EXPLICIT CONSTRUCTIONS IN DISCRETE VALUATIONS

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Abstract. In this paper we give an explicit construction of a system of parametric equations describing a discrete valuation over $k((X_1,\dots,X_n))$. This amounts to finding a parameter and a field of coefficients. We devote section 1 to finding an element of value 1, that is, the parameter. The field of coefficients is the residue field of the valuation, and it is explicitly constructed in section 2. As a consequence of this construction, at the end of this section, we prove that $k((X_1,\dots,X_n))$ can be embedded into a field $L((Y_1,\dots,Y_n))$, where the *"extended valuation" is as close as posible to the usual order function.*

Terminology and preliminaries

Let k be an algebraically closed field of characteristic 0, $R_n = k[\![X_1,\dots,X_n]\!]$, $M_n = (X_1,\dots,X_n)$ its maximal ideal and $K_n = k((X_1,\dots,X_n))$ its quotient field. Let v be a rank-one discrete valuation of $K_n|k$, R_v the valuation ring, $\mathfrak{m}_v$ the maximal ideal and Δ_v the residue field of v. The center of v in R_n is $\mathfrak{m}_v \cap R_n$. Throughout this paper "discrete valuation of $K_n|k$" means "rank-one discrete valuation of $K_n|k$ whose center in R_n is the maximal ideal M_n". The dimension of v is the transcendence degree of Δ_v over k. We shall assume that the group of v is $\mathbb{Z}$.

Let $\widehat{K}_n$ be the completion of K_n with respect to v, $\widehat{v}$ the extension of v to $\widehat{K}_n$, $R_{\widehat{v}}$, $\mathfrak{m}_{\widehat{v}}$ and $\Delta_{\widehat{v}}$ the ring, maximal ideal and the residue field of $\widehat{v}$, respectively (see [5]). We know that Δ_v and $\Delta_{\widehat{v}}$ are isomorphic ([1]). Let $\sigma : \Delta_{\widehat{v}} \to R_{\widehat{v}}$ be a $k-$section of the natural homomorphism $R_{\widehat{v}} \to \Delta_{\widehat{v}}$, $\theta \in R_{\widehat{v}}$ an element of value 1 and t an indeterminate. We consider the $k-$isomorphism

$$\Phi = \Phi_{\sigma,\theta} : \Delta_{\widehat{v}}[\![t]\!] \to R_{\widehat{v}}$$

given by

$$\Phi\left(\sum \alpha_i t^i\right) = \sum \sigma(\alpha_i)\theta^i,$$

and denote also by Φ its extension to the quotient fields. We have a $k-$isomorphism Φ^{-1} which, when composed with the usual order function on $\Delta_{\widehat{v}}((t))$, gives the valuation $\widehat{v}$. This is the situation we shall consider throughout this paper, and we will freely use it without new explicit references.

We shall use two basic transformations in order to find an element of value 1 and construct the residue field:

1. *Monoidal transformation:*

$$\begin{array}{rcl} k[\![X_1,\dots,X_n]\!] & \longrightarrow & k[\![Y_1,\dots,Y_n]\!] \\ X_1 & \longmapsto & Y_1 \\ X_2 & \longmapsto & Y_1Y_2 \\ X_i & \longmapsto & Y_i,\ i=3,\dots,n. \end{array}$$

with $v(X_2) > v(X_1)$.

2. *Change of coordinates:*

$$\begin{array}{rcl} k[\![X_1,\dots,X_n]\!] & \longrightarrow & L[\![Y_1,\dots,Y_n]\!] \\ X_1 & \longmapsto & Y_1 \\ X_i & \longmapsto & Y_i+\sigma(b_i)Y_1,\ i=2,\dots,n, \end{array}$$

   where $L\subset\sigma(\Delta_{\widehat{v}})$ is an extension field of k.

