{P}}_j$. Now we can apply

Chevalley's Theorem ([Z-S], Vol. 2, Chap. VIII, §5, Th. 13, p. 270), which asserts that there exists an integer-valued function $s(n)$ tending to infinity with n and such that for each valuation ideal $\hat{\mathcal{P}}_j$, we have $\hat{\mathcal{P}}_j \subset \hat{m}^{s(j)}$. This, added to the fact that the $\hat{\mathcal{P}}_j$ are primary for $\hat{m}$, proves that in the ring $\hat{R}^m/H$ the $\hat{\nu}$-adic topology coincides with the $\hat{m}$-adic topology, so that it is complete for both, and therefore has to be equal to $\hat{R}^\nu$.

Now coming back to the case where the center of ν is not necessarily m, this shows that in R we have the inclusions $\mathcal{P}_j \subset \mathbf{p}^{(s(j))}$, where $\mathbf{p}^{(k)}$ denotes the symbolic power $\mathbf{p}^{(k)} = \mathbf{p}^k R_{\mathbf{p}} \cap R$. Consider the prime ideals $\mathbf{q}$ containing $\mathbf{p}$. If we know that for all such ideals having height $\leq k-1$ the $\mathbf{q}^{(n)}$-topology coincides with the $\mathbf{q}^n$ topology, then by ([Z-S], Chap VIII, §5, Cor. to Theorem 14, page 276), the $\mathbf{q}^{(n)}$-completion of R is a subring of the m-adic completion $\hat{R}^m$, which is a local nœtherian integral domain by assumption, and so is an integral domain. Now we may apply ([Z-S], Chap VIII, §5, Cor. 5 p. 275) to prove that for all prime ideals $\mathbf{q}$ having height $\leq k$ the $\mathbf{q}^{(n)}$-topology coincides with the $\mathbf{q}^n$-topology. In the case where the height of $\mathbf{p}$ is one, by Abyankar's inequality (see [V]), the valuation is necessarily of height one so that its value group is $\mathbf{Z}$ and the topology which it defines on $R_{\mathbf{p}}$ is the same as the $\mathbf{p}R_{\mathbf{p}}$-adic topology of that ring and so the topology it defines on R is that of the $\mathbf{p}^{(n)}$. But because of the archimedian property this topology coincides with that of the $\mathbf{p}^n$. By this induction we see that since R is analytically irreducible, the topology defined by the symbolic powers of $\mathbf{p}$ coincides with the $\mathbf{p}$-adic topology. This gives the result in the general case. ◌

Remark. 1) This shows that the ring $\hat{R}/H$ to which Spivakovsky extends the valuation ν in the special case of a valuation of height one such that R_ν dominates R is nothing but $\hat{R}^\nu$.

2) In particular if R is local nœtherian, its completion $\hat{R}^{\mathbf{p}}$ for the $\mathbf{p}$-adic topology is a nœtherian ring ([B3], Chap. III, §3, No.4, Prop.8), and so if R is analytically irreducible, for every rank one valuation ν of the field of fractions of R whose valuation ring contains R, the completion $\hat{R}^\nu$ of R for the ν-adic valuation is nœtherian; as a topological ring it is in fact a Zariski ring with respect to the center of the valuation $\hat{\nu}$.

Proposition 5.2. *Let R be an analytically irreducible nœtherian local ring contained in a valuation ring R_ν of its field of fractions. The completion $\hat{R}^\nu$ is a nœtherian ring, which is isomorphic to the completion $\hat{R}^{\nu_1}$ of R with respect to the height one valuation ν_1 with which the valuation ν is composed, and is a quotient of the* $\mathbf{p}$*-adic completion of R, where $\mathbf{p}$ is the center of ν_1 in R.*

Proof. We saw above that is true for valuations of height one. Because of Abhyankar's inequality, the valuation ν is of finite height. Let us assume that $h(\nu) > 1$. By assumption we have a surjective monotone map $\lambda\colon \Phi \to \Phi_1$ with Φ_1 of height one and $\mathrm{Ker}\lambda$ of height $h-1$, a maximal isolated subgroup of Φ.

Denote by ν_1 the valuation $\lambda \circ \nu$. Then we have the inclusion $\mathcal{P}_\phi \subset \mathcal{P}_{\lambda(\phi)}$. Whenever $\phi_1 > \lambda(\phi)$, we have $\mathcal{P}_{\phi_1} \subset \mathcal{P}_\phi \subset \mathcal{P}_{\lambda(\phi)}$, which shows that the ν-adic topology on R coincides with the ν_1-adic topology, hence the equality of the completions. We are thus reduced to the case of height one treated above, and we see that the completion $\hat{R}^\nu$ is a quotient of the completion of R with respect to the $(m_{\nu_1} \cap R)$-adic filtration, hence is a nœtherian ring. ○

Remark. 1) In a nœtherian domain R with a non negative valuation ν, the ideal of elements of R which are topologically nilpotent with respect to the ν-adic topology is the center in R of the rank one valuation with which ν is composed.

2) The equality of the topologies determined on a nœtherian ring by two valuations, one of which is composed with the other, is well known; a more precise result can be found in [Z-S], Vol. 2, Lemma 4 of Appendix 3, p. 344.

Corollary. *Given a local nœtherian analytically irreducible ring R and a valuation ν of its field of fractions, non negative on R, if ν is composed with a valuation ν_1, the natural map of completions*

$$\hat{R}^\nu \longrightarrow \hat{R}^{\nu_1}$$

induced by the inclusions $\mathcal{P}_\phi \subset \mathcal{P}_{\lambda(\phi)}$ is an isomorphism of topological nœtherian rings. In particular, the valuation ν_1 extends to a valuation $\hat{\nu}_1$ on $\hat{R}^\nu$.

Corollary. *If R is an analytically irreducible excellent local ring, for any valuation ν of its field of fractions which is non negative on R, the ring $\hat{R}^\nu$ is excellent.*

Proof. By Proposition 5.2, the ν-adic completion $\hat{R}^\nu$ is a quotient of the completion $\hat{R}^{\mathbf{P}}$ of the local ring R with respect to an ideal, so it is excellent as a quotient of an excellent ring (see [Ro1], [Ro2]). ○

Corollary. *Abhyankar's inequality can be made slightly more precise; with the notations of section 3, we have:*

$$\mathrm{r}(\nu) + \mathrm{t}_{k_R}(\nu) \leq \dim \hat{R}^\nu \leq \dim R.$$

When trying to understand how to complete some non-nœtherian rings, essentially semigroup algebras, one encounters questions such as this

Question. (Compare with [Ka]) Let A be a commutative ring and Γ the value semigroup of a valuation of height one on a nœtherian ring R. Is it true that the completion of the semigroup algebra $A[t^\Gamma]$ with respect to the topology deduced from the canonical valuation $\nu(\sum a_\phi t^\phi) = \min\{\phi | a_\phi \neq 0\}$ is the ring of power series $A[[t^\Gamma]]$ in the sense of [McL-S]? What about height > 1?
Remark that $A[[t^\Gamma]]$, which a priori is only a group, is a ring, since in Γ there are only finitely many ways to write an element as a sum of other elements (because Γ is well ordered by Proposition 3.8; see [Ka], [B2], [C-G], Theorem 1), so that we can multiply two series.

Example. 1) Take the usual example of a rank one valuation not satisfying Abhyankar's equality (see above and [Z-S], Chap VI, §15, Example 2, [S1], [S2]): let k be a field, and take $R = k[u,v]_{(u,v)}$ so that $\hat{R} = k[[u,v]]$. Let $f(u,v) = u - \sum a_i v^i \in \hat{R}$ be such that it is transcendental over R. Define a valuation ν on R by means of the injective map $R \to k[[t]]$ determined by $v \mapsto t$, $u \mapsto \sum a_i t^i$. The residual transcendence degree of ν is 0 and its rational rank is one. The completion $\hat{R}^\nu$ is isomorphic to the quotient $k[[u,v]]/(f)$; it is one-dimensional.
This shows that the injective map $R \to \hat{R}^\nu$ is not flat in general. Note that we could have taken an irreducible element $f \in k[[u,v]]$ in the maximal ideal and not algebraic over $k[u,v]$ and the composed map $k[u,v]_{(u,v)} \to k[[u,v]]/(f) \to k[[t]]$ where the last is the normalization.

2) On the same ring R, consider the valuation ν with value group $\mathbf{Z}^2$ ordered lexicographically, which attributes value $(1,0)$ to u, value $(0,1)$ to v, and such that the valuation of a sum of monomials is the infimum of the valuations of these monomials. There is a natural monotone surjective map $\lambda = \mathrm{pr}_1 \colon \mathbf{Z}^2 \to \mathbf{Z}$, hence a valuation ν_1, by the u-adic order. The ideal $\mathcal{P}_{(i,j)}$ is the product ideal $u^i(u, v^j)$ of R, while the ideal $\mathcal{P}_i$ for ν_1 is (u^i). Since $(u^{i+1}) \subset u^i(u,v^j) \subset (u^i)$, these two filtrations define the same topology on R, so that in this case,

$$\hat{R}^\nu = \hat{R}^{\nu_1}.$$

Remark that the residual ideal $(\mathcal{P}_{(i,j)} : \mathcal{P}_i) = (u, v^j)$ is a simple ν-ideal in R in the sense of Zariski's theory (see [Z-S], Vol.2, Appendix 3).
In this case, the completions of $k[u,v]_{(u,v)}$ with respect to ν and ν_1 are isomorphic to

$$\widehat{k[u,v]_{(u,v)}}^{\nu} = \varprojlim_{i \geq 0} k[u,v]_{(u,v)}/(u)^i;$$

the ideal $\mathbf{p}$ is $(u)k[u,v]_{(u,v)}$ and the ideal H is zero.
3) (a non-discrete case) Let $\Phi \subset \mathbf{Q}$ be a group ordered by the order of $\mathbf{Q}$. Consider the formal power series ring $k[[t^{\Phi_+}]]$ with coeficients in k and exponents in Φ_+, such that the set of their exponents is a well ordered subset of Φ_+ (see [Ka]). This ring is naturally equipped with the t-adic valuation. Choose series without constant term $y_i(t) \in k[[t^{\Phi_+}]]$, $1 \leq i \leq r$ and let R be the sub k-algebra of $k[[t^{\Phi_+}]]$ generated by $y_1(t), \ldots, y_r(t)$. Since R is a nœtherian ring, the semigroup $\Gamma = \nu(R \setminus \{0\})$ is well ordered (Proposition 3.8), so that we can find a countable minimal system of generators for the semigroup Γ (see section 6 below):

$$\Gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_i, \ldots \rangle.$$

Since R is the image of a morphism $k[Y_1, \ldots, Y_r] \to k[[t^{\Phi_+}]]$ determined by $Y_i \mapsto y_i(t)$, we may attribute to Y_i the valuation $\nu(y_i(t))$, and define ideals $\tilde{\mathcal{P}}_\phi \subset k[Y_1, \ldots, Y_r]$ consisting of those elements whose image in $k[[t^{\Phi_+}]]$ has valuation $\geq \phi$. The sequence of these ideals is indexed by the γ_i. Our map extends, by substitution in formal power series, to a map $k[[Y_1, \ldots, Y_r]] \to k[[t^{\Phi_+}]]$, with a certain kernel H, which describes the formal relations between the $y_i(t)$. By lemma 5.1, the completion $\hat{R}^\nu$ is the image of this map.

Let R be as in Proposition 5.2, and let $\iota\colon R \to \hat{R}^\nu$ be the completion morphism. Let ν_1 be a valuation with which ν is composed, and denote by $\hat{\nu}_1$ its extension to $\hat{R}^\nu$ according to the first Corollary. Denote by $\overline{\nu}$ the residual valuation on $R/(m_{\nu_1} \cap R)$. After subsection 3.4, we know that the $\overline{\nu}$-adic topology on $R/(m_{\nu_1} \cap R)$ is the quotient of the ν-adic topology of R. Then we have:[5]

Proposition 5.3. *The morphism*

$$R/(m_{\nu_1} \cap R) \to \hat{R}^\nu/(m_{\hat{\nu}_1} \cap \hat{R}^\nu)$$

induced by ι is the completion morphism of the first ring with respect to the $\overline{\nu}$-adic topology, quotient of the ν-adic topology of R.

Proof. Since R is nœtherian, the value semigroup Γ of ν on R is well ordered (Proposition 3.8), so that the the ν-adic topology may be defined by a decreasing chain of ideals. Moreover, if δ_1 denotes the smallest non-zero element of $\Gamma_1 = \nu_1(R \setminus \{0\})$, which is also well ordered, the ideal $m_{\nu_1} \cap R$ is equal to $\{x \in R \mid \nu_1(x) \geq \delta_1\}$, so that its closure in $\hat{R}^\nu$ is $m_{\hat{\nu}_1} \cap \hat{R}^\nu$. Now we can apply Theorem 8.1, page 56 of [Ma], which states the equality we seek. ◯

Proposition 5.4. *Let $R_\nu \subset R_{\nu_1} \subset \cdots \subset R_{\nu_s}$ be a sequence of consecutive composed valuations, all composed with ν. Given a local subring $R \subset R_\nu$ with the same field of fractions, let $\hat{\nu}_i$ be the extension to $\hat{R}^\nu$ of the valuation ν_i (see the first Corollary to Proposition 5.2) and let*

$$m_{\hat{\nu}_s} \cap \hat{R}^\nu \subseteq \cdots \subseteq m_{\hat{\nu}_1} \cap \hat{R}^\nu \subseteq m_{\hat{\nu}}$$

be the corresponding sequence of prime ideals of $\hat{R}^\nu$. Each pair $(\hat{R}^\nu, m_{\hat{\nu}_i} \cap \hat{R}^\nu)$ is henselian. In particular, the local ring $(\hat{R}^\nu, \hat{m})$ is henselian.

Proof. We may extend the sequence of consecutive valuation rings to the right of R_{ν_s} until we reach the ring of the valuation of height one ν_{h-1} with which ν is composed.
Notice first the by definition of restricted power series, for $m_{\hat{\nu}_j} \cap \hat{R}^\nu \subseteq m_{\hat{\nu}_{j-1}} \cap \hat{R}^\nu$, the natural map of polynomial rings extends to a map of rings of restricted power series

$$\big(\hat{R}^\nu/(m_{\hat{\nu}_j} \cap \hat{R}^\nu)\big)\{X\} \to \big(\hat{R}^\nu/(m_{\hat{\nu}_{j-1}} \cap \hat{R}^\nu)\big)\{X\}.$$

We prove that $(\hat{R}^\nu,\ m_{\hat{\nu}_i} \cap \hat{R}^\nu)$ satisfies Hensel's Lemma in the sense of Bourbaki (*cf* [B3], Chap. 3, §4, No.3), which we now recall.

[5] In the classical theory of the completion of valued fields, it is the ν-adic completions of the valuation rings which play a role, more or less explicitely. They are not nœtherian in general, and a result such as this Proposition is not true; I believe this led to the introduction of *complétion par étages*; see [Ri].

Let $F \in \hat{R}^\nu\{X\}$ be a restricted power series, and assume that the image of F modulo $m_{\hat{\nu}_i} \cap \hat{R}^\nu$, denoted by $\overline{F}$, admits a factorization $\overline{F} = \overline{P}.\overline{Q}$, where $\overline{P} \in \big(\hat{R}^\nu/(m_{\hat{\nu}_i} \cap \hat{R}^\nu)\big)[X]$ is a unitary polynomial, and $\overline{Q}$ a restricted power series $\overline{Q} \in \big(\hat{R}^\nu/(m_{\hat{\nu}_{j-1}} \cap \hat{R}^\nu)\big)\{X\}$, with $\overline{P}$, $\overline{Q}$ strongly relatively prime. Then they lift in a unique way to a unitary polynomial $P \in \hat{R}^\nu[X]$ and a restricted power series $Q \in \hat{R}^\nu\{X\}$, strongly relatively prime, such that $F = P.Q$. If F is a polynomial, so is Q.

To prove this, note that the ideals $m_{\hat{\nu}_k} \cap \hat{R}^\nu$ are closed for the $\hat{\nu}_k$-adic topologies, which all coincide. By Theorem 1 of the quoted paragraph of [B3], the result is true for ν_{h-1}, which is a valuation of height one, since the elements of $m_{\hat{\nu}_{h-1}} \cap \hat{R}^\nu$ are topologically nilpotent as we saw in Remark 1) following Proposition 5.2.
To prove the general case, it suffices to note that the statement is amenable to induction on the height and that if $\mathrm{h}(\nu_{j-1}) > 1$, there is a unique (up to isomorphism) valuation ν_j of R which is the immediate successor of ν_{j-1}. The valuation induced by $\hat{\nu}_{j-1}$ on the quotient ring $\hat{R}^\nu/(m_{\hat{\nu}_j} \cap \hat{R}^\nu)$ is of height at most one, according to Section 3.5. Assuming that we have a decomposition in $\big(\hat{R}^\nu/(m_{\hat{\nu}_{j-1}} \cap \hat{R}^\nu)\big)\{X\}$ as in Hensel's lemma, by using the case of height one, we can lift it to a decomposition in $\big(\hat{R}^\nu/(m_{\hat{\nu}_j} \cap \hat{R}^\nu)\big)\{X\}$, and continue this until we have a decomposition in $\big(\hat{R}^\nu/\hat{\mathbf{p}}\big)\{X\}$, where $\hat{\mathbf{p}}$ is the center of the valuation $\hat{\nu}_{h-1}$ of height one with which $\hat{\nu}$ is composed, and we are reduced to the case of height one. The last sentence is proved by applying this result to a valuation ν which dominates (R, m), which is possible by subsection 3.6. ♁

Another consequence of the nœtherianity of $\hat{R}^\nu$ is the existence of a field of representatives for the completions $\hat{R}^\nu$ in the equicharacteristic case.

