ON THE DIMENSION OF DISCRETE VALUATIONS OF $k((X_1, \ldots, X_n))$

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Abstract

Let v be a rank-one discrete valuation of the field $k((X_1, \ldots, X_n))$. We know, after [1], that if $n = 2$ then the dimension of v is 1 and if v is the usual order function over $k((X_1, \ldots, X_n))$ its dimension is $n - 1$. In this paper we prove that, in the general case, the dimension of a rank-one discrete valuation can be any number between 1 and $n - 1$.

1 TERMINOLOGY AND PRELIMINARIES

Let k be an algebraically closed field of characteristic 0, $R_n = k[\![X_1, \ldots, X_n]\!]$, $M_n = (X_1, \ldots, X_n)$ the maximal ideal and $K_n = k((X_1, \ldots, X_n))$ the quotient field. Let v be a rank-one discrete valuation of $K_n|k$, R_v its valuation ring, $\mathfrak{m}_v$ its maximal ideal and Δ_v its residual field of v. The center of v in R_n is $\mathfrak{m}_v \cap R_n$. Throughout this paper "discrete valuation of $K_n|k$" will mean "rank-one discrete valuation of $K_n|k$ whose center in R_n be the maximal ideal M_n". The dimension of v is the transcendence degree of Δ_v over k. We shall suppose that the group of v is $\mathbb{Z}$.

Let $\widehat{K}_n$ be the completion of K_n with respect to v (see [2]), $\widehat{v}$ the extension of v to $\widehat{K}_n$, $R_{\widehat{v}}$, $\mathfrak{m}_{\widehat{v}}$ and $\Delta_{\widehat{v}}$ the ring, maximal ideal and the residual field of $\widehat{v}$, respectively.

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2 THE DIMENSION OF v

Fix a number m between 1 and $n-1$. We are featuring a constructive method in order to obtain examples of valuations with dimension m.

Let us consider the following homomorphism:

$$\begin{array}{rrcl} \varphi: & k[\![X_1,\dots,X_n]\!] & \longrightarrow & \overline{k(u)}[\![t]\!] \\ & X_1 & \longmapsto & t \\ & X_2 & \longmapsto & ut \\ & X_i & \longmapsto & \sum_{j\geq 1} u^{1/p_i^j} t^j \end{array}$$

where $\overline{k(u)}$ stands for the algebraic closure of $k(u)$ and $2 < p_3 < \ldots < p_n$ are prime numbers.

Lemma 1 *The homomorphism φ is one to one.*

PROOF: Let us take the fields $K_2 = k(u)$ and

$$K_i = k(u, \{u^{1/p_3^j}, j \geq 1\}, \dots, \{u^{1/p_i^j}, j \geq 1\})$$

for all $i \geq 3$.

Let us suppose that φ is not one to one, then $\ker(\varphi) \neq \{0\}$. So let f be a non-zero element of $M = (X_1, \dots, X_n)$ such that $f \in \ker(\varphi)$. Let m be the higher index such that $f \in k[\![X_1, \dots, X_m]\!]$.

If $\underline{m = 1 \text{ or } 2}$, trivially we have a contradiction.

If $\underline{m = 3}$, let us take

$$\overline{f} = f(\varphi(X_1), \varphi(X_2), X_3) \in K_2[\![t, X_3]\!],$$

and consider the homomorphism

$$\begin{array}{rrcl} \psi: & K_2[\![t,X_3]\!] & \longrightarrow & \overline{k(u)}[\![t]\!] \\ & t & \longmapsto & t \\ & X_3 & \longmapsto & \sum_{j\geq 1} u^{1/p_3^j} t^j. \end{array}$$

We know that $\overline{f} \in \ker(\psi)$ and this kernel is a prime ideal because ψ is an homomorphism between integral domains. We can write $\overline{f} = t^r g$, with $r \geq 0$ and t doesn't divide to g. This forces g to have some non-trivial terms in X_3. Let $s > 0$ be the minimum such that αX_3^s is one of these terms. By the Weierstrass preparation theorem we have $g = Ug'$, where $U(t, X_3)$ is a unit and

$$g' = X_3^s + a_1(t) X_3^{s-1} + \ldots + a_s(t).$$

Since U is a unit, $g' \in \ker(\psi)$ and

$$\psi(g') = g'\left(t, \sum_{j\geq 1} u^{1/p_3^j} t^j\right) = 0.$$

This leads to a contradiction because the roots of g' are in $K_2[\![t^{1/q}]\!]$, with $q \in \mathbb{Z}$, by the Puiseux theorem.