For both transformations we have the following facts:

1. The transformations are one to one: In the case of the monoidal transformations this property is well known. In the other case it is a consequence of [6] (corollary 2, page 137).
2. New variables Y_i lie in $R_{\widehat{v}}$, so we can put $(\Phi_{\sigma,\theta})^{-1}(Y_i)=\sum a_{ij}t^j$.
3. Let $\varphi : K_n\to\Delta_v((t))$ be the restriction of $(\Phi_{\sigma,\theta})^{-1}$ to K_n. Let us denote by $\varphi' : L_n = L((Y_1,\dots,Y_n))\to\Delta_v((t))$ the natural extension of φ to the field L_n. Then $v=\nu_t\circ\varphi'_{|K_n}$, with ν_t the usual order function over $\Delta_v((t))$. Therefore, if φ' is injective we can extend the valuation v to the field L_n and the extension is $v'=\nu_t\circ\varphi'$.

From now on transformation will mean monoidal transformation, change of coordinates, variables interchanges or finite compositions of these.

1. Construction of an element of value 1

Lemma 1.1. *Let $\alpha_i=v(X_i)$ for all $i=1,\dots,n$. By a finite number of monoidal transformations we can find n elements $Y_1,\dots,Y_n\in\widehat{K}_n$ such that $\widehat{v}(Y_i)=\alpha=$ $\gcd(\alpha_i)$ for all $i=1,\dots,n$.*

Proof. We can suppose that $v(X_1)=\alpha_1=\min\{\alpha_i|1\le i\le n\}$ and consider the following two steps:

Step 1.- If there exists $n_i\in\mathbb{Z}$ such that $\alpha_i=n_i\alpha_1$ for all $i=2,\dots,n$, then for each i we apply n_i-1 monoidal transformations

$$\begin{array}{rcl} k[\![X_1,\dots,X_n]\!] & \longrightarrow & k[\![Y_1,\dots,Y_n]\!] \\ X_1 & \longmapsto & Y_1 \\ X_i & \longmapsto & Y_1Y_i \\ X_j & \longmapsto & Y_j,\ j\neq i. \end{array}$$

Trivially $\widehat{v}(Y_i)=\alpha_1$ for all $i=1,\dots,n$.

Step 2.- Assume there exists i, with $2\le i\le n$, such that $v(X_1)=\alpha_1$ does not divide to $v(X_i)=\alpha_i$. We can suppose that $i=2$ with no loss of generality and then $\alpha_2=q\alpha_1+r$. So we apply q times the monoidal transformation

$$\begin{array}{rcl} k[\![X_1,\dots,X_n]\!] & \longrightarrow & k[\![Y_1,\dots,Y_n]\!] \\ X_1 & \longmapsto & Y_1 \\ X_2 & \longmapsto & Y_1Y_2 \\ X_i & \longmapsto & Y_i,\ i=3,\dots,n \end{array}$$

to obtain a new ring $k[\![Y_1,\dots,Y_n]\!]$ where $\widehat{v}(Y_2)=r>0$ and Y_2 is the element of minimum value.

As the values of the variables are greater than zero, in a finite number of steps 2 we come to the situation of step 1. In fact, this algorithm is equivalent to the "euclidean algorithm" to compute the greatest common divisor of $\alpha_1,\dots,\alpha_n$.

$\square$

Theorem 1.2. *We can construct an element of value 1 applying a finite number of monoidal transformations and changes of coordinates.*

Proof. We call $Y_{1r}, \ldots, Y_{nr}$ the elements found after r transformations.

We can suppose that we have applied the previous lemma to obtain $Y_{11}, \ldots, Y_{n1}$ such that $v(Y_{i1}) = \alpha$ for all $i = 1, \ldots, n$. Let us prove that there exists $b_i \in \Delta_{\widehat{v}}$ for each $i = 2, \ldots, n$ such that $\widehat{v}(Y_{i1} - \sigma(b_i)Y_{11}) > \alpha$. We can take

$$(\Phi_{\sigma,\theta})^{-1}(Y_{i1}) = \sum_{j \geq \alpha} a_{ij}t^j = \omega_i(t),\ a_{ij} \in \Delta_{\widehat{v}},\ a_{i\alpha} \neq 0,$$

and so it suffices taking $b_i = a_{i\alpha}/a_{ij}$.