Proposition 5.5. *Let R be a nœtherian equicharacteristic analytically irreducible local ring, and ν a valuation of its field of fractions, non negative on R. The completion of R for the ν-adic topology admits a field of representatives.*

Proof. We keep the notations introduced above, but now we use the results of subsection 3.6 to produce a sequence of consecutive specializations of the valuation ν having a center in R, of the form

$$R \subset R_{\nu_0} \subset R_{\nu_1} \subset \cdots \subset R_{\nu_j} = R_\nu$$

and such that R_{ν_0} dominates R. We can extend it to the right until we reach the valuation ring of the valuation of height one ν_{h-1} with which ν is composed. We have a sequence of quotient rings

$$\hat{R}^\nu/(m_{\hat{\nu}_{h-1}} \cap \hat{R}^\nu) \to \cdots \to \hat{R}^\nu/(m_{\hat{\nu}_1} \cap \hat{R}^\nu) \to \hat{R}^\nu/(m_{\hat{\nu}_0} \cap \hat{R}^\nu) = k_R,$$

where some of the maps may be isomorphisms. We need the following

Lemma 5.6. *Let S be a nœtherian equicharacteristic local ring and I an ideal of S. Assume that the quotient S/I admits a field of representatives and that S is complete for the I-adic topology. Then S admits a field of representatives.*

Proof. This follows from Cohen's Theorem as found in [Z-S], Vol. 2, Chap. VIII, §12 after making the following remark.
Let I be an ideal of S such that S/I admits a field of representatives. Let $\phi\colon S \to S/I$ be the natural map. Set $B = \phi^{-1}(k)$, where $k \subset S/I$ is a field of representatives. We have $I = m \cap B$, where m is the maximal ideal of S.
It follows from this that B is closed in S for the I-adic topology, so that it is complete for its $m_B = m \cap B$-adic topology, and by Cohen's Theorem it has a field of representatives, which is also a field of representatives for S. ○

We apply the lemma to the maps

$$\hat{R}^\nu/(m_{\hat{\nu}_j} \cap \hat{R}^\nu) \to \hat{R}^\nu/(m_{\hat{\nu}_{j-1}} \cap \hat{R}^\nu).$$

Each of them satisfies the hypotheses of the Lemma since by subsection 3.5 the valuation induced by $\hat{\nu}_{j-1}$ on $\hat{R}^\nu/(m_{\hat{\nu}_j} \cap \hat{R}^\nu)$ is of height at most one, and so the residual topology, for which this ring is complete, coincides with the $(m_{\hat{\nu}_{j-1}} \cap \hat{R}^\nu)/(m_{\hat{\nu}_j} \cap \hat{R}^\nu)$-adic topology. So we can step by step lift k to a field of representatives of $\hat{R}^\nu/(m_{\hat{\nu}_{h-1}} \cap \hat{R}^\nu)$. Since the valuation ν_{h-1} is of height one, the ring $\hat{R}^\nu$ is complete for the $\hat{\mathbf{p}}$-adic topology, where $\hat{\mathbf{p}} = m_{\hat{\nu}_{h-1}} \cap \hat{R}^\nu$ and we can apply the lemma one last time to lift the field of representatives to $\hat{R}^\nu$. ○

Question. Can one extend this result to the case where R is not equicharacteristic, to prove the existence of an injection $W(k) \to \hat{R}^\nu$ for a nœtherian analytically irreducible local ring R of unequal characteristic with a perfect residue field k and a valuation ν, where $W(k)$ is the ring of Witt vectors relative to the residue field k of S ([B3], Chap. IX, §§1,2, [Ei], Chap.7)?

Let us go back to the notations of the beginning of this subsection, and write $\hat{R}^\nu = \hat{R}^{\mathbf{p}}/H$ according to Proposition 5.2. We have:

Proposition 5.7. (Spivakovsky [S2]) *If the local ring R is excellent, the localization $\hat{R}^{\mathbf{p}}_H$ is regular.*

Proof. Since the map $R \to \hat{R}^\nu$ is injective, we have $H \cap R = (0)$ so that $\hat{R}^{\mathbf{p}}_H$ is a localization of the ring $\hat{R}^{\mathbf{p}} \otimes_R K$, where K is the field of fractions of R. It suffices to prove that this tensor product is regular. By ([EGA], Scholie 7.8.3, (v)), if R is excellent, the map $\mathrm{Spec}\hat{R}^{\mathbf{p}} \to \mathrm{Spec}R$ is regular. Our tensor product corresponds to the generic fiber of this map, so it is regular. ○

5.2 The $\nu_{\mathcal{A}}$-adic completion of $\mathcal{A}_\nu(R)$

The ν-adic completion will play an important role in the sequel, for the reasons explained in the introduction. In addition, we need to specialize the ring $\hat{R}^\nu$ to its associated graded ring in a way which is controlled by the generators of the graded algebra. The tool to do this is the completion of $\mathcal{A}_\nu(R)$ with respect to the filtration given by $\nu_{\mathcal{A}}$. It is described by

Proposition 5.8. *In the situation of 5.1, the completion* $\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}$ *is the closure of* $\mathcal{A}_{\hat{\nu}}(\hat{R}^\nu)$ *in the algebra of restricted power series* $\hat{R}^\nu\{v^\Phi\}$ *defined by*

$$\hat{R}^\nu\{v^\Phi\} = \varprojlim_{\phi\in\Phi_+} (R/\mathcal{P}_\phi)[v^\Phi].$$

Proof. This follows from the definitions, the fact that $\mathcal{S}_\delta = \mathcal{P}_\delta R[v^\Phi]\cap\mathcal{A}_\nu(R)$ and [B4], Chap II, §3, No. 9, Cor.1. ◯

Note that the closure in $\hat{R}^\nu$ of $\mathcal{P}_\phi$ is $\hat{\mathcal{P}}_\phi$ and that, with the notations of 1.3, we have:

$$\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}} = \varprojlim_{\delta\in\Phi_+} \mathcal{A}_\nu(R)/\mathcal{S}_\delta = \varprojlim_{\delta\in\Phi_+} \bigoplus_{\phi\in\Phi} \big(\mathcal{P}_\phi(R)/\mathcal{P}_\phi(R)\cap\mathcal{P}_\delta(R)\big)v^{-\phi}.$$

So we may view $\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}$ as the subring of the $\hat{R}^\nu$-module $\hat{R}^\nu[[v^\Phi]]$ of formal power series $\sum x_\phi v^{-\phi}$, consisting of power series $\sum x_\phi v^{-\phi}$ such that $\hat{\nu}(x_\phi)\geq\phi$ and which are restricted, i.e., such that for any $\psi\in\Phi_+$, all but a finite number of the coefficients x_ϕ are in $\mathcal{P}_\psi$. This last condition implies that when we formally multiply two series

$$\big(\sum x_\phi v^{-\phi}\big)\big(\sum x_\psi v^{-\psi}\big) = \sum_{\eta\in\Phi}\big(\sum_{\phi+\psi=\eta} x_\phi y_\psi\big)v^{-\eta},$$

the coefficient of $v^{-\eta}$ is a convergent series in $\hat{R}^\nu$.

Lemma 5.9. *The closure*

$$\overline{(v^{\Phi_+})\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}}$$

in

$$\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}$$

of the ideal $(v^{\Phi_+})\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}$ *is the set of series*

$$\sum x_\phi v^{-\phi}\in\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}} \quad \text{such that} \quad \hat{\nu}(x_\phi)>\phi.$$

Proof. The proof follows immediately from 5.8. ◯

Definition. With the notations of §1.4, let us define on the model of ([B4], exerc. 27 for §2) the algebra

$$\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}\{(v^{\Phi_+})^{-1}\} = \varprojlim_{\delta\in\Phi_+} (v^{\Phi_+})^{-1}\mathcal{A}_\nu(R)/\mathcal{S}_\delta = \widehat{\mathcal{A}_{\hat\nu}(\hat R^\nu)}^{\hat\nu_{\hat{\mathcal{A}}}}\{(v^{\Phi_+})^{-1}\}.$$

This ring is also the Hausdorff completion of $(v^{\Phi_+})^{-1}\mathcal{A}_\nu(R)$ for the topology having as a fundamental system of neighborhoods the $(v^{\Phi_+})^{-1}\mathcal{P}_\delta$. We follow Bourbaki's convention according to which writing $A\{S^{-1}\}$ entails the completion of A, i.e., $A\{S^{-1}\} = \hat A\{S'^{-1}\}$ where S' is the image of S in $\hat A$. The inclusion $\mathcal{A}_\nu(R) \longrightarrow \mathcal{A}_\nu(R)\{(v^{\Phi_+})^{-1}\}$ is universal with respect to continuous maps $\mathcal{A}_\nu(R) \to B$ for linearly topologized Hausdorff and complete rings B such that the images in B of the elements of the multiplicative set (v^{Φ_+}) are invertible.
We also consider the Hausdorff completion $\widehat{\mathrm{gr}_\nu} R$ of the graded algebra $\mathrm{gr}_\nu R$ with respect to the valuation ν_{gr}.
According to the general properties of completions of linearly topologized modules ([Ma], Prop. in 23.I, p.167), the completion of $\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}/(v^{\Phi_+})\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}$ for the quotient topology is $\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}/\overline{(v^{\Phi_+})\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}}$; it is also the completion, for the quotient topology, of $\mathcal{A}_\nu(R)/(v^{\Phi_+})\mathcal{A}_\nu(R)$. We can now state

Proposition 5.10. *a) The natural map*

$$\mathcal{A}_\nu(R) \longrightarrow \mathrm{gr}_\nu R$$

defined by

$$x_\phi v^{-\phi} \mapsto x_\phi \mod.\mathcal{P}^+_\phi(R)$$

induces an isomorphism of filtered rings

$$\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}/\overline{(v^{\Phi_+})\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}} \longrightarrow \widehat{\mathrm{gr}_\nu} R.$$

b) The natural inclusion $R[v^{\Phi_+}] \longrightarrow \mathcal{A}$ obtained by considering only the part of negative degree (i.e., $\phi\in\Phi_-$) of $\mathcal{A}$ induces an isomorphism

$$R[v^{\Phi_+}]\{(v^{\Phi_+})^{-1}\} = \hat R^\nu\{v^{\Phi_+}\}\{(v^{\Phi_+})^{-1}\} \longrightarrow \widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}\{(v^{\Phi_+})^{-1}\}.$$

Here, as above, $\hat R^\nu\{v^{\Phi_+}\}$ denotes the ring of restricted power series in v^ϕ, $\phi\in\Phi_+$ and coefficients in $\hat R^\nu$.
c) Given a homomorphism $\phi\mapsto c(\phi)$ from Φ to the multiplicative group of units of R, it induces a surjection $\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}} \to \hat R^\nu$ defined by $\sum x_\phi v^{-\phi} \mapsto \sum x_\phi c(-\phi)$. The kernel of this surjection is the closure of the ideal

$$((v^\phi - c(\phi))_{\phi\in\Phi_+})\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}.$$

Proof. Part a) follows from 5.8 and the general facts on completion, and Part b) follows from 2.1 and the remark following definition 5.2. Part c) follows from 2.1 which contains the fact that the corresponding result for $\mathcal{A}_\nu(R)$ is true. $\circlearrowleft$

Corollary. *If ν is composed with a valuation ν_1, according to a surjective monotone non-decreasing group homomorphism $\lambda\colon \Phi \to \Phi_1$, setting $\Psi = \mathrm{Ker}\lambda$, we have:*

$$\widehat{\mathcal{A}_{\nu_1}(R)}^{\nu_{1,\mathcal{A}}} = \widehat{\mathcal{A}_\nu(R)}^{\nu_\mathcal{A}} / \overline{((v^\psi - 1)_{\psi\in\Psi_+})},$$

where as usual the bar denotes the closure.

Proof. This follows from Proposition 5.8 and the fact that in view of 5.1 the continuous map $\hat{R}^\nu\{v^\Phi\} \to \hat{R}^{\nu_1}\{v^{\Phi_1}\}$ is surjective and its kernel is the ideal $\overline{((v^\psi - 1)_{\psi\in\Psi_+})}$. $\circlearrowleft$

In fact this corollary is the generalization to valuation algebras of the first corollary of Proposition 5.2.
In spite of the fact that none of this applies when ν_1 is the trivial valuation, we have:

$$\widehat{\mathcal{A}_\nu(R)}^{\nu_\mathcal{A}} / \overline{((v^\phi - 1)_{\phi\in\Phi_+})} = \hat{R}^\nu,$$

because the natural map $\mathcal{A}_\nu(R) \longrightarrow R$ with kernel $(v^\phi - 1)_{\phi\in\Phi_+}$ is continuous not only when R has the trivial topology given by the trivial valuation, but also when R has the ν-adic topology.
Assuming now, as we did in subsection 2.3, that R is a k-algebra for some field k, we have:

Proposition 5.11. **The $k[v^{\Phi_+}]$-algebra $\widehat{\mathcal{A}_\nu(R)}^{\nu_\mathcal{A}}$ is faithfully flat.**

Proof. This is proved exactly like Proposition 2.2 but we have to accept infinite sums $\sum y_{k\psi} v^{-\psi}$ and an infinite-dimensional vector space V in $\hat{R}^\nu$ but everything is convergent in $\widehat{\mathcal{A}_\nu(R)}^{\nu_\mathcal{A}}$. $\circlearrowleft$

We remark here that since ν is trivial on k, we have $k[v^{\Phi_+}] = k\{v^{\Phi_+}\}$.

In any case, we shall see below why it is for the family

$$\mathrm{Spec}\widehat{\mathcal{A}_\nu(R)}^{\nu_\mathcal{A}} \to \mathrm{Spec}k[v^{\Phi_+}]$$

that we may indeed hope to have simultaneous *embedded* uniformization; the $k[v^{\Phi_+}]$-algebra $\widehat{\mathcal{A}_\nu(R)}^{\nu_\mathcal{A}}$ is a quotient of the completion of $k[v^{\Phi_+}][(u_i)_{i\in I}]$, which corresponds to a simultaneous embedding of its fibers. This will put us in a position to apply the implicit function theorem to extend the uniformization of ν_{gr} on $\mathrm{gr}_\nu R$ to $\hat{R}^\nu$.

5.3 The specialization of $\hat{R}^\nu$ to $\widehat{\mathrm{gr}_\nu} R$

In this subsection, we assume that R_ν is the ring of a rational valuation of an analytically irreducible equicharacteristic nœtherian local ring R. If we fix a field of representatives k of $\hat{R}^\nu$ according to Proposition 5.5 and take a system of generators $(\overline{\xi}_i)_{i\in I}$ of the k-algebra $\mathrm{gr}_\nu R = \mathrm{gr}_{\hat{\nu}} \hat{R}^\nu$, where $k = R/m = R/(m_\nu \cap R)$, we wish to lift them to generators of $\hat{R}^\nu$, in the following sense

Lemma 5.12. *Assume that the valuation ν is of height one. Lift the $\overline{\xi}_i$ to elements $\xi_i \in \hat{R}^\nu$. The k-algebra $k[(\xi_i)_{i\in I}]$ is dense in $\hat{R}^\nu$ for the $\hat{\nu}$-adic topology.*

Proof. This follows from Lemma 5.1; we retain its notations, remarking that our assumption implies that ν is trivial on k. If we pick $x \in \hat{R}^\nu$, its $\hat{\nu}$-adic initial form is in $\mathrm{gr}_\nu R$, and therefore $\mathrm{in}_{\hat{\nu}} x = P_1(\overline{\xi}_i)$, $P_1 \in k[(X_i)_{i\in I}]$. Abbreviating $(\xi_i)_{i\in I}$ to (ξ), we set $x^{(0)} = x$, $x^{(1)} = x - P_1(\xi)$, and iterate this process. We have for all $j \geq 1$ the inequality $\hat{\nu}(x^{(j+1)}) > \hat{\nu}(x^{(j)})$ and thus $\hat{\nu}(x^{(j+1)} - x^{(j)}) = \hat{\nu}(x^{(j)})$. Now remark that since $\hat{R}^\nu$ is nœtherian by Lemma 5.1, the valuation $\hat{\nu}(\hat{m})$ is > 0, and since $\hat{\nu}$ is of height one, for any $\phi \in \Phi_+$ there is an integer $N(\phi)$ such that $\hat{m}^{N(\phi)} \subset \mathcal{P}_\phi$. Then there are only finitely many distinct $\mathcal{P}_{\phi'}$ containing $\mathcal{P}_\phi$, so that in a finite number of steps we have $\hat{\nu}(x^{(j+1)}) > \phi$, which proves that the sequence $x^{(j)}$ converges to x, and the Lemma. ◯

Remark. A priori, it could happen that the valuations of the differences $x - \sum_{j=1}^r P_j(\xi)$ increase at each step of the construction but remain bounded in the semigroup Φ_+.

Assume now that the height of ν is finite. We assume until the end of this subsection that R is local and complete for the ν-adic topology.

Let us consider a valuation ν_1 of height $\mathrm{h}(\nu) - 1$ with which ν is composed. We keep the notations of Lemma 3.10. Let us choose a system of generators $(\overline{\xi}_i)_{i\in I}$ of the k-algebra $\mathrm{gr}_\nu R$. We may choose them in such a way that those among them whose degree is in Ψ_+ generate the k-algebra $\mathrm{gr}_{\overline{\nu}} \overline{R}_1$, where $R_1 = R/(m_{\nu_1} \cap R)$. Note that if $m_{\nu_1} \cap R = m_\nu \cap R$, there are no such generators, and $\overline{R}_1 = \overline{R}$.
Let $\xi = (\xi_i)_{i\in I}$ be a system of representatives in R of the $(\overline{\xi}_i)_{i\in I}$.
Given $\phi_1 \in \Phi_{1,+}$, we have, whenever $\lambda(\phi) = \phi_1$ (see Lemma 3.10)

$$\mathcal{P}^+_{\phi_1} \subset \mathcal{P}_\phi \subset \mathcal{P}_{\phi_1},$$

and because $\overline{\nu}$ is of height one, it is shown in [Z-S], Appendix 3, lemmas 4 and its corollary, that there are only finitely many ideals $\mathcal{P}_{\phi'}$ between $\mathcal{P}_{\phi_1}$ and $\mathcal{P}_\phi$. Moreover, the ideals $\mathcal{P}_\phi$ between $\mathcal{P}_{\phi_1}$ and $\mathcal{P}^+_{\phi_1}$ form at worst (i.e., if the centers of ν and ν_1 are different) a simple infinite sequence, and the ideals $m^q.\mathcal{P}_{\phi_1}$ are cofinal in this sequence, where m is the maximal ideal of R.