If $\underline{m > 3}$ let us take

$$\overline{f} = f(\varphi(X_1), \ldots, \varphi(X_{m-1}), X_m) \in K_{m-1}[\![t, X_m]\!]$$

and consider the homomorphism

$$\begin{array}{rrcl} \psi: & K_{m-1}[\![t, X_m]\!] & \longrightarrow & \overline{k(u)}[\![t]\!] \\ & t & \longmapsto & t \\ & X_m & \longmapsto & \sum_{j \geq 1} u^{1/p_m^j} t^j. \end{array}$$

As in the previous case we can write $\overline{f} = t^r h$, where $h \in \ker(\psi)$. So we have $h = Uh'$, where $U(t, X_m)$ is a unit and

$$h' = X_m^r + b_1(t) X_m^{r-1} + \ldots + b_r(t) \in \ker(\psi),$$

so

$$h'\left(t, \sum_{j \geq 1} u^{1/p_m^j} t^j\right) = 0.$$

But this is again a contradiction by the Puiseux theorem: since $\ker(\psi)$ is a prime ideal, we can suppose that h' is an irreducible element of the ring $K_{m-1}[\![t]\!][X_m]$. In this situation the Puiseux theorem says that to obtain the coefficients of a root of $h' = 0$, like a Puiseux series in t with coefficients in $k(u)$, we have to resolve *a finite number* of algebraic equations of degree greater than 1 in K_{m-1}. Inside K_{m-1} we can not obtain u^{1/p_m} and, with a finite number of algebraic equations, we can obtain *a finite number* of powers of u^{1/p_m^j} but not all. So this proves the lemma. □

We shall extend to the quotient fields this injective homomorphism for giving an example of a rank-one discrete valuation of $k((X_1, \ldots, X_n))$ of dimension 1.

Lemma 2 *There exists a rank-one discrete valuation of $k((X_1, \ldots, X_n))$ of dimension 1.*

Proof: We know that the homomorphism

$$\varphi : k[\![X_1, \ldots, X_n]\!] \to \overline{k(u)}[\![t]\!]$$

previously defined is one to one. So we can take the valuation $v = \nu \circ \varphi$, where ν is the usual order function over $\overline{K(u)}((t))$ in t and φ is the natural extension to the quotient fields. Let α be the residue $X_2/X_1 + \mathfrak{m}_v \in \Delta_v$. Hence, to obtain the lemma we have to prove that $\alpha \notin k$ and Δ_v is an algebraic extension of $k(\alpha)$.

Let us suppose that $\alpha \in k$. Then there must exist $a \in k$ such that $X_2/X_1 + \mathfrak{m}_v = a + \mathfrak{m}_v$, so

$$\frac{X_2 - aX_1}{X_1} \in \mathfrak{m}_v.$$

This means that $v(X_2 - aX_1) > 1$. On the other side we have

$$\varphi(X_2 - aX_1) = (u - a)t,$$

so $v(X_2 - aX_1) = 1$ and we have a contradiction. Hence $\alpha \notin k$.

Let us prove that Δ_v is an algebraic extension of $k(\alpha)$. We can consider each element of $k[\![X_1, \ldots, X_n]\!]$ like a sum of forms with respect to the usual degree. If f_r is a form of degree r, then $\varphi(f_r) = t^r P$, with P a polynomial in u and a finite number of elements u^{1/p_i}.

Let us take $f,\ g \in k[\![X_1, \ldots, X_n]\!]$ such that $g \neq 0$ and $v(f/g) = 0$. Then $\varphi(f/g) = h_0 + th_1$, where h_0 is a rational fraction in u and a finite number of elements u^{1/p_i}. So h_0 is algebraic over $k(u)$. Let us consider

$$P(u, Z) = c_0(u)Z^m + c_1(u)Z^{m-1} + \ldots + c_{m-1}(u)Z + c_m(u) \in k[u][Z]$$

a polynomial satisfied by h_0, where $c_i(u) \in k[u]$ for all i and $c_0 \neq 0$. Let β be the element

$$\beta = P\left(\frac{X_2}{X_1}, \frac{f}{g}\right) = c_0\left(\frac{X_2}{X_1}\right)\left(\frac{f}{g}\right)^m + \ldots + c_m\left(\frac{X_2}{X_1}\right).$$