The following two steps defines an algorithm to obtain an element of value 1:

Step 1.- We apply the coordinate change

$$\begin{array}{rcl} k[\![Y_{11}, \ldots, Y_{n1}]\!] & \longrightarrow & L[\![Y_{12}, \ldots, Y_{n2}]\!] \\ Y_{11} & \longmapsto & Y_{12} \\ Y_{i1} & \longmapsto & Y_{i2} + \sigma(b_i)Y_{12},\ i = 2, \ldots, n. \end{array}$$

With this transformation the values of the new variables are not equals to $\widehat{v}(Y_{12})$.

Step 2.- We apply lemma 1.1 to equalize the values of elements and go to step 1. Obviously, the minimum of the values of the elements does not increase, because the greater common divisor of the values does not exceed the minimum of the values. Moreover the first variable does not change.

If we obtain an element of value 1, and we are finished.

We have to show that the algorithm produces an element of value 1 in a finite number of transformations. The only way for the process to be infinite is that, in step 2, the minimum of the values of the elements does not decrease. This means that, in step 1, the value of the first variable divides the values of the new variables.

The composition of steps 1 and 2 is the transformation

$$\begin{array}{rcl} k[\![Y_{1,r}, \ldots, Y_{n,r}]\!] & \longrightarrow & L[\![Y_{1,r+1} \ldots, Y_{n,r+1}]\!] \\ Y_{1,r} & \longmapsto & Y_{1,r+1} \\ Y_{i,r} & \longmapsto & Y_{i,r+1} + \sigma(b_i)Y_{1,r+1}^{m_i},\ i = 2, \ldots, n. \end{array}$$

If we use steps 1 and 2 infinitely many times, we have an infinite sequence of transformations

$$\begin{array}{rcl} k[\![Y_1, \ldots, Y_n]\!] & \longrightarrow & L[\![Y_{1,j}, \ldots, Y_{n,j}]\!] \\ Y_1 & \longmapsto & Y_{1,j} \\ Y_i & \longmapsto & Y_{i,j} + \sum_{k=1}^{j} \sigma(b_{i,k})Y_{1,j}^{m_{i,k}},\ i = 2, \ldots, n. \end{array}$$

Then we can obtain an infinite sequence of variables

$$\begin{array}{rcl} Y_{1,j} & = & Y_{1,j} \\ Y_{i,j} & = & Y_i - \sum_{k=1}^{j} \sigma(b_{i,k})Y_1^{m_{i,k}},\ i = 2, \ldots, n, \end{array}$$

with $\widehat{v}(Y_{i,j}) > \widehat{v}(Y_{i,j-1})$ for all $i,\ j$. So any sequence of partial sums of the series

$$Y_i - \sum_{k=1}^{\infty} \sigma(b_{i,k})Y_1^{m_{i,k}},\ \forall i = 2, \ldots, n$$

have increasing values. Then these series converge to zero in $R_{\widehat{v}}$, so

$$Y_i = \sum_{k=1}^{\infty} \sigma(b_{i,k})Y_1^{m_{i,k}},\ \forall i = 2, \ldots, n.$$

Let $f(Y_1, \ldots, Y_n) \in K_n$, then

$$v(f) = \widehat{v}\left(f\left(Y_1, \sum_{k=1}^{\infty} \sigma(b_{2,k}) Y_1^{m_{2,k}}, \ldots, \sum_{k=1}^{\infty} \sigma(b_{n,k}) Y_1^{m_{n,k}}\right)\right) = m \cdot v(Y_1).$$

In this situation, the group of v is $v(X_1) \cdot \mathbb{Z}$ (see [2]) but as the group is assumed to be $\mathbb{Z}$, $\widehat{v}(Y_1) = 1$. □

Example 1.3. Let us consider the embedding

$$\begin{array}{rcl} \Psi : \mathbb{C}[\![X_1, X_2, X_3]\!] & \longrightarrow & \mathbb{C}[\![t, T_2, T_3]\!] \\ X_1 & \longmapsto & t^2 \\ X_2 & \longmapsto & T_2 t^4 + T_2 t^6 \\ X_3 & \longmapsto & T_2 t^2 + T_3 t^5 \end{array}$$

with t, T_2 and T_3 variables over $\mathbb{C}$. We are going to denote its extension to the quotient fields by Ψ as well. The composition of this injective homomorphism with the order function in t gives a discrete valuation of $\mathbb{C}((X_1, X_2, X_3))|\mathbb{C}$, $v = \nu_t \circ \Psi$. If we apply the procedure given in this section we construct the follwing element of value 1:

$$\frac{X_3 - \sigma(b_2) X_1}{X_1^2},$$

where $b_2 \in \Delta_{\widehat{v}}$ such that $\Psi(\sigma(b_2)) = T_2$.