Given $x \in R$, we define inductively
$x^{(1)} = x - P_0((\xi)_{i\in I})$, where $\mathrm{in}_\nu(x) = P_0((\overline{\xi}_i)_{i\in I})$ and

$x^{(j+1)} = x^{(j)} - P_i((\xi)_{i\in I})$, where $\text{in}_\nu(x^{(j)} = P_i(\overline{\xi})$.
By construction, we have $x - \sum_{k=0}^{j} P_k(\xi) = x^{(j+1)}$ and $\nu(x^{(j+1)}) > \nu(x^{(j)})$. But this does not imply that the series $\sum_{k=0}^{j} P_k(\xi)$ converges to x in the ν-adic topology, since it may well happen that $\nu(x^{(j+1)})$ remains bounded in Φ as $j \to \infty$.
To find a sequence $z^{(j)}$ of elements in $k[(\xi)]$ such that $\nu(x - z^{(j)})$ tends to infinity with j, it suffices to show that $\nu_1(x - z^{(j)})$ tends to infinity in Φ_{1+}. Let us consider the case where $\nu(x - z^{(j)})$ increases at each step but $\nu_1(x - z^{(j)})$ is constant, say equal to ϕ_1. We look at the initial forms of $x - z^{(j)}$ in $(\text{gr}_{\nu_1} R)_{\phi_1}$. Their index in the $\overline{\mathcal{P}}(\phi_1)$-filtration increases at each step, and therefore so does the index of $z^{(j+1)} - z^{(j)}$. Because $\overline{\nu}$ is of height one, it follows from the results quoted above that the $\text{in}_{\nu_1}(z^{(j)})$ converge in $(\text{gr}_{\nu_1} R)_{\phi_1}$ for its $\overline{m}$-adic topology as an $\overline{R}_1$-module, which coincides with the topology corresponding to the $\overline{\mathcal{P}}(\phi_1)$-filtration. Indeed, we assume that R is complete for the ν-adic topology, so $\overline{R}_1$ is also complete by Proposition 5.3 for its m-adic topology and the $\overline{\nu}$-adic topology, which coincide, and $(\text{gr}_{\nu_1} R)_{\phi_1}$ is an $\overline{R}_1$-module of finite type, so it is also complete ([B3], Chap. III, §2, No. 12, Cor.1). Take $z^{(j)} = x - x^{(j+1)}$ and let $\overline{z} \in (\text{gr}_{\nu_1} R)_{\phi_1}$ be the limit of this sequence. Let us pick a representative $z \in R$ of $\overline{z}$. We have that $\text{in}_{\nu_1}(x - z) = \text{Lim}_{i\to\infty}\text{in}_{\nu_1}(x - z^{(j)}) = 0$, so that $\nu_1(x - z) > \nu_1(x)$. Let us now choose generators $(\overline{\zeta}_k)_{k\in K}$ for the $\overline{R}_1$-algebra $\text{gr}_{\nu_1} R$ and lift them to R as $(\zeta_k)_{k\in K}$. Since the valuation $\overline{\nu}$ on $\overline{R}_1$ is of rank one, we know that if we choose generators $(\tilde{\eta}_\ell)_{\ell\in L}$ in $\overline{R}$ whose images in $\text{gr}_{\overline{\nu}}\overline{R}$ generate it as a k-algebra, the subalgebra $k[(\tilde{\eta}_\ell)_{\ell\in L}]$ is dense in $\overline{R}_1$ for the $\overline{\nu}$-adic topology, which coincides with the $\overline{m}$-topology.
Note moreover that because ν_{gr} is a monomial valuation (see 3.7), the k-algebra R_1 is also the completion of $k[(\tilde{\eta}_\ell)_{\ell\in L}]$ for the topology corresponding to the monomial valuation giving value $\nu(\eta_\ell)$ to η_ℓ. Thus, we can approximate x for the ν-adic topology by a sequence of elements $z_h \in R$, each of which is obtained as above by an infinite process from a polynomial algebra. We call this situation "density in scales"[6] for the polynomial ring $k[(\eta_\ell)_{\ell\in L}; (\zeta_k)_{k\in K}]$, and it suggests a redefinition of completion in the general case.

Example. Let k be a field, let R be a complete nœtherian local k-algebra with residue field k, and let $f \in R$ generate a prime ideal. We go back to example 3.4 and its notations. Let $x \in R$ be of the form

$$x = \sum_{i=\ell}^{\infty} a_i f^i \quad \text{with} \quad a_i \notin (f).$$

We have $\nu_1(x) = \ell$ and its ν-initial form is $(\text{in}_{\overline{\nu}}\, \overline{a_\ell})\xi^\ell$. In general the a_i will not be polynomials in the liftings (ξ_i) to R of the generators of $\text{gr}_{\overline{\nu}}\overline{R}$, so that we will

[6] F.-V. Kuhlmann has shown to me that this idea was related to Kaplansky's pseudo-Cauchy sequences ([Ka]) and Ribenboim's notion of "complet par étages" ([Ri]) in the theory of completion of valued fields. The problem is the same, but I did not compare the two definitions; there is a major difference in that I work with nœtherian rings.

have, denoting by $\tilde{a}_i$ a lifting of $\mathrm{in}_{\overline{\nu}}\overline{a}_i$ which is a polynomial in the (ξ_i),

$$x - \tilde{\mathrm{in}}_\nu x = (a_\ell - \tilde{a}_\ell)f^\ell + \cdots$$

and the ν_1-value will not have changed. However, if we can approximate $\overline{a}_\ell$ in $\overline{R}$ by a sequence of polynomials in $k[(\overline{\xi}_i)_{i\in I_1}]$, we will find as limit element z of a sequence of polynomials, with the notations of Remark 5.3 above, the element $a_\ell f^\ell$. Then we have to repeat the process with the next term.

Let R be an analytically irreducible nœtherian local ring with a field of representatives k, and ν a k-valuation of finite height of its field of fractions, positive on R and whose center is the maximal ideal m of R. Let $(\xi_i)_{i\in I}$ be a set of elements of R whose images in $\mathrm{gr}_\nu R$ generate it as a k-algebra. Consider the set of isolated subgroups of the value group Φ of ν

$$(0) = \Psi_h \subset \Psi_{h-1} \subset \cdots \subset \Psi_k \subset \cdots \subset \Psi_1 \subset \Psi_0 = \Phi,$$

and the corresponding valuations ν_k with value group Φ/Ψ_k indexed by their height. Let

$$\mathbf{p}_t \subset \mathbf{p}_{t-1} \subset \cdots \subset \mathbf{p}_1 \subset m$$

be the sequence of the distinct centers of the valuations ν_k in R. Let us partition the set I as follows: $I = \bigcup_{k=1}^h I_k$ where

$$I_k = \{i \in I \mid \nu(\xi_i) \in \Psi_{h-k} \setminus \Psi_{h-k+1}\},$$

remarking that some of the sets I_k may be empty. Let us also define the following sequence of rings

$$\hat{R}_1 = k[(\xi_i)_{i\in I_1}]^{\widehat{\ }\tilde{\nu}_1}, \ldots, \hat{R}_k = \hat{R}_{k-1}[(\xi_i)_{i\in I_k}]^{\widehat{\ }\tilde{\nu}_k}, \ldots, \hat{R}_h = \hat{R}_{h-1}[(\xi_i)_{i\in I_h}]^{\widehat{\ }\tilde{\nu}_h},$$

where $\tilde{\nu}_k$ is the monomial valuation on $\hat{R}_{k-1}[(\xi_i)_{i\in I_k}]$ with values in Ψ_{h-k}/Ψ_{h-k+1} defined by $\tilde{\nu}_k(\sum a_\alpha \xi^\alpha) = \min(\tilde{\nu}_k(\xi^\alpha))$, and the hat means the completion.
We may identify $\hat{R}_1$ with a subring of $\hat{R}^\nu$, namely the closure of $k[(\xi_i)_{i\in I_1}]$ in $\hat{R}^\nu$ for the topology defined by the filtration by the ideals $(\mathcal{P}_\psi)_{\psi\in\Psi_{h-1}}$. Indeed, we can check that an element of $\hat{R}^\nu/(m_{\hat{\nu}_k} \cap \hat{R}^\nu)$, which is in the closure of $k[(\overline{\xi}_i)_{i\in I_1}]$, lifts uniquely to an element of this closure in $\hat{R}^\nu$, and we have seen in Proposition 3.12 that the valuation $\overline{\nu}_1$ is monomial with respect to the $(\overline{\xi}_i)_{i\in I_1}$. We can continue and identify inductively $\hat{R}_k$ to the closure of $\hat{R}_{k-1}[(\xi_i)_{i\in I_k}]$ in $\hat{R}^\nu$ for the topology defined by the filtration by the ideals $(\mathcal{P}_\psi)_{\psi\in\Psi_{h-k}}$. So we may identify in a natural way all the $\hat{R}_k$ with subrings of $\hat{R}^\nu$.

Definition. We say that $k[(\xi_i)_{i\in I}]$ is dense-in-scales in $\hat{R}^\nu$ for the $\hat{\nu}$-adic topology if $\hat{R}_h$ is equal to $\hat{R}^\nu$.

Proposition 5.13. *Let R_ν be the ring of a rational valuation of an analytically irreducible equicharacteristic nœtherian local ring R. If we fix a field of representatives k of $\hat{R}^\nu$ according to Proposition 5.5 and take a system of generators $(\overline{\xi}_i)_{i\in I}$ of the k-algebra $\mathrm{gr}_\nu R = \mathrm{gr}_{\hat{\nu}}\hat{R}^\nu$, where $k = R/m = R/(m_\nu \cap R)$ and if the images in $\mathrm{gr}_\nu R$ of the $(\xi_i)_{i\in I}$ generate it as a k-algebra, the algebra $k[(\xi_i)_{i\in I}]$ is dense-in-scales in $\hat{R}^\nu$ for the $\hat{\nu}$-adic topology.*

Proof. We have to prove that given $x \in \hat{R}^\nu$, there is a sequence $(r_j)_{j\in\mathbf{N}}$ of elements of $\hat{R}_{h-1}[(\xi_i)_{i\in I_h}]$ converging to x in the $\tilde{\nu}_h$-adic topology. Define the sequence of $x^{(j)}$ and $P_j(\xi_i)$ as in the proof of Lemma 5.1, and let us build from it a sequence $(r_j)_{j\in\mathbf{N}}$ of elements of $\hat{R}_{h-1}[(\xi_i)_{i\in I_h}]$ such that the valuations $\nu(x - r_j)$ exceed, for large j, any $\phi \in \Phi_+$ given in advance. Let s be the least integer such that for infinitely many values of the integer j we have $\nu_s(x^{(j+1)}) > \nu_s(x^{(j)})$. If $s = 1$, since the valuation ν_1 is of height one, for any $\phi_1 \in \Phi_{1+}$, after finitely many steps we have $\nu_1(x^{(j)}) > \phi_1$, hence for any $\phi \in \Phi_+$, after finitely many steps we have $\nu(x^{(j)}) > \phi$, so we may take $r_j = x^{(j)}$ and get $\nu(x - \sum_{r=0}^{j-1} P_r(\xi_i)) > \phi$ and this shows that $x \in \hat{R}_h$. If $s > 1$, by definition, there exists a j_0 such that $\nu_{s-1}(x^{(j+1)}) = \nu_{s-1}(x^{(j)})$ for $j \geq j_0$. By substracting from x a polynomial in the ξ_i, we may assume that $j_0 = 1$.
Note that the center of ν_{s-1} is necessarily distinct from the center of ν_s, since if the two centers are equal, there are only finitely many distinct ν_s-ideals between two consecutive ν_{s-1}-ideals of R, according to [Z-S], Vol. II, Appendix 3, Corollary p.345.
We consider the initial forms $\mathrm{in}_{\nu_{s-1}} x^{(j)} \in \mathrm{gr}_{\nu_{s-1}} R$; for $j \geq j_0$, they all have the same degree, say ϕ_{s-1}. Since the residual valuation $\overline{\nu}_s$ on $R/(m_{\nu_{s-1}} \cap R)$ is of rank one, with group Ψ_{s-1}/Ψ_s, and each homogeneous component of $\mathrm{gr}_{\nu_{s-1}} R$ is a $R/(m_{\nu_{s-1}} \cap R)$-module of finite type, the sequence of the initial forms $\mathrm{in}_{\nu_{s-1}}(x - x^{(i)})$ converges in $(\mathrm{gr}_{\nu_{s-1}} R)_{\nu_{s-1}(x)}$ to a limit $\overline{x}_1^{(1)}$, which we may lift to an element $x_1^{(1)} \in \hat{R}_h$. This element has the property that for all $j \geq 1$, we have $\nu_{s-1}(x_1^{(1)}) = \nu_{s-1}(x^{(j)})$ and $\nu_{s-1}(x - x_1^{(1)}) > \nu_{s-1}(x - x^{(j)})$. Replacing now x by $x - x_1^{(1)}$, and so on, we build a sequence of elements $x_1^{(k)} \in R_h$ such that $\nu_{s-1}(x - x_1^{(k)})$ increases at each step. We may now repeat the whole process, replacing the sequence $x^{(j)}$ by $x_1^{(k)}$; s is replaced by $s - 1$ and ultimately we are reduced to the case where $s = 1$, where as we have seen, the problem is solved by the nœtherianity of $\hat{R}^\nu$. This proves the Proposition. ◯

Recall from subsection 2.4 that we have a graded isomorphism

$$\mathrm{gr}_{\nu_{\mathcal{A}}}\mathcal{A}_\nu(R) \to (\mathrm{gr}_\nu R) \otimes_{\overline{R}} \overline{R}[v^{\Phi_+}].$$

If we take a system of generators $\overline{\xi}_i$ of the $\overline{R}$-algebra $\mathrm{gr}_\nu R$, we obtain generators $\overline{v}_i = \overline{\xi}_i \otimes 1$ of the $\overline{R}[v^{\Phi_+}]$-algebra $\mathrm{gr}_{\nu_{\mathcal{A}}}\mathcal{A}_\nu(R) = \mathrm{gr}_{\hat{\nu}_{\mathcal{A}}}\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}$ and we can lift them to elements $v_i \in \mathcal{A}_\nu(R) \subset \widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}$.

Note that according to Proposition 2.5, each element υ_i is of the form $\xi_i v^{-\nu(\xi_i)}$ where $\xi_i \in R$ has initial form $\overline{\xi}_i$.
Let us begin by assuming that R is local and that the residual extension $k \to R/m$ is trivial. Then we may consider the morphism of $k[v^{\Phi_+}]$-algebras

$$k[v^{\Phi_+}][(u_i)_{i\in I}] \to \mathcal{A}_\nu(R)$$

determined by $u_i \mapsto \upsilon_i = \xi_i v^{-\nu(\xi_i)}$. We can consider the completion-in-scale of $k[v^{\Phi_+}][(u_i)_{i\in I}]$ defined as follows: define, with the notations introduced above, the sequence of rings

$$\hat{S}_1 = k[v^{\Phi_+}][(u_i)_{i\in I_1}]^{\widehat{\ }\tilde{\nu}_1}, \dots, \hat{S}_k = \hat{S}_{k-1}[(u_i)_{i\in I_k}]^{\widehat{\ }\tilde{\nu}_k}, \dots, \hat{S}_h = \hat{S}_{h-1}[(u_i)_{i\in I_h}]^{\widehat{\ }\tilde{\nu}_h}$$

where $\tilde{\nu}_k$ is the monomial valuation on $\hat{S}_{k-1}[(u_i)_{i\in I_k}]$ whose values lie in Ψ_{h-k}/Ψ_{h-k+1} defined by $\tilde{\nu}_k(\sum a_\alpha u^\alpha) = \min(\tilde{\nu}_k(\xi^\alpha))$, and where the hat means the completion.

Definition. We can inductively define a valuation $\tilde{v}$ on $\hat{S}_h$ as follows: it is trivial on $k[v^{\Phi_+}]$ and, assuming that it has been defined on $\hat{S}_{k-1}$, for an element $\sum a_\alpha u^\alpha \in \hat{S}_k$ we set

$$\tilde{v}(\sum a_\alpha u^\alpha) = \min_\alpha\{\tilde{v}(a_\alpha) + \nu(\xi_\alpha)\},$$

which is well defined because the semigroup $\Gamma = \nu(R\setminus\{0\})$ is well ordered. It extends the monomial valuation on the polynomial ring with the $(u_i)_{i\in I}$ as system of generators and $\tilde{v}(u_i) = \xi_i$.

We will denote by $\widehat{k[v^{\Phi_+}][(u_i)}_{i\in I}]$ the ring $\hat{S}_h$ endowed with the topology determined by the $\tilde{v}$-adic filtration, and call it the completion-in-scales of the ring $k[v^{\Phi_+}][(u_i)_{i\in I}]$.