Then we have

$$\varphi(\beta) = c_0(u)(h_0 + th_1)^m + \ldots + c_m(u),$$

so $v(\beta) = \nu \circ \varphi(\beta) > 0$ and $\beta \in \mathfrak{m}_v$. Subsequently,

$$0 + \mathfrak{m}_v = \beta + \mathfrak{m}_v = P\left(\alpha, \frac{f}{g} + \mathfrak{m}_v\right).$$

This proves that $f/g + \mathfrak{m}_v$ is an algebraic element over $k(\alpha)$ and, a fortiori, the lemma. □

Lemma 3 *The dimension of a rank-one discrete valuation of $k((X_1, \ldots, X_n))$ is between 1 and $n - 1$.*

PROOF: We know, after [1], that the dimension of a rank-one discrete valuation of $k((X_1, \ldots, X_n))$ is minor or equal than $n - 1$. So we have to prove that there exists a transcendental residue in Δ_v.

Let us suppose that $v(X_i) = n_i$ for all $i = 1, \ldots, n$. Then the value of $X_2^{n_1}/X_1^{n_2}$ is zero, so $0 \neq (X_2^{n_1}/X_1^{n_2}) + \mathfrak{m}_v \in \Delta_v$. If this residue lies in k then there exists $a_{21} \in k$ such that

$$\frac{X_2^{n_1}}{X_1^{n_2}} + \mathfrak{m}_v = a_{21} + \mathfrak{m}_v.$$

This implies

$$\frac{X_2^{n_1}}{X_1^{n_2}} - a_{21} = \frac{X_2^{n_1} - a_{21}X_1^{n_2}}{X_1^{n_2}} \in \mathfrak{m}_v,$$

and then

$$v\left(\frac{X_2^{n_1}-a_{21}X_1^{n_2}}{X_1^{n_2}}\right)>0.$$

So we have $v(X_2^{n_1}-a_{21}X_1^{n_2})=m_1>n_1n_2$. Then

$$v\left(\frac{(X_2^{n_1}-a_{21}X_1^{n_2})^{n_1}}{X_1^{m_1}}\right)=0.$$

If the residue of this element lies too in k, then there must exist $a_{22}\in k$ such that $v((X_2^{n_1}-a_{21}X_1^{n_2})^{n_1}-a_{22}X_1^{m_1})=m_2>n_1m_1$. We can repeat this operation.

The previous procedure is finite: if it didn't stop we would construct the power series

$$X_2^{n_1}-\sum_{i=1}^{\infty}b_{2i}X_1^i$$

such that the sequence of partial sums has increasing values. Since $\widehat{K_n}$ is a complete field, then this series amounts to zero in contradiction with X_1 and X_2 being formally independent. So the procedure must stop and there exists a transcendental element over k in Δ_v. □

Theorem 4 *Let m be a fixed number between 1 and $n-1$, then there exists a rank-one discrete valuation of $k((X_1,\ldots,X_n))$ of dimension m.*

PROOF: Let us consider the one to one (the proof of injectivity parallels that of lemma 1) homomorphism

$$\begin{array}{rrcl}
\varphi & k[\![X_1,\ldots,X_n]\!] & \longrightarrow & \overline{k(u)}[\![t_1,\ldots,t_m]\!] \\
 & X_1 & \longmapsto & t_1 \\
 & X_2 & \longmapsto & ut_1 \\
 & X_i & \longmapsto & \left\{\begin{array}{l} t_i \text{ if } i\le m+1 \\ \sum_{j\ge 1} u^{1/p_i^j}t_1^j \text{ if } i>m+1. \end{array}\right.
\end{array}$$

We can take the valuation $v:=\nu\circ\varphi$, with ν the usual order function in $\overline{k(u)}[\![t_1,\ldots,t_m]\!]$ and φ the natural extension to the quotient fields. We know (lemma 2) that the residue $X_2/X_1+\mathfrak{m}_v$ is transcendental over k. Trivially the residue $X_i/X_1+\mathfrak{m}_v$ for all $i=3,\ldots,m+1$ are transcendental over $k(X_2/X_1+\mathfrak{m}_v,\ldots,X_{i-1}+\mathfrak{m}_v)$ because t_i are formally independent variables. Any element $f/g+\mathfrak{m}_v\in\Delta_v$ is algebraic over $k(X_2/X_1+\mathfrak{m}_v,\ldots,X_{m+1}+\mathfrak{m}_v)$ parallels that of lemma 2. So the dimension of v is m. □

References

[1] E. Briales, *Constructive theory of valuations.*, Comm. Algebra **17** (1989), no. 5, 1161–1177.

[2] J. P. Serre, *Corps locaux.*, Hermann, Paris, 1968.