2. Construction of the residue field

In this section we give a finite procedure to construct the residue field Δ_v of a discrete valuation of $K_n|k$, as a transcendental extension of k.

Remark 2.1. Preliminary transformations.
1) We can suppose that we have applied lemma 1.1 to obtain $Y_1 \ldots, Y_n$ such that $v(Y_i) = \alpha$ for all $i = 1, \ldots, n$.
2) In this situation $v(Y_i/Y_j) = 0$, so $0 \neq (Y_i/Y_j) + \mathfrak{m}_v \in \Delta_v$. If this residue lies in k then there exists $a_{i1} \in k$ such that

$$\frac{Y_i}{Y_j} + \mathfrak{m}_v = a_{i1} + \mathfrak{m}_v,$$

so

$$\frac{Y_i}{Y_j} - a_{i1} = \frac{Y_i - a_{i1} Y_j}{Y_j} \in \mathfrak{m}_v,$$

and then

$$v\left(\frac{Y_i - a_{i1} Y_j}{Y_j}\right) > 0.$$

So we have $v(Y_i - a_{i1} Y_j) = \alpha_1 > \alpha$. If α divides to α_1 then $\alpha_1 = r_1 \alpha$ with $r_1 \geq 2$ and

$$v\left(\frac{Y_i - a_{i1} Y_j}{Y_j^{r_1}}\right) = 0.$$

If the residue of this element lies too in k, then exist $a_{ir_1} \in k$ such that

$$v(Y_i - a_{i1} Y_j - a_{ir_1} Y_j^{r_1}) = \alpha_2 > \alpha_1.$$

If α divides to α_2 then $\alpha_2 = r_2 \alpha$ with $r_2 > r_1$ and we can repeat this operation.
3) The above procedure is finite for some pair (i, j). We know ([2]) that any discrete valuation of $k((X_1, X_2))$ has dimension 1, so the restriction, v', of our valuation

v to the field $k((X_1, X_2))$ is a valuation with dimension 1, and the dimension of v is greater or equal than 1, because a transcendental residue of v' over k is a transcendental residue of v too. If the procedure never ends for all (i, j) then all the residues of v are in k, so the dimension of v is 0 and there is a contradiction. So we can suppose that the above procedure ends for $(1, 2)$ by reordering the variables if necessary.

4) We can apply the procedure of item **2)** to the variables Y_1, Y_2, then we have the transformations:

$$Z_i = Y_i,\ i \neq 2$$
$$Z_2 = Y_2 - \sum_{i=1}^{s_2} a_{2i} Y_1^i,$$

such that one of the following two situation occurs:

a) $v(Y_1)$ divides $v(Z_2)$ and the residue of Z_2/Y_1^r is not in k with $v(Z_2) = r \cdot v(Y_1)$.

b) $v(Y_1)$ does not divide $v(Z_2)$.

In case a), we make the transformation

$$Z_2 = Y_2 - \sum_{i=1}^{s_2} a_{2i} Y_1^i,$$

and apply lemma 1.1 to obtain elements with the same values. We note these elements by $Y_1, \ldots, Y_n$ again in order not to complicate the notation. So, after this procedure, we have a transcendental element $u_2 = \sigma(Y_2/Y_1 + \mathfrak{m}_{\widehat{v}})$ over k.

In case b) we make the same transformation, apply the lemma 1.1 and go back to the item **2)** of this remark.

Anyway this procedure stops, because the value of the variables are greater or equal than 1.

5) Then we can suppose that, after a finite number of transformations, we have n elements $Y_1, \ldots, Y_n$ such that $v(Y_i) = v(Y_1) = \alpha$ and the residue $Y_2/Y_1 + \mathfrak{m}_{\widehat{v}}$ is not in k.