The map $k[(u_i)_{i\in I}] \to R$ defined by $u_i \mapsto \xi_i v^{-\nu(\xi_i)}$ induces a map

$$k[v^{\Phi_+}][(u_i)_{i\in I}] \to \mathcal{A}_\nu(R); \quad v^{-\phi} \mapsto v^{-\phi}, \quad u_i \mapsto \xi_i v^{-\nu(\xi_i)},$$

which in turn induces a surjective map of $k[v^{\Phi_+}]$-algebras

$$\widehat{k[v^{\Phi_+}][(u_i)}_{i\in I}] \longrightarrow \widehat{\mathcal{A}_\nu(R)}^{\nu_A},$$

where the first hat designates the completion-in-scales defined above. The surjectivity follows from Proposition 5.13 or rather from repeating its proof in this situation where k is replaced by $k[v^{\Phi_+}]$. This map induces, modulo the *closures* of the ideals generated by (v^{Φ_+}), the surjective map

$$k[\widehat{(U_i)_{i\in I}}] \to \widehat{\mathrm{gr}_\nu R}$$

which extends to the completions the original map $U_i \mapsto \overline{\xi}_i$.
We can summarize this as follows

Proposition 5.14. * *Let R be an analytically irreducible nœtherian equicharacteristic local ring with residue field k, and ν a rational valuation of R with value group Φ. Let $(\xi_i)_{i\in I}$ be representatives in R of a system of homogeneous generators $(\overline{\xi}_i)_{i\in I}$ of the k-algebra $\mathrm{gr}_\nu R$. Then $\hat{R}^\nu$ admits a field of representatives $k\subset \hat{R}^\nu$ and the application $k[v^{\Phi_+}][(u_i)_{i\in I}]\to \mathcal{A}_{\hat\nu}(\hat{R}^\nu)$ determined by the injection $k[v^{\Phi_+}]\subset \mathcal{A}_{\hat\nu}(\hat{R}^\nu)$ deduced from $k\subset \hat{R}^\nu$ and the application $u_i\mapsto \xi_i v^{-\nu(\xi_i)}$ extends to a continuous surjective map of topological $k[v^{\Phi_+}]$-algebras*

$$\widehat{k[v^{\Phi_+}][(u_i)_{i\in I}]}\to \widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}},$$

which induces, modulo the closures in each ring of the ideal (v^{Φ_+}), the map

$$\widehat{k[(U_i)_{i\in I}]}\to \widehat{\mathrm{gr}_\nu R}.*$$

(I write capital U's for the initial forms of the u's and $v(u)$ for the $\nu_{\mathcal{A}}$-valuation of the image of u in $\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}$).
The kernel of this last map is the closure in $\widehat{k[(U_i)_{i\in I}]}$ of the binomial ideal $(U^n-\lambda_{mn}U^m)k[(U_i)_{i\in I}]$ defining $\mathrm{gr}_\nu R$; it is the ideal consisting of possibly infinite sums $\sum_{m,n}A_{m,n}(U)(U^n-\lambda_{mn}U^m)\in \widehat{k[(U_i)_{i\in I}]}$, with $A_{m,n}(U)\in \widehat{k[(U_i)_{i\in I}]}$. Using the fact that $\mathcal{A}_\nu(R)$ is a flat $k[v^{\Phi_+}]$-algebra, one can show

Proposition 5.15. * *There exist elements of the form*

$$F_{mn}=u^n-\lambda_{mn}u^m+\sum_p c_p^{(mn)}(v^\phi)u^p\in \widehat{k[v^{\Phi_+}][(u_i)_{i\in I}]},$$

with, for all p appearing in F_{mn}, $v(u^p)>v(u^n)=v(u^m)$ and $c_p^{(mn)}$ a term (=constant times a monomial) in the v^ϕ, and such that the closure in $\widehat{k[v^{\Phi_+}][(u_i)_{i\in I}]}$ of the ideal generated by the F_{mn} is the kernel of the surjection

$$\widehat{k[v^{\Phi_+}][(u_i)_{i\in I}]}\to \widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}.*$$

By the remark following Proposition 2.2, and with its notations, after replacing R by $\hat{R}^\nu$, the $k[v^{\Phi_+}]$-algebra $\mathcal{B}(S)$, which is the image of the map of $k[v^{\Phi_+}]$-algebras

$$k[v^{\Phi_+}][(u_i)_{i\in I}]\to \mathcal{A}_\nu(R),\quad u_i\mapsto \xi_i v^{-\nu(\xi_i)}$$

is faithfully flat. It follows that, denoting by J the kernel of this map, we can lift the generators $U^m-\lambda_{mn}U^n$ of the kernel of the map $k[(U_i)_{i\in I}]\to \mathrm{gr}_\nu R$ to elements $F_{mn}\in k[v^{\Phi_+}][(u_i)_{i\in I}]$ generating an ideal $J'\subseteq J$ such that $J\subseteq J'+(v^{\Phi_+}).J$. The equality $(v^{\Phi_+})\cap J=(v^{\Phi_+}).J$ holds by flatness since it is equivalent to

$$\mathrm{Tor}_1^{k[v^{\Phi_+}][(u_i)_{i\in I}]}(k[v^{\Phi_+}][(u_i)_{i\in I}]/(v^{\Phi_+}),k[v^{\Phi_+}][(u_i)_{i\in I}]/J)=0.$$

It means that we can write in $\mathcal{A}_\nu(R)$ generators of J which begin with

$$u^m - \lambda_{mn} u^n + \sum_i v^{\phi_i} A_i (u^{m^i} - \lambda_{m^i n^i} u^{n^i} + \cdots).$$

Now we observe that since this sum must give zero in $\widehat{\mathcal{A}_\nu(R)}^{\nu_A}$, the v-valuation of $\sum_i v^{\phi_i} A_i (u^{m^i} - \lambda_{m^i n^i} u^{n^i} + \cdots)$ must be greater than that of u^m. Following through the process of completion in scales, one sees that the closure in $k[v^{\Phi+}]\widehat{[(u_i)_{i\in I}]}$ of the ideal J' is the kernel of the surjection

$$k[v^{\Phi+}]\widehat{[(u_i)_{i\in I}]} \to \widehat{\mathcal{A}_\nu(R)}^{\nu_A}.$$

Corollary. *Given a valuation ν_0 of R, and a rational specialization ν of ν_0, let c be a character of Φ i.e., an homomorphism $c\colon \Phi \to k^*$; replacing v^ϕ by $c(\phi)$ gives a continuous surjection*

$$\hat{c}\colon k\widehat{[(u_i)_{i\in I}]} \to \hat{R}^\nu = \hat{R}^{\nu_0},$$

where the first hat means completion in scales. The kernel of this map is the closure of the ideal generated by the elements

$$c(F_{mn}) = u^n - \lambda_{mn} u^m + \sum_p c_p^{(mn)}(c(\phi)) u^p.$$

In particular, we may take $c(\phi) = 1 \ \ \forall \phi \in \Phi$.

This may be thought of as a valuative analogue of Cohen's theorem stating that every complete equicharacteristic nœtherian local ring is a quotient of a power series ring in finitely many variables over its residue field. The advantage here is that, when the residue field is algebraically closed, we know that the kernel of the map is generated, up to closure, by deformations of binomials.
One may summarize a part of my approach by saying that the elements $\xi_i \in \hat{R}^\nu$ representing generators $\overline{\xi}_i$ of the k-algebra $\mathrm{gr}_\nu R$ form a system of (topological) generators of the maximal ideal of $\hat{R}^\nu$ in which the valuation $\hat{\nu}$ is monomial in the sense of subsection 3.7. The generators $c(F_{mn})$ may be thought of as an infinite standard, or Gröbner, basis for the kernel of the map $k\widehat{[(u_i)_{i\in I}]} \to \hat{R}^\nu$ with respect to the filtration induced by the valuations of the images of the elements u_i.

Problem. Is it possible that the completion $\hat{R}^\nu$ of a nœtherian analytically irreducible equicharacteristic local ring R should always be excellent, just because it is a nœtherian quotient of the generalized power series ring $k\widehat{[(u_i)_{i\in I}]}$?

5.4 An example: complex plane branches

Example. Let R be the analytic algebra of a complex analytic plane branch (i.e., germ of analytically irreducible curve) $(X, 0)$; then it is dominated by only one valuation, the t-adic valuation ν induced by the injection $R \subset \mathbf{C}\{t\}$ of R into its normalization, and the valuation ring is $\mathbf{C}\{t\}$. Let $\Gamma = \nu(R \setminus \{0\}) \subset \mathbf{N}$ be the semigroup of values of ν on R; it has $g+1$ generators $(\overline{\beta}_0, \ldots, \overline{\beta}_g)$ and if we choose for each $0 \leq i \leq g$ an element $\xi_i(t) \in R$ with valuation $\overline{\beta}_i$, then the completion $\hat{\mathcal{A}}(R)$ of $\mathcal{A}(R)$ with respect to the (t)-adic valuation (our $\nu_{\mathcal{A}}$) (resp. the analyticization $\mathcal{A}(R)^{an}$ of $\mathcal{A}(R)$) is isomorphic to the (t)-adic completion $\hat{\mathcal{A}}$ (resp. the analyticization) of the sub $\mathbf{C}[v]$-algebra of $R[v, v^{-1}]$ generated by the elements $\xi_i(t)v^{-\overline{\beta}_i}$. This follows from ([T1]) and what we have seen above. Anyway, this is the algebra of a family M, parametrized by v, of branches, each fiber $M(v)$ for $v \neq 0$ being isomorphic to the branch $(X, 0)$, and the special fiber being isomorphic to the monomial curve described parametrically by $u_i = t^{\overline{\beta}_i}$; $0 \leq i \leq g$. There is a natural map which resolves its singularities; it is the map $\mathcal{A}^{an} \to \mathbf{C}\{v, t'\}$ given by

$$\xi_i(t)v^{-\overline{\beta}_i} \mapsto \xi_i(vt')v^{-\overline{\beta}_i}.$$

Indeed, the natural resolution of singularities for $\mathrm{Spec}\mathcal{A}(R)$ is $\mathrm{Spec}\mathcal{A}(\mathbf{C}\{t\})$. This last ring is a genuine Rees algebra for the ideal (t) in $\mathbf{C}\{t\}$

$$\mathcal{A}(\mathbf{C}\{t\}) = \mathbf{C}\{t\}[v] \bigoplus \oplus_{n\in\mathbf{N}} (t)^n v^{-n},$$

and its analytization maps isomorphically to $\mathbf{C}\{v, t'\}$ by $v \mapsto v,\ t \mapsto vt'$.
This shows that the parametric specialization described in ([T1], 1.10, p. 167) and ([G-T], §3) coincides with the specialization given by the valuation algebra.
It was shown in [T1] that the binomial equations

$$u_i^{n_i} - u_0^{\ell_0^{(i)}} \ldots u_{i-1}^{\ell_{i-1}^{(i)}} = 0 \quad , 1 \leq i \leq g$$

defining the monomial curve in $\mathbf{C}^{g+1}$ extend to equations defining the family M in $\mathbf{C} \times \mathbf{C}^{g+1}$, which are of the form

$$\begin{array}{ll}
u_1^{n_1} - u_0^{\ell_0^{(1)}} + b_2 v u_2 + v \sum_{w(p) > n_1\overline{\beta}_1} c_p^{(1)}(v) u^p & = 0 \\
u_2^{n_2} - u_0^{\ell_0^{(2)}} u_1^{\ell_1^{(2)}} + b_3 v u_3 + v \sum_{w(p) > n_2\overline{\beta}_2} c_p^{(2)}(v) u^p & = 0 \\
\vdots & \vdots \\
u_{g-1}^{n_{g-1}} - u_0^{\ell_0^{(g-1)}} \cdots u_{g-2}^{\ell_{g-2}^{(g-1)}} + b_g v u_g + v \sum_{w(p) > n_{g-1}\overline{\beta}_{g-1}} c_p^{(g-1)}(v) u^p & = 0 \\
u_g^{n_g} - u_0^{\ell_0^{(g)}} \cdots u_{g-1}^{\ell_{g-1}^{(g)}} & = 0
\end{array}$$

where the $c_p^{(j)}(v)$ are in $(v)\mathbf{C}\{v\}$, $w(p) = \sum_0^g \overline{\beta}_j p_j$ is the weight of the monomial u^p with respect to the weight vector $w = (\overline{\beta}_0, \ldots, \overline{\beta}_g)$, and with all $b_j \neq 0$ in $\mathbf{C}$, which corresponds to the fact that our branch is plane. The other u^p appearing in the j-th equation are different from u_{j+1}. Note that since $n_i\overline{\beta}_i < \overline{\beta}_{i+1}$ (see [T1]), we add to each binomial only terms of higher weight, one of which is linear.

We see that we have a surjective map

$$\pi\colon \mathbf{C}\{v\}[[u_o,\ldots,u_g]] \to \hat{\mathcal{A}}(R),$$

and its kernel is generated by the equations just written, which illustrates Proposition 5.15.
It was shown in [G-T] that the toric map $\pi\colon Z \to \mathbf{C}^{g+1}$ resolving the monomial curve gives extends a toric map $\mathrm{Id}_{\mathbf{C}} \times \pi\colon \mathbf{C}\times Z \to \mathbf{C}\times\mathbf{C}^{g+1}$ inducing a resolution of M which is a strong simultaneous embedded resolution along $\mathbf{C}\times\{0\}$.
In this example, the valuation $\tilde{\nu}$ of subsection 3.2 has values in the group $\mathbf{Z}\oplus\mathbf{Z}$ ordered lexicographically, with the values $\tilde{\nu}(v) = (1,0)$, $\tilde{\nu}(t) = (1,1)$, this last value corresponding to $\tilde{\nu}(t') = (0,1)$ on $\mathbf{C}\{v,t'\}$, and so to the natural rank two valuation on this last ring.

6 Toric modifications

In this section, I begin to show how to uniformize, by a combinatorial process, the valuation ν_{gr} on $\mathrm{gr}_\nu R$ when ν is a rational valuation of the nœtherian local ring R. In fact I show how to resolve singularities of an irreducible variety defined by a binomial ideal in finite-dimensional space. As far as I know, existing literature deals only with toric varieties associated to fans, which are in particular normal.

6.1 The graded algebra of a rational valuation is essentially toric

Proposition 6.1. *Assume that a valuation ring R_ν birationally dominates a local ring R and that the residual extension $k_R = R/m \to R_\nu/m_\nu = k_\nu$ is trivial. Then each homogeneous component of the graded k_R-algebra $\mathrm{gr}_\nu R$ is a vector space of dimension ≤ 1 over k_R.*

Proof. Indeed, by definition of the ideals $\mathcal{P}_\phi(R)$, we have the graded inclusion

$$\mathrm{gr}_\nu R \subset \mathrm{gr}_\nu R_\nu$$

of k_R-algebras and the result is true for the second algebra since the initial forms of two elements of R_ν with the same valuation differ by a factor which is the initial form of a unit, i.e., an element of k_ν^*; each homogeneous component of $\mathrm{gr}_\nu R_\nu$ is a k_ν-vector space of dimension one. ◯

From Proposition 6.1 above follows by a direct generalization of a result of Korkina ([Ko], see also [E-S]).

Proposition 6.2. *In the situation of Proposition 6.1, assuming that* $\mathrm{r}(\nu)$ *is finite, the graded k_R-algebra* $\mathrm{gr}_\nu R$ *is the quotient of a polynomial ring in countably many indeterminates $k_R[(U_j)_{j\in J}]$ by a binomial ideal.*

Proof. By assumption, Φ is of finite rational rank, and therefore countable. Let $(\overline{\xi}_j)_{j\in J}$ be countable system of nonzero homogeneous generators for the k_R algebra $\mathrm{gr}_\nu R$ (note that the number of these generators may be finite); consider the kernel K of the surjective homomorphism

$$k_R[(U_j)_{j\in J}] \to \mathrm{gr}_\nu R$$

determined by $U_j \mapsto \overline{\xi}_j$. It is a homogeneous ideal for the Φ_+-grading giving to each U_j the degree of $\overline{\xi}_j$. Let $P = \sum c_m U^m$ be a homogeneous generator of K, say of degree ϕ. All the monomials occurring in P have their images in the one dimensional k-vector space $(\mathrm{gr}_\nu R)_\phi$ and the sum of these images is zero. By induction on the number of monomials of P, we see that P is a sum of binomial terms $c_{m,m'}(U^m - \lambda_{m,m'}U^{m'})$ with $c_{m,m'}, \lambda_{m,m'} \in k$.
The ideal K is generated by binomials of the form $U^m - \lambda_{m,m'}U^{m'}$ and there is one such for each (m, m') such that u^m and $u^{m'}$ have the same degree. Moreover, all the $\lambda_{m,m'}$ are $\neq 0$. ◌

Corollary. *For any valuation ring R_ν with countable value group, the graded algebra $\mathrm{gr}_\nu R_\nu$ is a quotient of a polynomial algebra in countably many variables over $k_\nu = R_\nu/m_\nu$ by a binomial ideal.*

Remark. This corollary is used in the proof of Proposition 4.3, but does not tell us anything precise about the form of the binomial equations.

Assume now that R is nœtherian.

Let $\nu(R \setminus \{0\}) = \Gamma \subset \Phi_+$ be the semigroup of R. Since R is nœtherian, the semigroup Γ is well ordered (Proposition 3.8), and therefore we can build a countable system of generators

$$\Gamma = \langle \gamma_1, \gamma_1, \ldots, \gamma_j, \ldots \rangle \quad \text{with } \gamma_j \in \Phi_+ \ , \gamma_j < \gamma_{j+1},$$

where γ_1 is the smallest non zero element of Γ, ..., γ_j the smallest element of Γ not contained in the semigroup $\langle \gamma_1, \gamma_2, \ldots, \gamma_{j-1} \rangle$ generated by the previous ones. Note that in the plane branch case, the generators are denoted by $\overline{\beta}_i$ to underline their kinship with Puiseux exponents, according to a tradition initiated by Zariski; we followed that tradition in [G-T] and in the example treated above. Anyway, to this situation is associated the map of groups

$$\mathbf{Z}^{\mathbf{N}} \to \Phi \qquad \text{determined by} \quad \sum \lambda_j e_j \mapsto \sum \lambda_j \gamma_j,$$

where e_j is the j-th base vector, such that the image of $(\mathbf{Z}_{\geq 0})^{\mathbf{N}}$ is Γ. It defines a map of k_ν-algebras

$$k_\nu[(U_j)_{j\in J}] \to k_\nu[t^{\Phi_+}] \quad \text{determined by} \quad U_j \mapsto t^{\gamma_j}.$$

The image of this map is the semigroup algebra $k_\nu[t^\Gamma] \subset k_\nu[t^{\Phi_+}]$, and its kernel is the binomial ideal obtained from that of of Proposition 6.2 by setting all $\lambda_{m,m'}$ equal to 1.

Remark. We note that by definition of the γ_i, none of the binomial equations is of the form

$$U_i - \lambda_i U^{m(i)} = 0.$$

Proposition 6.3. *Let Λ denote the multiplicative subset of $k_\nu[(\Lambda_{m,m'})_{(m,m')}]$ generated by the $\Lambda_{m,m'}$. The natural map of k_ν-algebras*

$$\Lambda^{-1}k_\nu[(\Lambda_{m,m'})_{(m,m')}] \to \Lambda^{-1}k_\nu[(\Lambda_{m,m'})_{(m,m')}][(U_j)_{j\in J}]/(U^m - \Lambda_{m,m'}U^{m'})_{(m,m')}$$

is faithfully flat and corresponds to a flat family of schemes having one fiber equal to $\mathrm{Spec}k_\nu[t^\Gamma]$ *and another to* $\mathrm{Specgr}_\nu R$.

Indeed, if the second algebra is graded by giving to U_j the degree γ_j, then by definition of the binomial relations, each of its homogeneous components becomes a free module of rank one over $\Lambda^{-1}k_\nu[(\Lambda_{m,m'})_{(m,m')}]$.

Moreover, this family parametrized by the $\Lambda_{m,n}$ has simultaneous resolution in any reasonable sense; it is locally analytically trivial. We shall see below that it has simultaneous embedded resolution by a toric map.

6.2 An example

Assume now that $k = k_R$ is algebraically closed and that ν is a valuation of rank $d = \dim R$ dominating R. By Abhyankar's inequality, the extension $k \to k_\nu$ is trivial, so we are in the situation of Proposition 6.2, and the ordered group Φ is isomorphic to $\mathbf{Z}^d$ with the lexicographic order (see the text before Proposition 3.8).

Then we can write

$$\Gamma = \langle \gamma_1, \gamma_1, \dots, \gamma_j, \dots \rangle \text{ with } \gamma_j \in \mathbf{Z}^d_{\geq 0}.$$

The semigroup Γ is the image of $\mathbf{N}^{\mathbf{N}}$ by a map

$$b \colon \mathbf{Z}^{\mathbf{N}} \to \mathbf{Z}^d$$

defined by sending the i-th basis vector to γ_i. This defines a map

$$k[(U_j)_{j\in J}] \to k[t_1^{\pm 1}, \dots, t_d^{\pm 1}] \text{ determined by } \; U_j \mapsto t^{\gamma_j}$$

having $k[t^\Gamma]$ for image. This describes $\mathrm{Spec}k[t^\Gamma]$ as the closure of an orbit of the torus $(k^*)^d$ in the N-dimensional affine space $\mathbf{A}^{\mathbf{N}}(k)$.
Note that since R and R_ν have the same field of fractions, the semigroup Γ generates Φ as a group; here it means that there is a finite index i_0 such that the finitely generated semigroup $\langle \gamma_0, \dots, \gamma_{i_0}\rangle$ generates $\mathbf{Z}^d$ as a group. Note also that we have a difficulty which does not appear when $\Phi = \mathbf{Z}$; the γ_j do not have all their coordinates positive, so that our orbit closure is not the image of a map $\mathbf{A}^d(k) \to \mathbf{A}^{\mathbf{N}}(k)$, but it should be the image of a map $W \to \mathbf{A}^{\mathbf{N}}(k)$ where W is a toric modification of $\mathbf{A}^d(k)$.

By dualizing the map b, we obtain a map

$$w\colon \check{\mathbf{Z}}^d \to \check{\mathbf{Z}}^{\mathbf{N}}.$$

If the right-hand group was finite dimensional, say $\check{\mathbf{Z}}^N$, we would know how to resolve the singularities of $k[t^\Gamma]$ by a single toric map, as is explained in the next section.

6.3 Embedded toric resolution of orbit closures and binomial ideals in finite dimensions

I describe here the resolution of singularities of torus orbit closures (in finite dimensions) as a generalization of what is done in [G-T] for monomial curves. I assume that the reader is aware of the basics of toric geometry, and especially of the fact that any fan can be refined to a regular simplicial fan, by iterated stellar subdivisions. I refer to David Cox's notes in this volume and ([E], VI, No.8, p. 253), but I use the description of orbit closures given by Sturmfels in [St1], [St2]; see [Cox], §4. I keep the notations introduced above, so that our orbit closure is described by a map $b\colon \mathbf{N}^N \to \mathbf{N}^d$, where N is finite, and therefore an exact sequence

$$0 \to \mathrm{Ker} b \to \mathbf{Z}^N \to \mathbf{Z}^d \to \mathrm{Coker} b \to 0;$$

Since any subgroup of a free abelian group is free we may assume that the vectors $\gamma_1, \dots, \gamma_N$ generate $\mathbf{Z}^d$ i.e., $\mathrm{Coker} b = 0$. Choosing generators $(m^\ell - n^\ell)_{\ell \in L}$ for $\mathrm{Ker} b$ gives us an exact sequence

$$\mathbf{Z}^{|L|} \to \mathbf{Z}^N \to \mathbf{Z}^d \to 0 \ .$$

Note that the generators $m^\ell - n^\ell$ are necessarily primitive vectors in $\mathbf{Z}^N$, and we may choose them in such a way that m^ℓ and n^ℓ both have non-negative coordinates (see [St]). We assume from now on that they are chosen in this way.
By dualizing, we obtain an exact sequence

$$0 \to \check{\mathbf{Z}}^d \to \check{\mathbf{Z}}^N \to \check{\mathbf{Z}}^{|L|},$$

where the image of the $i-th$ basis vector of $\check{\mathbf{Z}}^d$ is the vector

$$w^i = (\gamma_{1i}, \gamma_{2i}, \dots, \gamma_{Ni}).$$

Choosing one of the $m^\ell - n^\ell$ gives us an injective map $\mathbf{Z} \to \mathbf{Z}^N$, and a surjective $\check{\mathbf{Z}}^N \to \check{\mathbf{Z}}$ having kernel $H_\ell = \{a \in \check{Z}^N \mid \langle a, m^\ell - n^\ell \rangle = 0\}$.
The intersection $\bigcap_{\ell \in L} (H_\ell \otimes_{\mathbf{Z}} \mathbf{Q})$ is the $\mathbf{Q}$-linear span W of the w^i.
By construction, we have $< w^i, m^\ell - n^\ell >= 0$ for $i = 1, 2, \dots, d,$ and $\ell \in L$, hence also $< w, m^\ell - n^\ell >= 0$ for $\ell \in L$ and all vectors $w \in W$. Conversely, W is the dual of the space generated by the $m^\ell - n^\ell$ for $\ell \in L$. The complement in $\check{\mathbf{R}}_+^N$ of the union of the hyperplanes H_ℓ is a union of rational N-dimensional convex

cones, and the union of hyperplanes itself is the reunion of rational convex cones of smaller dimensions. Note that H_ℓ meets the interior of $\check{\mathbf{R}}_+^N$ since the coordinates of the $m^\ell - n^\ell$ are not all of the same sign.
We can always find a regular fan Σ with support $\check{\mathbf{R}}_+^N$ which is not only compatible with the w^i but also with the union of the hyperplanes H_ℓ in the sense that any $\sigma \in \Sigma$ meets any one of the H_ℓ along one of its faces. Then, Σ is also compatible with the intersection W of the H_ℓ. We add to the w^i the primitive vectors of this fan which are contained in the convex cone generated by the w^i. Let us say that we obtain in this way vectors $w^1, \dots, w^d, w^{d+1}, \dots, w^{d+s}$. This corresponds to adding to the γ_j new vectors of the form;

$$\gamma_{N+t} = \sum a_j^t \gamma_j \text{ with } a_j^t \in \mathbf{N} \cup \{0\}.$$

Let $(U^{m^\ell} - U^{n^\ell})_{\ell \in L}$ be the generators of the kernel of the map

$$k[b] \colon k[U_1, \dots, U_N] \to k[t_1, \dots, t_d] \text{ given by } \quad U_i \mapsto t^{\gamma_i},$$

where as usual $U^m = U_1^{m_1} \cdots U_N^{m_N}$. If we take a regular simplicial cone of maximum dimension $\sigma = \langle a^1, \dots, a^N \rangle \in \Sigma$, associated to Σ by definition of the toric map $\pi(\Sigma) \colon Z(\Sigma) \to \mathbf{A}^N(k)$, in the affine chart $U_\sigma \subset Z(\Sigma)$, with coordinates $Y_1, \dots, Y_N$, we have the following expression for $\pi(\Sigma) | U_\sigma \colon U_\sigma \to \mathbf{A}^N(k)$

$$U_i \mapsto Y_1^{a_i^1} \cdots Y_N^{a_i^N} \quad \text{and so} \quad U^m \mapsto Y_1^{<a^1, m>} \cdots Y_N^{<a^N, m>}.$$

Let us compute the transform by this map of one of our binomial generators, denoted by $U^m - U^n$. We may assume that $a^1, \dots, a^t$ are those among the a^j which lie on the hyperplane H dual to $m - n$, i.e., such that $\langle a^j, m-n \rangle = 0$, $1 \le j \le t$. Because our fan is compatible with H, all the other $\langle a^j, m-n \rangle$ are of the same sign, say $\langle a^j, m-n \rangle > 0$. We have then

$$U^m - U^n \mapsto Y_1^{\langle a^1, n \rangle} \cdots Y_N^{\langle a^N, n \rangle} (Y_{t+1}^{\langle a^{t+1}, m-n \rangle} \cdots Y_N^{\langle a^N, m-n \rangle} - 1).$$

Compare with [Z1], Th. 2, p.863 and [S2]. This essentially means that in toric geometry, there is no need for Hironaka's game; it is replaced by the existence of a fan compatible with the H_ℓ as above. Hironaka's game comes into play when one wants to dominate our toric map by the composition of a sequence of blowing-ups with non singular centers.
The exceptional divisor of the map $\pi(\sigma)$ is the union of the $y_j = 0$ for those j such that a^j is not a basis vector of $\check{\mathbf{Z}}^N$ (see [G-T]). For simplicity we will assume that none of the a^j is a basis vector, so that the exceptional divisor is $Y_1 \dots Y_N = 0$. If $t = 0$, the strict transform $Y_{t+1}^{\langle a^{t+1}, m-n \rangle} \cdots Y_N^{\langle a^N, m-n \rangle} - 1 = 0$ does not meet the exceptional divisor; therefore the strict transform of $U^m - U^n = 0$ meets the exceptional divisor only in those charts U_σ for which at least one of the primitive vectors a^i is in H.

If the strict transform by $\pi(\Sigma)$ of the subspace $X \subset \mathbf{A}^N(k)$ defined by the ideal $(U^{m^\ell} - U^{n^\ell})_{\ell \in L}$ is to be non empty in the chart $\pi(\sigma)\colon U_\sigma \to \mathbf{A}^N(k)$, there must be a j such that $\langle a^j, m^\ell - n^\ell \rangle = 0$ for $\ell \in L$, i.e., $a^j \in W$; (this generalizes the result of [G-T], 5.2).

The space $\check{W}$ generated by the $m^\ell - n^\ell$ is of dimension $N-d$; let us choose $N-d$ vectors $m-n$ generating $\check{W}$, say $m^1 - n^1, \ldots, m^{N-d} - n^{N-d}$. We now have to check that whenever the strict transform of the subspace of $\mathbf{A}^N(k)$ defined by

$$U^{m^1} - U^{n^1} = \cdots = U^{m^{N-d}} - U^{n^{N-d}} = 0$$

is not empty it is non singular and transverse to the exceptional divisor, and that the strict transforms of all the other $U^{m^\ell} - U^{n^\ell} = 0$ vanish on it.

The strict transforms of the equations have the form

$$\begin{array}{rcl} Y_1^{\langle a^1, m^1 - n^1 \rangle} \cdots Y_N^{\langle a^N, m^1 - n^1 \rangle} - 1 & = & 0 \\ Y_1^{\langle a^1, m^2 - n^2 \rangle} \cdots Y_N^{\langle a^N, m^2 - n^2 \rangle} - 1 & = & 0 \\ \vdots & \vdots & \vdots \\ Y_1^{\langle a^1, m^{N-d} - n^{N-d} \rangle} \cdots Y_N^{\langle a^N, m^{N-d} - n^{N-d} \rangle} - 1 & = & 0 \end{array}$$

Note that this strict transform meets the component $Y_j = 0$ of the exceptional divisor if and only if $a^j \in W$; let us renumber the a^j so that $a^1, \ldots, a^t$ are those who lie in W, i.e., $\langle a^i, m^s - n^s \rangle = 0$ for $1 \leq i \leq t$ and all s, and assume $t \geq 1$, otherwise the strict transform does not meet the chart U_σ. As in ([G-T], §4), we can compute the jacobian matrix J of these equations by logarithmic differentiation, and find the equality of $N \times (N-d)$ matrices

$$Y_{t+1} \ldots . Y_N J = Y_{t+1}^{\sum_s \langle a^{t+1}, m^s - n^s \rangle} \ldots Y_N^{\sum_s \langle a^N, m^s - n^s \rangle} (\langle a^j, m^s - n^s \rangle).$$

Since $\dim W = d$, we must have $t \leq d$; since the vectors a^j form a basis of $\mathbf{Q}^N$, and the space $\check{W}$ generated by the $m^s - n^s$ is of dimension $N-d$, the rank of the matrix $(\langle a^j, m^s - n^s \rangle)$ is $N-d$. So we see that the rank of J is $N-d$ everywhere on the strict transform, and that this strict transform is transversal to the exceptional divisor. For any other $m^\ell - n^\ell = \sum_{s=1}^{N-d} b_s^\ell (m^s - n^s)$, we have:

$$\langle a^j, m^\ell - n^\ell \rangle = \sum_{s=1}^{N-d} b_s^\ell \langle a^j, m^s - n^s \rangle, \text{ with } b_s^\ell \in \mathbf{Q}.$$

Now the signs of the nonzero $\langle a^j, m^\ell - n^\ell \rangle$for $t+1 \leq j \leq N$ are the same by our assumption and we see that the strict transform of $U^{m^\ell} - U^{n^\ell} = 0$ is of the form

$$Y_{t+1}^{\sum_s \pm b_s^\ell \langle a^{t+1}, m^s - n^s \rangle} \ldots Y_N^{\sum_s \pm b_k^\ell \langle a^N, m^s - n^s \rangle} - 1.$$

Using the irreducibility of our orbit closure X, and the fact that all our equations vanish at the point $Y_{t+1} = \cdots = Y_N = 1$, we see that it vanishes whenever all the strict transforms of $U^{m^k} - U^{n^k}$, which are

$$Y_{t+1}^{\langle a^{t+1}, m^s - n^s\rangle} \cdots Y_N^{\langle a^N, m^s - n^s\rangle} - 1,\ 1 \leq k \leq N - d,$$

vanish. This shows that the strict transform of our orbit closure, in a chart as above where it is not empty, is defined by the $N - d$ equations written above. The charts corresponding to regular cones $\sigma \in \Sigma$ of dimension $< N$ are open subsets of those which we have just studied, so they contribute nothing new.
We also have to show that our toric map induces an isomorphism outside of the singular locus of our binomial variety, so that it induces a resolution and not only a pseudo-resolution in the sense of [G-T].

Proposition 6.4. ** The singular locus of an irreducible and reduced binomial variety over a perfect field is the union of some of the intersections of this variety with the coordinate hyperplanes. The toric resolution can be chosen to be an isomorphism outside of the singular locus.**

Proof. Let d be the dimension of our binomial variety. An easy computation shows that the jacobian determinant $J_{I,L'}$ of rank $c = N - d$ of the generators $(u^{m^\ell} - \lambda_{m^\ell n^\ell} u^{n^\ell})_{\ell \in L}$ of our prime binomial ideal $P \subset k[u_1, \ldots, u_N]$ associated to $I = (i_1, \ldots, i_c)$ and a subset $L' \subseteq L$ of cardinality c satisfies the congruence

$$u_{i_1} \ldots u_{i_c}.J_{I,L'} \equiv \left(\Pi_{\ell \in L'} u^{m^\ell}\right)\left(\Pi_{\ell \in L'} \lambda_{m^\ell n^\ell}^{e_\ell}\right) \mathrm{Det}_{I,L'}\left(\left(\langle m - n\rangle\right)\right) \quad \mathrm{mod}.P,$$

where $\left(\langle m - n\rangle\right)$ is the matrix of the vectors $(n^\ell - n^\ell)_{\ell \in L}$, $\mathrm{Det}_{I,L'}$ indicates the minor in question, and the e_ℓ are exponents in $\mathbf{Z}$. Since k is perfect, by Zariski's jacobian criterion (see [Z 4]), the rank of the image of the jacobian matrix in $k^{N \times |L|}$ is equal to c. The only possibility for the intersection of our variety with $u_i = 0$ not to be in the singular locus is that all there exists such a minor such that all the m_i^ℓ for $\ell \in L'$ are zero except one, which is equal to one. In that case one sees that our binomial variety looks like the intersection with a hypersurface $u_i u'^r - u'^s = 0$ of a binomial variety defined by equations in $u' = (u_1, \ldots, \hat{u}_i, \ldots, u_N)$, so that there is a toric resolution which is an isomorphism outside of the singular locus. This proves the result. ○

Example. Zariski's jacobian criterion implies that the $N-d$-minors of the matrix $\left(\langle m - n\rangle\right)$ are coprime integers. In the case where $d = 1$ and the monomial curve corresponding to the semigroup $\Gamma = \langle \overline{\beta_0}, \ldots, \overline{\beta_g}\rangle$ is a complete intersection, as in the example of subsection 5.4 the minors of the matrix coincide, up to sign, with the generators of the semigroup, and so they are coprime. This argument solves the minor difficulty mentionned at the end of §5 of [G-T].