The proof of the following lemma is straightforward from ([2], theorem 2.4):

Lemma 2.2. *Let v be a discrete valuation of $K_n|k$. If v is such that $v(f_r) = r\alpha$ for all forms f_r of degree r with respect to the usual degree, then the group of v is $\alpha \cdot \mathbb{Z}$.*

So we have

Theorem 2.3. *In the case $n = 2$, the extension of the valuation v to the field $k((Y_1, Y_2))$ is the usual order function.*

Proof. After a finite number of transformations we are in the situation of 2.1.**5)**. Obviously, if $n = 2$, $k((Y_1, Y_2)) \subset R_{\widehat{v}}$ so v can be extended to a valuation v' over $k((Y_1, Y_2))$ such that $\Delta_{v'} = \Delta_v = \Delta_{\widehat{v}}$. We denote the extension by v for simplifying. Let $\sigma : \Delta_{\widehat{v}} \to R_{\widehat{v}}$ a $k-$section of $R_{\widehat{v}} \to \Delta_{\widehat{v}}$, $u_2 = \sigma(Y_2/Y_1 + \mathfrak{m}_{\widehat{v}})$, $h \neq 0$ a form of degree r and $\gamma = Y_2 - u_2 Y_1$. From the construction procedure of u_2 we know that $\widehat{v}(\gamma) > \alpha$ (remember $\alpha = v(Y_1)$). Then

$$h(Y_1, Y_2) = h(Y_1, u_2 Y_1 + \gamma) = Y_1^r h(1, u_2) + \gamma',$$

where γ' is such that $v(\gamma') > r\alpha$. As $u_2 \notin k$, u_2 is transcendental over k, so $h(1, u_2) \neq 0$ and $v(h) = r\alpha$. By the previous lemma, the group of v is $\alpha \cdot \mathbb{Z}$, so $\alpha = 1$ and v is the usual order function. ◻

Remark 2.4. We will construct the residue field of v as an extension of k. We are going to check all the variables searching those residues which generate the extension. Then we will have to move between $R_{\widehat{v}}$ and $\Delta_{\widehat{v}}$ by the $k-$section σ and the natural homomorphism $\Delta_{\widehat{v}} \to R_{\widehat{v}}$. In this sense we have to do the following remarks:

1) Let us consider the diagram

$$\begin{tikzcd}
R_{\widehat{v}} \arrow[r, shift right, "\varphi"'] & \Delta_{\widehat{v}} \arrow[l, shift right, "\sigma"'] \\
\mathbb{F} \arrow[u] \arrow[r, shift right, "\varphi"'] & \mathbb{F}' \arrow[u] \arrow[l, shift right, "\sigma"'] \\
k \arrow[u] \arrow[r, leftrightarrow, "id"] & k \arrow[u]
\end{tikzcd}$$

where $\mathbb{F}$ and $\mathbb{F}'$ are subfields of $R_{\widehat{v}}$ and $\Delta_{\widehat{v}}$ respectively. Let $\omega \in R_{\widehat{v}}$ an element such that $\widehat{v}(\omega) = 0$. The question is: if $\omega + \mathfrak{m}_{\widehat{v}}$ is transcendental over $\mathbb{F}'$, is $\sigma(\omega + \mathfrak{m}_{\widehat{v}})$ transcendental over $\mathbb{F}$? What happens in the algebraic case?

So we suppose $\omega + \mathfrak{m}_{\widehat{v}}$ to be transcendental over $\mathbb{F}'$. Let $f(X) \in \mathbb{F}[X]$ be a non-zero polynomial. Let us put

$$f(X) = \sum_{i=0}^{n} \sigma(a_i')X^i, \ a_i' \in \mathbb{F}'.$$

Then

$$f(\sigma(\omega + \mathfrak{m}_{\widehat{v}})) = \sum_{i=0}^{n} \sigma(a_i')\sigma(\omega + \mathfrak{m}_{\widehat{v}})^i = \sigma\left(\sum_{i=0}^{n} a_i'(\omega + \mathfrak{m}_{\widehat{v}})^i\right) \neq 0$$

because $\omega + \mathfrak{m}_{\widehat{v}}$ is transcendental over $\mathbb{F}'$. So we have proved that $\sigma(\omega + \mathfrak{m}_{\widehat{v}})$ is transcendental over $\mathbb{F}$ if $\omega + \mathfrak{m}_{\widehat{v}}$ is transcendental over $\mathbb{F}'$