Remember that a field of characteristic zero or an algebraically closed field of positive characteristic are perfect.
Finally we have shown:

Proposition 6.5. *An embedded resolution of the orbit closure $X \subset \mathbf{A}^N(k)$ corresponding to N vectors γ_j in $\mathbf{Z}^d$, giving a map $b\colon \mathbf{Z}^N \to \mathbf{Z}^d$, is obtained by taking the toric map $\pi(\Sigma)\colon Z(\Sigma) \to \mathbf{A}^N(k)$ associated to a regular fan Σ with support $\check{\mathbf{R}}^N_+$ compatible with the hyperplanes in $\check{\mathbf{R}}^N$ dual to a system of generators of* Ker$b$.

Let us now suppose that we have an irreducible subspace of k^N defined by binomials $U^{m^\ell} - \lambda_{m^\ell n^\ell} U^{n^\ell}$ for $\ell \in L$, where as usual $U^m = U_1^{m_1} \cdots U_N^{m_N}$ and λ_{mn} is invertible in k. By ([E-S], Cor. 2.3), at least if our field k is algebraically closed, the subspace defined by the binomials obtained by changing each λ_{mn} to 1 is a torus orbit closure, and has an embedded resolution by a toric map $\pi(\Sigma)\colon X(\Sigma) \to \mathbf{A}^N(k)$ as above. In one of the charts of $X(\Sigma)$, the ideal $(U^m - U^n)$ becomes the ideal generated (up to permutation of the coordinates) by

$$Y^{e(m_k,n_k)}(Y_{t+1}^{\langle a^{t+1}, m^k - n^k\rangle} \cdots Y_N^{\langle a^N, m^k - n^k\rangle} - 1); \qquad 1 \le k \le N-d.$$

The nature of the computation shows that we have:

$$U^m - \lambda_{mn} U^n \mapsto Y^{e(m,n)}(Y_{t+1}^{\langle a^{t+1}, m-n\rangle} \cdots Y_N^{\langle a^N, m-n\rangle} - \lambda_{mn}).$$

The relations between the λ_{mn} associated to linear relations between the $m-n$, of the form (with the notations used above)

$$\lambda_{m^\ell n^\ell} = \Pi_{k=1}^{N-d} \lambda_{m^k n^k}^{b_k^\ell},$$

are exactly what is needed to ensure that once we have chosen a system of generators $m^k - n^k$ as above, the transforms of the remaining binomials $U^m - \lambda_{mn} U^n$ will vanish on the strict transform defined by the equations

$$Y_{t+1}^{\langle a^{t+1}, m^k - n^k\rangle} \cdots Y_N^{\langle a^N, m^k - n^k\rangle} - \lambda_{m^k n^k} = 0\ ; \qquad 1 \le k \le N-d,$$

so that the subspace defined by our binomial ideal is also resolved by $\pi(\Sigma)$. Keeping the notations just introduced, we see that we have proved

Proposition 6.6. *Assuming that the field k is algebraically closed, let $X \subset \mathbf{A}^N(k)$ be an irreducible binomial variety, defined by $(U^{m^\ell} - \lambda_{m^\ell n^\ell} U^{n^\ell})_{\ell \in L}$, $\lambda_{m^\ell n^\ell} \in k^*$. For any regular simplicial fan Σ with support $\mathbf{R}^N_+$ compatible with each of the hyperplanes H_ℓ dual to $m^\ell - n^\ell$, the corresponding toric map $Z(\Sigma) \to \mathbf{A}^N(k)$ gives an embedded resolution of $X \subset k^N$.*

Remark. We could also try to generalize the argument of [G-T]: we seek a parametrization of our orbit closure, say with algebra $k[t^\Gamma]$, where Γ is the sub-semigroup of $\mathbf{Z}^d$ generated by the vectors $(\gamma_1, \ldots, \gamma_N)$. The algebra $k[t^\Gamma]$ is given parametrically by

$$\begin{aligned} U_1 = &\quad t_1^{\gamma_{11}} \ldots t_d^{\gamma_{1d}} \\ \vdots\ = &\quad \vdots \\ U_i = &\quad t_1^{\gamma_{i1}} \ldots t_d^{\gamma_{id}} \\ \vdots\ = &\quad \vdots \\ U_N = &\quad t_1^{\gamma_{N1}} \ldots t_d^{\gamma_{Nd}} \end{aligned}$$

where the γ_{ij} need not be all positive, and we seek a toric modification described by a fan. Each cone of this fan should be generated by vectors $(a^j \in \mathbf{Z}^N)$, and the corresponding toric map described by

$$\begin{aligned} U_1 = &\quad y_0^{a_1^0} y_1^{a_1^1} \dots y_N^{a_1^N} \\ \vdots = &\qquad\quad \vdots \\ U_i = &\quad y_0^{a_i^0} y_1^{a_i^1} \dots y_N^{a_i^N} \\ \vdots = &\qquad\quad \vdots \\ U_N = &\quad y_0^{a_N^0} y_1^{a_N^1} \dots y_N^{a_N^N} \end{aligned}$$

Note that in order for this to make sense, if we let N tend to infinity, the following condition must be satisfied by the vectors a^k

$$\forall\ i \geq 0, \text{ there exists } k_0(i) \text{ such that } a_i^k = 0 \ \text{ for } k \geq k_0(i).$$

We seek a parametrization of the "strict transform" of Spec$k[t^\Gamma]$ of the form

$$y_i = t_1^{s_{i1}} \dots t_d^{s_{id}}$$

such that if we substitute this parametrization in the expression of the y_k we get our original parametrization for Spec$k[v^\Gamma]$. This means that we must have the equality

$$\gamma = A.S$$

where γ is the $N \times d$ matrix (γ_{ij}), A is the $N \times N$ matrix $(a_{ik} = a_i^k)$, and S is the $N \times d$ matrix (s_{kj}). We may wish the a_{ik} to be non negative but allow the s_{ij} to be negative, or we may insist that the s_{ij} should be positive, and then must allow some negative a_{ik}. In this last case, the vectors $(w^i \in N^N)_{1 \leq i \leq N}$ with coordinates

$$w^i = (\gamma_{1i}, \gamma_{2i}, \dots, \gamma_{Ni})$$

are in the convex hull of the a^j since they are written

$$w^i = \sum_k s_{ki} a^k$$

and we must choose our fan so that this implies, as much as possible, that the w^i are among the a^j (see [G-T], 5.2).

However, the natural approach is to consider the map $p \colon W \to \mathbf{A}^N(k)$ having our orbit closure as an image, mentionned above, and rewrite the map in terms of the coordinates on W. Then the corresponding a_{ik} will all be nonnegative, and our problem to build a toric modification is to find the a_i^k as the edges of a "regular " fan, and such that the parametrization $y_i = t_1^{s_{i1}} \dots t_d^{s_{id}}$ describes a non singular orbit closure. One way is that S contains a $d \times d$ identity matrix, up to permutation of lines or of columns, but it is by no means the only one. If we impose that the

s_{ij} be non negative, that each of the d vectors w^i is in the convex cone generated by the a^k. Therefore if we choose a fan Σ in $\mathbf{R}_+^N$ which is compatible with each vector w^i in the sense that if a cone $\sigma \in \Sigma$ contains one of the w^i it must contain it in one of its edges (one-dimensional faces), we will have that each w^i must be one of the a^k. Indeed, each w^i must be an integral multiple of one of the a^k, but since $\gamma_1 \ldots, \gamma_N$ generates $\mathbf{Z}^d$ as a group, each of the vectors w^i must be primitive. They must also be distinct since a $d \times d$ minor of the matrix formed by the w^i is not zero.

This implies that the matrix s_{ij} contains, after permutation of the rows and columns, a $d \times d$ identity matrix, and zeros elsewhere so that the strict transform of our torus orbit by the toric map $\mathbf{A}^N(k) \to \mathbf{A}^N(k)$ defined by σ is parametrized by

$$y_1 = t_1, \ldots, y_d = t_d, y_{d+1} = 1, \ldots, y_N = 1.$$

It is indeed non singular and transverse to the exceptional divisor of the toric map.

Of course, given the vectors $\gamma_1, \ldots, \gamma_N$ and therefore the w^i, these need not form part of a basis of the integral lattice, and we have to add new vectors to the w^i as above.

However, we note that this description starts with the assumption that the strict transform of our orbit closure is again an orbit closure of the orbit of the point $(1, \ldots, 1)$ for an action with non negative exponents in each chart, for which there is no ground when $d > 1$. This means that we must accept negative s_{ij}'s and then the method we use above is simpler.

6.4 An example: branches revisited

Let R be a one-dimensional excellent equicharacteristic analytically irreducible local ring, i.e., the local ring of a branch. Let us assume that the residue field k of R is algebraically closed. The ring R has only one non trivial valuation, whose valuation ring is the integral closure R_ν of R in its field of fractions. Since R is excellent, R_ν is a finite R-module ([EGA], Scholie 7.8.3, vii)). Let $\Gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_N \rangle$ be the semigroup of values of ν on $R \setminus \{0\}$. Since k is algebraically closed, the valuation ν is rational, and the completion $\hat{R}^m$ of R is equal to $\hat{R}^\nu$ and is a subring of $k[[t]]$, the valuation $\hat{\nu}$ being induced by the t-adic valuation. Let us chose elements $(\xi_i)_{1 \leq i \leq N}$ in R such that $\nu(\xi_i) = \gamma_i$. Their initial forms for the ν-adic valuation generate the graded ring

$$\mathrm{gr}_\nu R = k[t^{\gamma_1}, \ldots, t^{\gamma_N}] \subset k[t] = \mathrm{gr}_\nu R_\nu.$$

By Proposition 5.14, the map $u_i \mapsto \xi_i v^{-\nu(\xi_i)}$ determines a surjective map

$$k[v][[u_1, \ldots, u_N]] \to \widehat{\mathcal{A}_\nu(R)} \subset k[v][[t]].$$

The kernel of this map is generated by polynomials

$$F_{mn} = u^m - u^n + \sum c_p^{(mn)}(v) u^p$$

which are deformations of binomial generators $u^m - u^n$ of the kernel of the map

$$k[u_1, \dots, u_N] \to k[t^{\gamma_1}, \dots, t^{\gamma_N}], \quad u_i \mapsto t^{\gamma_i}$$

Denoting by $w = (\gamma_1, \gamma_2, \dots, \gamma_N) \in \mathbf{N}^N$ the weight vector, for each of these equations, we add to the binomial $u^m - u^n$ only terms $c_p^{(mn)}(v)u^p$ such that $w(p) > w(m) = w(n)$. If now we choose a regular fan Σ in $\mathbf{R}_+^N$ compatible with the dual hyperplanes of the vectors $m-n$, and so in particular with the vector w, we obtain a toric map

$$\pi(\Sigma)\colon Z(\Sigma) \to \mathbf{A}^N(k)$$

which is a toric embedded resolution of the monomial curve $C^\Gamma \subset \mathbf{A}^N(k)$ according to what we saw in the previous section. The map $\mathrm{Id}_{\mathbf{A}^1(k)} \times \pi(\Sigma)\colon \mathbf{A}^1(k) \times Z(\Sigma) \to \mathbf{A}^1(k) \times \mathbf{A}^N(k)$ then induces an embedded resolution of singularities for the formal subspace defined in $k[v][[u_1, \dots, u_N]]$ by the ideal generated by the F_{mn}, which is a simultaneous resolution for all the fibers of the projection to $\mathbf{A}^1(k)$. This is checked exactly as in [G-T]. It implies that the map $\pi(\Sigma)$ gives an embedded resolution of the singularities of the image of the formal completion of our curve embedded in $\mathbf{A}^N(k)$ by using the elements $\xi_i \in R \subset \hat{R}^\nu$ as coordinates. Now since $\pi(\Sigma)$ is a monomial map, it corresponds to adding to $\hat{R}^\nu$ certain monomials in the ξ_i, with some negative exponents, and then localizing the $\hat{R}^\nu$-algebra which they generate. This operation has a meaning in R itself, and builds a local ring R' which is essentially of finite type over R and birationally equivalent to R. The local ring R' is excellent since R is, and its completion is regular since it is the transform of $\hat{R}^\nu$, so R' itself is regular.

6.5 Another example, in dimension three

Example. Let us give $\mathbf{Z}^2$ the lexicographic order and denote by $k((t^{\mathbf{Z}^2_{\mathrm{lex}}}))$ the field of Puiseux series associated to the ordered group $\mathbf{Z}^2_{\mathrm{lex}}$ and the field k as in [B2]; it is the field of formal series with exponents forming a well ordered subset of $\mathbf{Z}^2_{\mathrm{lex}}$. It is naturally endowed with the t-adic valuation. Let us denote by $k[[t^{\mathbf{Z}^2_+}]]$ the corresponding valuation ring. Set $x = t^{(1,0)}$, $y = t^{(0,1)}$. Choose a sequence of pairs of positive integers $(a_i, b_i)_{i \geq 3}$ such that $a_{i+1} > a_i$, the series $\sum_{i \geq 3} x^{a_i}$ is not algebraic over $k[x]$, and the sequence of ratios $\frac{b_{i+1} - b_i}{a_i}$ increases strictly with i. Let R_0 be the k-subalgebra of $k[[t^{\mathbf{Z}^2_+}]]$ generated by

$$x = t^{(1,0)}, y = t^{(0,1)}, z = \sum_{i \geq 3} x^{a_i} y^{-b_i}.$$

There cannot be an algebraic relation between x, y, and z, so the ring $R_0 = k[x, y, z]$ is the polynomial ring in three variables. It inherits the t-adic valuation of $k[[t^{\mathbf{Z}^2_+}]]$. One checks that this valuation extends to the localization $R = k[x, y, z]_{(x,y,z)}$; it is a rational valuation of height two and rational rank two.

Let us try to compute the semigroup Γ of the values that it takes on R. We have $\gamma_1 = (0,1)$, $\gamma_2 = (1,0), \gamma_3 = (a_3, -b_3) \in \Gamma$. Set $\Gamma_3 = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$ and $u_1 = x$, $u_2 = y$, $u_3 = z \in R$. Then we have $u_2^{b_3} u_3 - u_1^{a_3} = \sum_{i \geq 4} u_1^{a_i} u_2^{b_3 - b_i} \in R$, so that $\gamma_4 = (a_4, b_3 - b_4)$ is in Γ. It is easy to see from our assumptions that *no multiple of* γ_4 *is in* Γ_3, and that is is the smallest element of Γ which is not in Γ_3. We set $u_4 = u_2^{b_3} u_3 - u_1^{a_3}$, and continue in the same manner: $u_2^{b_3 - b_4} u_4 - u_1^{a_4} = u_5$, ..., $u_2^{b_i - b_{i-1}} u_i - u_1^{a_i} = u_{i+1}$,... with the generators $\gamma_i = \nu(u_i) = (a_i, b_{i-1} - b_i)$ for $i \geq 4$. Finally we have

$$\Gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_i, \ldots \rangle,$$

the u_i are generators ξ_i lifting generators of the graded algebra $\mathrm{gr}_\nu R$, and the equations (setting $b_2 = 0$)

$$u_2^{b_i - b_{i-1}} u_i - u_1^{a_i} = u_{i+1}, \quad i \geq 3$$

above describe $\hat{R}^\nu$. In fact they even describe R; it is clear that from them we can reconstruct the value of u_3 as a function of u_1, u_2 by (infinite) elimination. The binomial equations for $\mathrm{gr}_\nu R$ are the

$$U_2^{b_i - b_{i-1}} U_i - U_1^{a_i} = 0, \quad i \geq 3$$

showing that all the U_i for $i \geq 3$ are rationally dependent on U_1, U_2. Using the remark following proposition 3.7, we see that the Krull dimension of $\mathrm{gr}_\nu R$ is two. The fact that R is regular of dimension three is due to the abyssal phenomenon that we met in the third example of subsection 4.2. Remark that in this example $\mathrm{gr}_\nu R$ is not regular if $a_3 > 1$.
From our assumption on the growth of the ratios we see moreover that *no multiple of* γ_i is in $\Gamma_{i-1} = \langle \gamma_1, \ldots, \gamma_{i-1} \rangle$. In fact γ_i is outside of the cone with vertex 0 generated by Γ_{i-1} in $\mathbf{R}^2$.
It is interesting to study in this case the construction of a system of regular fans which will resolve the singularities of $\mathrm{gr}_\nu R$.

Problem. Can the fact that even for large i no multiple of γ_i is in Γ_{i-1} occur in dimension two? Hint: use [A2].

7 The proof (speculative sketch)

...and he will be too dazzled by the light to look at the objects whose shadows he was seeing a moment ago...

Plato, *The Republic*, VII, 515

Assume that the local nœtherian excellent analytically irreducible local ring R has an algebraically closed residue field k.

Given a valuation ν_0 on the ring R, we may specialize ν_0 to a valuation ν whose ring R_ν dominates R with trivial residue field extension (see subsection 3.6, Proposition 3.11).
We first uniformize the valuation $\hat{\nu}$ on $\hat{R}^\nu$. In the last subsection we will sketch the descent from $\hat{R}^\nu$ to R.
According to the first Corollary to Proposition 5.10, which ensures the nice behaviour of the specialization above with respect to composition of valuations, the same toric map will also simultaneously uniformize the valuations in the fibers of the total space of the specialization of $\hat{R}^{\nu_0}$ to $\widehat{\mathrm{gr}_{\nu_0}}R$, which is the subspace of the preceding specialization defined by the closure of the ideal $(v^\psi - 1)_{\psi\in\Psi_+}$, and hence also uniformize $\hat{\nu}_0$ on $\hat{R}^{\nu_0} = \hat{R}^\nu$.

Since ν is rational, by Proposition 6.2, the graded algebra $\mathrm{gr}_\nu R$ is a quotient of a polynomial ring by a binomial ideal. The ν-adic completion $\hat{R}^\nu$ has a field of representatives $k \subset \hat{R}^\nu$ which we take as our base field. According to Propositions 5.14 and 5.15, we have a surjective map

$$k[\widehat{v^{\Phi_+}][(u_i)}_{i\in I}] \to \widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}},$$

the kernel of which is the closure in $k[\widehat{v^{\Phi_+}][(u_i)}_{i\in I}]$ of the ideal generated by the elements

$$F_{mn} = u^n - \lambda_{mn}u^m + \sum_p c_p^{(mn)}(v^\phi)u^p.$$

The idea for the proof is that if the k-algebra $\mathrm{gr}_\nu R$ is finitely generated, say with N generators one can proceed essentially as in [G-T]: we have in this case $\dim \hat{R}^\nu = \dim \mathrm{gr}_\nu R = \mathrm{r}(\nu)$, and a toric map $Z \to \mathbf{A}^N(k)$ which resolves $\mathrm{Specgr}_\nu R$ will also uniformize $\hat{\nu}$ on $\hat{R}^\nu$. If $\mathrm{gr}_\nu R$ is not finitely generated, we have to show that it suffices to resolve a finite approximation to $\mathrm{gr}_\nu R$, the subalgebra $\mathrm{gr}_\nu^{(h)} R$ generated by the images of $(U_1, \ldots, U_{N(h)})$ for sufficiently large h, where h is a level of approximation for $\mathrm{gr}_\nu R_\nu$ in the sense of Proposition 4.3. Here the key point is that for N sufficiently large, since Φ is of finite rational rank and $\hat{R}^\nu$ is nœtherian and henselian, all the equations F_{mn} whose binomial initial form involves a variable U_j with $j > N$ also involve u_{j+1} linearly, so that their only signification is to produce the abyssal phenomenon which we observed in the third example of subsection 4.2 and in subsection 6.5. This does not obstruct regularity, so that it is sufficient to resolve the singularities of the finitely generated approximating algebra.

7.1 The approximation process

sense we can resolve ν_{gr} by a single infinite dimensional toric modification of $k[(U_i)_{i\in I}]$, hence uniformize the completion $\hat{\nu}_{\mathrm{gr}}$ on $\widehat{\mathrm{gr}_\nu}R$. This toric modification should then uniformize $\hat{\nu}$ on the completion $\hat{R}^\nu$.
Let us keep the notations of the preceding section; the variables U_i are ordered by their valuations $\gamma_i \in \Gamma$. Let us consider the nested sequence of polynomial

algebras $k[t^{(h)}]=k[t_1^{(h)},\dots,t_r^{(h)}]$ approximating $\mathrm{gr}_\nu R_\nu$ according to Proposition 4.3, b). Remember that r is the rational rank of the value group Φ of ν. For each integer h, there is a largest integer $N(h)$ such that the images in $\mathrm{gr}_\nu R$ of the first $N(h)$ variables $(U_1,\dots,U_{N(h)})$ are in $k[t^{(h)}]$. Let $\mathrm{gr}_\nu^{(h)}R$ be the subalgebra of $\mathrm{gr}_\nu R$ generated by the images of the first $N(h)$ variables. The kernel of the map $k[(U_i)_{1\leq i\leq N(h)}]\to \mathrm{gr}_\nu^{(h)}R$ is generated by a binomial ideal, and corresponds to a torus orbit closure in $\mathbf{A}^{N(h)}(k)$ characterized by the map $\omega(h)\colon \mathbf{Z}^{N(h)}\to\mathbf{Z}^r$ sending the i-th base element to $\nu(\xi_i)\in\mathbf{Z}^r$. We apply the results of subsection 6.3 to this binomial ideal in finitely many variables, obtaining a regular fan in $\check{\mathbf{R}}_+^{N(h)}$. Going from h to $h+1$, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{Z}^{N(h)} \arrow[r] \arrow[d] & \mathbf{Z}^r \arrow[d] \\
\mathbf{Z}^{N(h+1)} \arrow[r] & \mathbf{Z}^r
\end{tikzcd}$$

and after dualizing

$$\begin{tikzcd}
\check{\mathbf{Z}}^r \arrow[r] \arrow[d] & \check{\mathbf{Z}}^{N(h+1)} \arrow[d, "p_h"] \\
\check{\mathbf{Z}}^r \arrow[r] & \check{\mathbf{Z}}^{N(h)}
\end{tikzcd}$$

Note that in the case of curves, $r=1$ and the image of $\check{\mathbf{Z}}$ in $\check{\mathbf{Z}}^{N(h)}=\check{\mathbf{Z}}^N$ is the weight vector. Although no approximation is needed, we can illustrate the procedure by setting $e_i=\gcd(\gamma_1,\dots,\gamma_i)$ and considering the nested sequence of subalgebras

$$k[t^{e_1}]\subset k[t^{e_2}]\subset\cdots\subset k[t^{e_{N-1}}]\subset k[t].$$

The construction shown above then builds a sequence of vector spaces $(\mathbf{R}^i\to\mathbf{R}^{i-1})_{2\leq i\leq N}$ and a weight vector in each. In the case of plane branches, where the relations between the γ_i are particularly simple, this suggests a procedure to build inductively a fan compatible with the hyperplanes corresponding to these relations; see subsection 5.4 and [G-T].

For each h we take a fan Σ_h in $\check{\mathbf{R}}_+^{N(h)}$ compatible with the hyperplanes dual to generators of the kernel of $\omega(h)$ and with the inverse image by p_h of Σ_{h-1} In this manner, letting h grow, we obtain a projective system of vector spaces indexed by the linearly ordered set I of generators of Γ, and a subordinate system of regular fans, which is what we would like to call a regular fan for the projective system of vector spaces.

We are led to the following

Definition of a possibly infinite dimensional fan:

Definition. Let P be a projective system of $\mathbf{R}$-linear maps between finite-dimensional real vector spaces, indexed by a filtering ordered set I.

$$P\colon \big((p_{i,j}\colon \mathbf{R}^{n(i)} \to \mathbf{R}^{n(j)})_{i,j\in I, i>j}\big),$$

where $n(i)$ is an integer valued function. A fan Σ subordinate to P is the datum for each $i \in I$ of a rational fan Σ_i in $\mathbf{R}^{n(i)}$ in such a way that for each $i > j$, the fan Σ_i is a refinement of $p_{i,j}^{-1}(\Sigma_j)$.
We say that the fan is of finite type if there exists a finite subset $F \subset I$ such that for $i, j \notin F$, $i > j$, the image of Σ_i is equal to Σ_j, and if we denote by $K_i \subset \mathbf{R}^{n(i)}$ the kernel of $p_{i,j}$, the map $\sigma \mapsto (p_{i,j}(\sigma), \sigma \cap K_i)$ is a bijection from the set of cones in $\Sigma_{n(i)}$ to the set of pairs consisting of a cone in Σ_j and one of the cones in $\Sigma_{n(i)}$ whose image by $p_{i,j}$ is 0. We say that Σ is simplicial (resp. regular) if all Σ_i are. We say that Σ is *rational* if there exists a finite subset $F \subset I$ such that for $i, j \notin F$, $i > j$, the map $p_{i,j}$ sends a basis of the integral lattice of $\mathbf{R}^{n(i)}$ to part of a basis of the integral lattice of $\mathbf{R}^{n(j)}$.

To such a system of fans is associated a system of toric maps $\pi(\Sigma_i)\colon Z(\Sigma_i) \to \mathbf{A}^{n(i)}(k)$; if we have a simple system of surjective maps $p_i\colon \mathbf{R}^{n(i+1)} \to \mathbf{R}^{n(i)}$, we get inclusions $\mathbf{A}^{n(i)}(k) \subset \mathbf{A}^{n(i+1)}(k)$, so that we get a compatible system of toric maps of an increasing sequence of affine spaces, which we may call an infinite-dimensional toric map. I leave it to the reader to interpret geometrically the various conditions defined for the system of fans.

7.2 Computations in the semigroup and transforms of $\hat{R}^\nu$

Using the specialization of $\hat{R}^\nu$ to $\widehat{\mathrm{gr}_\nu} R$, which we can write in a very explicit form (see 5.10) as the space associated to

$$k\widehat{[v^{\Phi_+}][(u_i)_{i\in I}]}/\overline{\big(u^n - \lambda_{mn}u^m + \sum_p c_p^{(mn)}(v^\phi)u^p\big)},$$

with $v(u^p) > v(u^n) = v(u^m)$. For any homomorphism $c\colon \Phi_+ \to k^*$, say $\phi \mapsto c(\phi)$, we have an isomorphism

$$\hat{c}\ :\ k\widehat{[v^{\Phi_+}][(u_i)_{i\in I}]}/\overline{\big(u^n - \lambda_{mn}u^m + \sum_p c_p^{(mn)}(v^\phi)u^p,\ (v^\phi - c(\phi))_{\phi\in\Phi_+}\big)} \xrightarrow{\simeq} \hat{R}^\nu.$$

We will denote by $c(F_{mn})$ the series

$$c(F_{mn}) = u^n - \lambda_{mn}u^m + \sum_p c_p^{(mn)}(c(v^\phi))u^p \in k\widehat{[(u_i)_{i\in I}]}.$$

Let us first assume that the k-algebra $\mathrm{gr}_\nu R$ is of finite type, generated say by $(\overline{\xi}_1, \ldots, \overline{\xi}_N)$. In this case, using the fact that completion in scales is a succession of completions with respect to valuations of rank one, and that these preserve dimension, one can check that $\dim\hat{R}^\nu = \mathrm{r}(\Phi) = \dim \mathrm{gr}_\nu R$. Indeed this argument shows that we can apply the semi-continuity of the dimension of fibers to the map

$$\mathrm{Spec}\widehat{\mathcal{A}_{\hat\nu}(\hat{R}^\nu)} \to \mathrm{Spec}k[v^{\Phi_+}]$$

so that $\mathrm{r}(\Phi) = \dim \mathrm{gr}_\nu R \geq \dim\hat{R}^\nu$. The converse inequality is Abhyankar's inequality applied to $\hat{R}^\nu$. Let us keep the notations of the preceding subsection, with $N = N(h)$ and the notations used in the proof of Proposition 6.6. We must compute the strict transforms of the equations F_{mn} under the toric map associated to a regular fan Σ as in 6.6. Let us denote by γ_i the degree of U_i (the valuation of ξ_i) for $1 \leq i \leq N$; by assumption it is in the free submonoid $\mathbf{N}^r \subseteq \Phi_+$ and we may consider the map

$$\mathbf{Z}^N \to \mathbf{Z}^r, \quad (n_1, \dots, n_N) \mapsto \sum_{i=1}^{N} n_i\gamma_i \in \mathbf{Z}^r.$$

The image $W \subset \check{\mathbf{Z}}^N$ of the dual map is the r-dimensional free submodule generated by the r elements $w^j = (\gamma_{ij})_{\ 1\leq i\leq N} \in \check{\mathbf{Z}}^N$.

We want to show that if u^p has value greater than u^n, after a toric modification corresponding to a regular fan compatible with the hyperplanes defining W, the transform of u^n divides the transform of u^p.

By construction, the assumption means that $\langle w^j, p-n\rangle > 0$ for $1 \leq j \leq r$. Since the F_{mn} are polynomials, after refining our fan Σ to a regular fan Σ' without changing its intersection with W, we may assume that these inequalities imply for each cone $\sigma = \langle a^1, \dots, a^N\rangle \in \Sigma$ such that $\sigma \cap W \neq \emptyset$ the inequalities $\langle a^k, p-n\rangle \geq 0, \ \ 1 \leq k \leq N$. These inequalities precisely mean that in the chart of $Z(\Sigma)$ corresponding to σ, the transform of u^n divides the transform of u^p (compare with [Z1], Th.2, p. 863). We may assume without loss of generality that the intersection of σ with W is of dimension r, so that $(a^1, \dots, a^r)$, say, are in W. Then in our monomial transformation

$$u_i = y_1^{a_i^1} \dots y_N^{a_i^N}, \quad 1 \leq i \leq N,$$

we find that the strict transform of F_{mn} is in that chart has the following expression, where I write $y^{\langle a,m\rangle}$ for $y_1^{\langle a^1,m\rangle} \dots y_N^{\langle a^N,m\rangle}$ except when I want to single out some of the y_i's

$$F_{mn} \circ \pi(\sigma) =$$
$$y_1^{\langle a^1,m\rangle} \dots y_r^{\langle a^r,m\rangle} \big(y_{r+1}^{\langle a^{r+1},n-m\rangle} \dots y_N^{\langle a^N,n-m\rangle} - \lambda_{mn} + \textstyle\sum_p c_p^{(mn)}(v^\phi) y^{\langle a,p-m\rangle}\big).$$

It shows that the strict transform of $\hat{R}^\nu$ is non singular, as a deformation of the strict transform of $\mathrm{gr}_\nu R$. Now the refinement of Σ to Σ' corresponds to a birational map $Z(\Sigma') \to Z(\Sigma)$ which induces an isomorphism outside the divisor $\Pi_{j=r+1}^N u_j = 0$, which the strict transform of our space does not meet, so that we may descend the result of simultaneous resolution from $Z(\Sigma')$ to $Z(\Sigma)$. This is the same process as in [G-T].

In the general case the first problem is to show that our valuation $\hat{\nu}$ on $\hat{R}^\nu$ can be uniformized after a toric modification which is sufficiently far in the projective system, but occurs after finitely many steps.
The key fact now is that $\hat{R}^\nu$ is nœtherian. The images by $\hat{c}$ of the u_i generate the maximal ideal of $\hat{R}^\nu$, and therefore there exists a finite subset $G \subset I$, which we may assume to consist of $\{1, 2, \dots, q\}$, such that the images of $(u_1, \dots, u_q)$ generate the maximal ideal of $\hat{R}^\nu$. This means that for $j \notin \{1, 2, \dots, q\}$, we have a relation

$$u_j - \sum_{k=1}^{q} R_k u_k \in \overline{(u^n - \lambda_{mn} u^m + \sum_p c_p^{(mn)}(v^\phi) u^p)}$$

in

$$k[\widehat{v^{\Phi+}][(u_i)}_{i \in I}],$$

and this implies that one of the equations $u^n - \lambda_{mn} u^m + \sum_p c_p^{(mn)}(v^\phi) u^p$ must contain u_j linearly (compare with example 5.4). Moreover, according to Remark 6.1, this u_j cannot appear in the binomial term of F_{mn}. By an adaptation of [B1], Chap. IV, using the fact that $\hat{R}^\nu$ is henselian, we may solve this equation for u_j in $\hat{R}^\nu$. So finally, in order to uniformize the valuation $\hat{\nu}$ on $\hat{R}^\nu$ it should suffice to resolve the singularities of a finite number of the equations F_{mn}, so that we may consider only finitely many elements of the projective system of toric blowing-ups. To check this, we must now compute the effect on $\hat{R}^\nu$ of the toric modifications associated to our regular fan.

Let

$$\Gamma = \langle \gamma_1, \gamma_2, \dots, \gamma_i, \dots \rangle \subset \Phi_+$$

be the semigroup of the values taken on $R \setminus \{0\}$ or $\hat{R}^\nu \setminus \{0\}$ by the valuation ν (resp. $\hat{\nu}$). For each $i \geq 1$, set $\Gamma_i = \langle \gamma_1, \gamma_2, \dots, \gamma_i \rangle$. There is a smallest integer p_0 such that the rational rank of the group generated by Γ_{p_0} is equal to the rational rank of Φ. This means that for each $i > p_0$ we have an expression $t_i \gamma_i = \sum_{1 \leq j \leq p_0} s_j^{(i)} \gamma_j$ with $s_j^{(i)} \in \mathbf{Z}$. However, it is much more convenient for us to have such an expression with non negative coefficients. We can state this as

Proposition 7.1. *For each $i > p_0$ there is a relation with non negative integral coefficients*

$$t_i \gamma_i + \sum_{1 \leq k \leq p_0} t_k^{(i)} \gamma_k = \sum_{1 \leq j \leq p_0} s_j^{(i)} \gamma_j.$$

Moreover, each relation between the γ_j has a multiple which is equivalent, modulo these relations, to a relation between the γ_j, $1 \leq j \leq p_0$.

Proof. we must check the second statement; let $\sum_r n_r \gamma_r = \sum_p m_p \gamma_p$ be a relation with non negative coefficients. Let s be the largest index of a γ_i appearing in the relation, and assume it appears in the left side of the relation. If $s > p_0$, we have a relation $t_s\gamma_s + \sum_{1\leq k\leq p_0} t_k^{(i)}\gamma_k \in \Gamma_{p_0}$ and after multiplying the original relation by t_s and adding $n_s(\sum_{1\leq k\leq p_0} t_k^{(i)}\gamma_k)$ to both sides, we can make γ_s disappear. ◌

It is often convenient to remember only the fact that for $j > p_0$ some multiple of γ_j to which we add a sum of the form $\sum_{1\leq k\leq p_0} t_k^{(j)}\gamma_k$ is in Γ_{j-1}; choosing the smallest such multiple, we will then write

$$n_j\gamma_j + \sum_{1\leq k\leq p_0} t_k^{(j)}\gamma_k = \sum_{1\leq k\leq j-1} \ell_k^{(i)}\gamma_k$$

The coefficients of each such relation have no common divisor.

With the notations of subsection 7.1, let us choose h such that $N(h)$ is greater than both the number q given by the nœtherianity of $\hat{R}^\nu$ and p_0. Remark that among the binomial relations defining $\mathrm{gr}_\nu R$ we have, for $j \geq N = N(h)$

$$(B_{j+1}) \quad U_j^{n_j} U_1^{t_1^{(j)}} \ldots U_{p_0}^{t_{p_0}^{(j)}} - \lambda_j U_1^{\ell_1^{(j)}} \ldots U_{j-1}^{\ell_{j-1}^{(j)}} = 0 \quad \text{in } \mathrm{gr}_\nu R \ \text{ with } \lambda_j \in k^*.$$

Let us denote by $\mathbf{B}$ the binomial equations between $(U_1, \ldots, U_N)$ which define, since $N = N(h)$, the graded algebra $\mathrm{gr}_\nu^{(h)} R \subset \mathrm{gr}_\nu R$. Let $\mathcal{L}_0$ be the lattice of exponents of $\mathbf{B}$. By [E-S], since $\mathbf{B}$ is prime, for any $j > N(h)$, the lattice $\mathcal{L}_0$ is a direct factor of the lattice of exponents of the $(U_i)_{1\leq i\leq j}$. The ideal $\mathcal{B}$ in $k[(U_i)_{i\in I}]$ which defines $\mathrm{gr}_\nu R$ contains the ideal $\mathcal{B}'$ generated by $\big(\mathbf{B}, (B_{j+1})_{j\geq N+1}\big)$.