2) In the algebraic case let us consider the next diagram:

$$\begin{tikzcd}
R_{\widehat{v}} & \Delta_{\widehat{v}} \\
\mathbb{F} \arrow[u] \arrow[r, shift right, "\varphi"'] & \mathbb{F}' \arrow[u] \arrow[l, shift right, "\sigma"'] \\
k \arrow[u] \arrow[r, leftrightarrow, "id"] & k \arrow[u]
\end{tikzcd}$$

Let $\alpha + \mathfrak{m}_{\widehat{v}} \in \Delta_{\widehat{v}}$ be an algebraic element over $\mathbb{F}'$, with $\widehat{v}(\alpha) = 0$ (i.e. $\alpha + \mathfrak{m}_{\widehat{v}} \neq 0$). Let

$$\overline{f}(X) = X^m + \beta_1 X^{m-1} + \cdots + \beta_m \in \mathbb{F}'[X]$$

be its minimal polynomial over $\mathbb{F}'$. Let us take the polynomial

$$f(X) = X^m + b_1 X^{m-1} + \cdots + b_m \in \mathbb{F}[X], \text{ with } b_i = \sigma(\beta_i).$$

By Hensel's Lemma ([6], corollary 1, page 279) we know that there exists $a \in R_{\widehat{v}}$ such that a is a simple root of $f(X)$ y $\varphi(a) = \alpha + \mathfrak{m}_{\widehat{v}}$. As $\varphi\sigma = id$, $f(X)$ is the minimal polynomial of a, so we can extend $\sigma : \mathbb{F}'[\alpha + \mathfrak{m}_{\widehat{v}}] \to \mathbb{F}[a]$. Then we have

$$\begin{tikzcd}
R_{\widehat{v}} & \Delta_{\widehat{v}} \\
\mathbb{F}(a) \arrow[u] \arrow[r, shift right, "\varphi"'] & \mathbb{F}'(\alpha + \mathfrak{m}_{\widehat{v}}) \arrow[u] \arrow[l, shift right, "\sigma"'] \\
\mathbb{F} \arrow[u] \arrow[r, shift right, "\varphi"'] & \mathbb{F}' \arrow[u] \arrow[l, shift right, "\sigma"'] \\
k \arrow[u] \arrow[r, leftrightarrow, "id"] & k \arrow[u]
\end{tikzcd}$$

Let us consider the set

$$\Omega = \{(\mathbb{F}_1, \sigma_1) | \mathbb{F}_1 \supset \mathbb{F} \text{ and } \sigma_1 \text{ extends } \sigma\}$$

partially ordered by

$$(\mathbb{F}_1, \sigma_1) < (\mathbb{F}_2, \sigma_2) \iff \mathbb{F}_1 \subset \mathbb{F}_2 \text{ and } \sigma_{2|\mathbb{F}_1} = \sigma_1.$$

By Zorn's Lemma there exists a maximal element $(\mathbb{L}, \sigma') \in \Omega$, and again by Hensel's Lemma ([6], corollary 2, page 280) we have $\varphi(\mathbb{L}) = \Delta_{\widehat{v}}$. So we can extend σ to a $k-$section σ' of φ in such a way that $a = \sigma'(\alpha + \mathfrak{m}_{\widehat{v}})$ is an algebraic element over $\mathbb{F}$.