Proposition 7.2. ** The ideals $\mathcal{B}$ and $\mathcal{B}'$ are equal.**

Proof. The basic facts are that $\mathcal{B}$ and $\mathbf{B}$ are prime ideals, and that the translation in terms of binomials the argument given above to show the second part of Proposition 7.1, shows that for $U^m - \lambda_{mn}U^n \in \mathcal{B}$, there exist an integer t and a monomial U^q in $(U_1, \ldots, U_{p_0})$ such that

$$U^q(U^{tm} - \lambda_{mn}^t U^{tn}) \in \mathcal{B}'.$$

Since $\mathbf{B}$ is a prime ideal, this implies that we have $U^{tm} - \lambda_{mn}^t U^{tn} \in \mathcal{B}'$. Now let us show that the lattice $\mathcal{L}'$ of exponents of $\mathcal{B}'$ is saturated in the sense of [E-S], which means that if for some integral point m we have $dm \in \mathcal{L}'$ for some integer d, then $m \in \mathcal{L}'$. To prove that $\mathcal{L}'$ is saturated, notice that it is obtained from $\mathcal{L}_0$ by adding, one at a time, primitive vectors of the form $m + pv$ where m is an integral point in the vector space generated by $\mathcal{L}_0$ and v is a unit vector corresponding to the new variable u_i.
If for some integer d we have $dm = m' + b(m + pv)$ with $m' \in \mathcal{L}_0$, because $\mathcal{L}_0$ is a direct factor and saturated ([E-S]), there is $m_0 \in \mathcal{L}_0$ such that $d(m - m_0) = b(m + pv)$; since $m + pv$ is primitive, this implies $m \in \mathcal{L}_0 \oplus \mathbf{Z}(m + pv)$, which is therefore saturated. From this one deduces that $\mathcal{L}'$ is saturated, and again by the arguments of [E-S], that $\mathcal{L}' = \mathcal{L}$.

Note also that if there are two linear integral relations between γ_i and the $(\gamma_j)_{1\leq j\leq N}$, this saturation implies that they must be equivalent modulo the relations between the $(\gamma_j)_{1\leq j\leq N}$. ◌

From now on, we take $\mathcal{B}'$ as system of generators for the kernel of the map

$$k[(U_i)_{i\in I}] \to \mathrm{gr}_\nu R.$$

Turning now to $\hat{R}^\nu$, we note that the result just proved gives us a natural ordering on the set of equations F_{mn} such that u_j appears in the initial binomial part, for $j > N(h) \geq N_0 = \sup(q, p_0)$; they can be written $(F_{j+1})_{j>N(h)}$.
Now we use the fact that u_j belongs in $\hat{R}^\nu$ to the maximal ideal, generated by $(u_1, \dots, u_N)$. By the Proposition, any relation such that u_j appears for the first time in the initial (binomial) part is one of the relations

$$F_{j+1} = u_j^{n_j} u_1^{t_1^{(j)}} \dots u_{p_0}^{t_{p_0}^{(j)}} - \lambda_j u_1^{\ell_1^{(j)}} \dots u_{j-1}^{\ell_{j-1}^{(j)}} + \sum_p c_p^{(j)}(v^\phi)u^p = 0,$$

with the condition that the weight of each u^p must be greater than the weight of $u_j^{n_j} u_1^{t_1^{(j)}} \dots u_{p_0}^{t_{p_0}^{(j)}}$, which is equal to that of $u_1^{\ell_1^{(j)}} \dots u_{j-1}^{\ell_{j-1}^{(j)}}$, so that this relation is the last one where u_{j+1} can appear. By induction on $j > N_0$, using the fact that $\hat{R}^\nu$ is henselian, we see that u_{j+1} must appear linearly in this relation, although it may appear also before. Argue as follows, with $N = N(h) \geq N_0$; in the equations F_{mn} whose initial forms depends on $(U_1, \dots, U_N)$ only, we may eliminate by the henselian property of $\hat{R}^\nu$ the variables u_k for $k > N$ since we know that they can be expressed, thanks to the other equations, in terms of $(u_1, \dots, u_N)$. The first equation containing U_{N+1} in its initial form then must contain u_{N+2} linearly, since u_{N+2} must appear linearly somewhere and cannot appear in any of the other equations containing u_j, $j > N+1$ in their initial form, since it may appear only in equations whose initial form is of v-degree less than γ_{N+2}. To summarize

Proposition 7.3. **For $j > N_0$, the equation F_{j+1} must contain a term of the form $c(v^\phi)u_{j+1}$, where $c(v^\phi)$ is a term in the v^ϕ and therefore does not vanish outside of the union of the hyperplanes $(v^\phi = 0)_{\phi\in\Phi_+}$; equivalently, with the notations of subsection 5.2, Proposition 5.10, $c(v^\phi)$ is invertible in $\widehat{\mathcal{A}_\nu(R)}\{(v^{\Phi_+})^{-1}\}$.**

Keeping the notations introduced at the beginning of this subsection, we have:

Corollary. *Let us denote by* $\mathbf{F}$ *the finite set consisting of those equations among the F_{mn} whose initial forms depend only on $(U_1, \dots, U_N)$, that is, belong to* $\mathbf{B}$*. Given an homomorphism $c\colon \Phi \to k^*$, the kernel of the map*

$$\hat{c}\colon k[\widehat{(u_i)_{i\in I}}] \to \hat{R}^\nu$$

is generated, up to closure, by the ideal $\big(c(\mathbf{F}), c(F_{j+1})_{j\geq N}\big)$.

This is a fundamental fact: all but finitely many of the equations defining $\hat{R}^\nu$ serve only to express u_{j+1}, for $j \geq N_0$, in terms of $(u_1, \ldots, u_j)$ and so ultimately in terms of $(u_1, \ldots, u_{N_0})$.

Corollary. *For $j > N_0$, we have the inequality*

$$\gamma_{j+1} > n_j \gamma_j .$$

Corollary. **For each $j > N_0$, the image in $\hat{R}^\nu$ of the element u_{j+1} can be written as a series in the images of $(u_1, \ldots, u_{N_0})$; such a series, viewed in $k\widehat{[(u_i)_{i\in I}]}$, may be called a j-th approximate root for $\hat{R}^\nu$. It also has in $\hat{R}^\nu$ an expression as a series in $(u_1, \ldots, u_j)$*

$$u_{j+1} = d_{j+1}\big(u_j^{n_j} u_1^{t_1^{(j)}} \ldots u_{p_0}^{t_{p_0}^{(j)}} - \lambda_j u_1^{\ell_1^{(j)}} \ldots u_{j-1}^{\ell_{j-1}^{(j)}} + \sum_p d_p^{(j)} u^p\big),$$

*with $d_{j+1} \in k^*$, obtained by solving for the $(u_{j+1})_{j\geq N_0+1}$ the equations $\big((c(F_k) = 0)_{k\geq N_0+1}\big)$.**

Proof. This requires an infinite-dimensional implicit function theorem in the spirit of ([B1], Chap. IV) to use the fact that all the $\partial_{u_{j+1}} c(F_{j+1})$ are invertible in $k\widehat{[(u_i)_{i\in I}]}$. ⟲

This suggests a way to generalizes the theory of approximate roots. From a valuation theoretic viewpoint the idea appears already in the "key polynomials" of MacLane (see [McL 1], [McL 2]), but in a different guise it was developped by Abhyankar-Moh, (see [A-M1], [A-M 2]), Cossart-Moreno (in preparation), Lejeune-Jalabert, (see [L]) (for curves, the idea being that of branches having "critical" contact with a given plane curve singularity), and by Spivakovsky (in [S1] for surfaces, and in [S2] for valuations of rank one).

Let us now consider the binomial ideal in $k[u_1, \ldots, u_N]$ which is the kernel of the map to $\mathrm{gr}_\nu^{(h)} R$ sending u_i to $\overline{\xi}_i$. It is a prime ideal, so that it corresponds to a torus orbit closure since our residue field is algebraically closed ([E-S]). We apply to it Proposition 6.6, and obtain a regular fan Σ of $\mathbf{R}_+^N$ compatible with the hyperplanes dual to the $m - n$ where $(U^m - \lambda_{mn} U^n)_{mn}$ is a system of generators of the binomial ideal. We may choose a simplex $\sigma \in \Sigma$ whose intersection with the space of weights $W \subset \mathbf{R}_+^N$ is of the largest possible dimension, that is, the rational rank r of Φ. This corresponds to a monomial map

$$u_i \mapsto y_1^{a_i^1} \ldots y_N^{a_i^N}, \quad 1 \leq i \leq N,$$

which we may extend by setting

$$u_k = y_k, \quad k > N.$$

Now we compute the transforms of the equations of $\hat{R}^\nu$ under this transformation. By construction, using the same argument as we used in the case where the graded algebra is finitely generated, the first equations F_{mn}, those whose initial forms involve only $(U_1, \ldots, U_N)$, become

$$\begin{aligned} y^{\langle a, m^1 - n^1 \rangle}(y^{\langle a, n^1 \rangle} - \lambda_{m^1 n^1} + \cdots) &= 0 \\ \vdots \qquad\qquad &\ \ \vdots \\ y^{\langle a, m^t - n^t \rangle}(y^{\langle a, n^t \rangle} - \lambda_{m^t n^t} + \cdots) &= 0, \end{aligned}$$

while the next ones become

$$\begin{aligned} u_{N+1}^{n_{N+1}} y^{t(N+1)} - \lambda_{N+1} u_1^{\ell_1^{(N+1)}} \ldots u_N^{\ell_N^{(N+1)}} + \cdots + c^{(N+1)}(v^\phi) u_{N+2} + \cdots &= 0 \\ u_{N+2}^{n_{N+2}} y^{t(N+2)} - \lambda_{N+2} u_1^{\ell_1^{(N+2)}} \ldots u_{N+1}^{\ell_{N+1}^{(N+2)}} + \cdots + c^{(N+2)}(v^\phi) u_{N+3} + \cdots &= 0 \\ \vdots \qquad\qquad &\ \ \vdots \\ u_{N+k}^{n_{N+k}} y^{t(N+k)} - \lambda_{N+k} u_1^{\ell_1^{(N+k)}} \ldots u_{N+k-1}^{\ell_{N+k-1}^{(N+k)}} + \cdots + c^{(N+k)}(v^\phi) u_{N+k+1} + \cdots &= 0 \\ \vdots \qquad\qquad &\ \ \vdots \end{aligned}$$

where $y^{t(N+k)}$ is a monomial which is the transform of the monomial in the variables $(u_1, \ldots, u_N)$ which affected u_{N+k} in the original relations. Note that one can compute the valuations of the y_k from those of the u_j. Some of the y_k, say for $k \in T$, will have valuation zero, so that for $k \in T$ there is a constant $c_k \in k_R^*$ such that $y_k - c_k$ has positive valuation; the equations $(y_k - c_k)_{k \in T}, (y_k)_{k \notin T}, (u_j)_{j > N}$ define the closed point which is the center of the valuation in the space obtained by the toric modification. By openness of transversality, if our toric map has resolved the $\mathrm{gr}_\nu^{(h)} R$, the first set of equations defines a regular ν-complete local ring at that point. By the nature of the equations, it will be the case for the whole set of equations and the strict transform of $\hat{R}^\nu$ will be regular.

Remark. If one compares with the case where $\mathrm{gr}_\nu R$ is finitely generated, as in the example of plane branches, one see that the effect of the "abyssal" phenomenon is also to send singularities away to infinity: if we stopped at any finite step, replacing u_{j+1} by 0 or a polynomial in $(u_1, \ldots, u_j)$ of higher weight than the binomial initial form, the new equation thus obtained from F_{j+1} would be singular.

Note that there are at least $\mathrm{r}(\nu)$ independent variables, and possibly more if $\dim \hat{R}^\nu > \mathrm{r}(\nu)$, because after substitution of the expression for the u_j, $j > N$, some of the equations may become identities.

At this point we have proved local uniformization by a finite dimensional toric map for the rational valuation $\hat{\nu}$ on $\hat{R}^\nu$. According to what we saw at the beginning of this section, the toric map will also uniformize the valuations $\hat{\nu}_j$ with which ν is composed.

To deduce from this local uniformization for R, the fact that R is excellent is crucial. Since the elements ξ_i which we used to generate topologically the k-algebra

$\hat{R}^{\nu}$ are in R, we may perform in R the trace of the toric modification described above; it consists in adding to R the monomials (with some negative exponents) in the ξ_i which are the y_i and localizing at the point specified by the values of the y_i. If we denote by $\hat{R'}^{\nu}$ the transform of $\hat{R}^{\nu}$ localized at the point picked by $\hat{\nu}$, this gives us a map $R' \to \hat{R'}^{\nu}$ where R' birationally dominates R and is contained in the valuation ring R_{ν}.

Proposition 7.4. ** This map is the ν-adic completion of R'.**

Let $\hat{R}^{\nu}$ be the completion of a nœtherian analytically irreducible excellent local ring. We know by 5.1 that $\hat{R}^{\nu}$ is a quotient of the completion $\hat{R}^{\mathbf{P}}$ by an ideal H. As remarked by Spivakovsky (Proposition 5.7), the assumption that R is excellent implies that the localization $(\hat{R}^{\mathbf{P}})_H$ is regular. If we have equality in Abhyankar's inequality, the map $\hat{R}^{\mathbf{P}} \to \hat{R}^{\nu}$ is an isomorphism and so is the map $\hat{R'}^{\mathbf{P}'} \to \hat{R'}^{\nu}$. Let m' be the maximal ideal of R'. By [EGA], Proposition 7.4.6, [B3], Chap. III, §3, No.4, we know that, denoting by $\hat{m}'$ the maximal ideal $m'\hat{R'}^{\mathbf{P}'}$ of $\hat{R'}^{\mathbf{P}'}$, the m'-adic completion $\hat{R}'$ of R' is isomorphic to the completion of the local ring $(\hat{R'}^{\mathbf{P}'})_{m'}$. This last ring is regular because $\hat{R'}^{\mathbf{P}'}$ is, so $\hat{R}'$ is regular, and since R' is excellent as a finitely generated R-algebra, it is regular.
For the general case I use the method of Spivakovsky ([S 2],1997, Lemma 6.4). We are reduced to the case where R is such that $\hat{R}^{\mathbf{P}}_H$ and $\hat{R}^{\mathbf{P}}/H$ are both regular, the first one by excellence and the second one because we have uniformized $\hat{\nu}$ on $\hat{R}^{\nu}$. We want to show that we can make $\hat{R}^{\mathbf{P}}$ (i.e., its strict transform) non singular by a toric modification defined in R. Then by excellence R itself will be made non singular. Let $(w_1, \ldots, w_n)$ be a system of generators of the maximal ideal $\hat{m}$ of $\hat{R}^{\mathbf{P}}$, such that $(w_1, \ldots, w_q)$ form a minimal system of generators of H and $(w_{q+1}, \ldots, w_n)$ induce a regular system of parameters of $\hat{R}^{\mathbf{P}}/H$. Let h be the height of the prime ideal H. If $h = q$, then $\hat{R}^{\mathbf{P}}$ is regular and we are done; since R is excellent, it has to be regular (see [EGA]). If $h < q$, there is a relation

$$\sum_{i=1}^{q} a_i w_i \in H^2, \quad \text{with the } \; a_i \text{ not all in } \; H.$$

If we consider the images $\tilde{a}_i$ in $\hat{R}^{\nu}$ of the a_i, by refining the method shown here, we should be able to show that after a toric modification we can obtain that the $\tilde{a}_i$ with the smallest valuation, say $\tilde{a}_1$, divides all the others. Then we can express $w_1 = \sum_{j=2}^{q} \beta_j w_j$ mod. H^2 in $\hat{R}^{\mathbf{P}}$, and we have reduced q by one. After finitely many steps we reach the situation where $q = h$ and we are done.

There will remain to translate the trace of the toric modification on the completion $\hat{R}$ (or $\hat{R}^{\nu}$) into a finite sequence of blowing ups with non singular centers coming from R. But one can prove a torus-equivariant version of Gruson-Raynaud's ([G-R]) and Hironaka's ([H]) proper flattening theorem and use it to show that a toric modification is dominated by a sequence of toric blowing ups (called *monomial blowing ups* in [S]). In fact a more general result is proved in [DC-P]; see also [O], Chap. 1, §1.7, p.39. We will also have to show that the centers of these blowing-ups are contained in the singular locus of Spec$R$, whereas all we know so far is that these centers are contained in the union of the coordinate hyperplanes (and their transforms), and that this union contains the singular locus of Spec$\hat{R}^\nu$ (see Proposition 6.4).

A last example. An example where $\mathrm{gr}_\nu R$ is finitely generated is obtained by taking the first example after Proposition 5.2. The completion $\hat{R}^\nu$ is the ring of a branch, the associated graded ring is that of the corresponding monomial curve which may be desingularized as in subsections 5.4 and 6.4 by a toric map. The transform R' of R by this map has the property that it is a two-dimensional local ring with maximal ideal m' such that the maximal ideal of $\hat{R}'$ contains an element f' such that $\hat{R}'/(f')$ is a regular one-dimensional local ring and $(\hat{R}')_{(f')}$ is regular. So $\hat{R}'$ is regular and, since R' is excellent as an R-algebra essentially of finite type, so is R. Essentially the same argument will work whenever $\dim\hat{R}^\nu = 1$, since once the singularity of this ring is resolved, we deal with a discrete valuation ring, where the argument to bring the number of generators of H down to its height presents no difficulty.

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