3) In the following remarks we are going to give an explicit construction of the residue field of v as a subfield of $R_{\widehat{v}}$. Notes **1)** and **2)** of this remark will prove to be useful to do it because, if $\omega + \mathfrak{m}_{\widehat{v}} \in \Delta_{\widehat{v}}$, $\widehat{v}(\omega) = 0$, is a transcendental (resp. algebraic) element over $\mathbb{F}'$, there exists a $k-$section of φ which extends σ and $\sigma(\omega + \mathfrak{m}_{\widehat{v}})$ is transcendental (resp. algebraic) over $\mathbb{F}$.

$$\begin{tikzcd}
R_{\widehat{v}} \arrow[r, shift right, "\varphi"'] & \Delta_{\widehat{v}} \arrow[l, shift right, "\sigma'"'] \\
\mathbb{F} \arrow[u] \arrow[r, shift right, "\varphi"'] & \mathbb{F}' \arrow[u] \arrow[l, shift right, "\sigma"'] \\
k \arrow[u] \arrow[r, leftrightarrow, "id"] & k \arrow[u]
\end{tikzcd}$$

Remark 2.5. Let us move to the general case. Assume that $n > 2$ and suppose we have applied the procedure of remark 2.1 to find $Y_1, \ldots, Y_n \in \widehat{K}$ such that

a) The value of these elements are $\alpha \in \mathbb{Z}$.
b) The residue of Y_2/Y_1 is transcendental over k.
1) Let $\Delta_2 = k(Y_2/Y_1 + \mathfrak{m}_v)$ a purely transcendental extension of k of transcendence degree 1. Let $\sigma_2 : \Delta_2 \to k(Y_2/Y_1)$ defined by

$$\sigma_2\left(\frac{Y_2}{Y_1} + \mathfrak{m}_v\right) = \frac{Y_2}{Y_1} = u_2.$$

We know that there exists a $k-$section σ which extends σ_2 in the sense of the previous remark.
3) Let us suppose that the residue of Y_3/Y_1 is algebraic over Δ_2, and let u_{31} be its image by σ. Then $v(Y_3 - u_{31}Y_1) = \alpha_1 > \alpha$. If α divides to α_1 then there exists $u_{3r} \in \operatorname{Im}(\sigma)$ and $r > 1$ such that $v(Y_3 - u_{31}Y_1 - u_{3r}Y_1^r) = \alpha_2 > \alpha_1$. Let us suppose that u_{3r} is algebraic over Δ_2 too and α divides to α_2. Then we can find ourselves in one of the three situations shown in the following items.
4) (Situation 1) After a finite number of transformations, we obtain a value α_s such that it is not divided by α. Then we make the transformation

$$Z_3 = Y_3 - \sum_{j=1}^{s} u_{3j}Y_1^j,$$

with u_{3j} algebraic over Δ_2 for all $j = 1, \ldots, s$. So we have to apply transformations to find elements with the same values and begin with all the procedure described in this section from remark 2.1. When this occurs, the values of the elements decrease, so we can suppose that after a finite number of transformations we have reached a strictly minimal value. In fact this value should be 1, because we are assuming that the values group is $\mathbb{Z}$. We shall denote these elements by $Y_1, \ldots, Y_n$ in order not to complicate the notation. So we can suppose that this situation will never occur again for any variable.
5) (Situation 2) After a finite number of steps, we have a transcendental residue of Δ_2. Let us denote this residue by u_3. This means

$$Z_3 = Y_3 - \sum_{j=1}^{s_3} u_{3j}Y_1^j,$$

where the elements $\{u_{3j}\}_{j=1}^{s_3}$ are algebraic over Δ_2 and $u_3 = \sigma(Z_3/Y_1^{\widehat{v}(Z_3)} + \mathfrak{m}_v)$ is transcendental over Δ_2. We shall note $\Delta_3 = k(u_2, \{u_{3j}\}_{j=1}^{s_3}, u_3)$.

In this situation, if $n = 3$ we can apply monoidal transformations to obtain elements with the same values. We will denote these elements again by $\{Y_1, Y_2, Y_3\}$. The extension of the valuation v to the field $L((Y_1, Y_2, Y_3))$ with $L = k(\{u_{3j}\}_{j=1}^{s_3})$, is the usual order function, for analogy with the case $n = 2$ (theorem 2.3).
6) (Situation 3) All the residues obtained are algebraic elements. Then we take $\Delta_3 = \Delta_2(\{u_{3j}\}_{j\geq 1})$, an algebraic extension of Δ_2.

Remark 2.6. Let us suppose that we have repeated the construction of remark 2.5 with each element $Y_4, \ldots, Y_{i-1}$, so we have a field

$$\Delta_{i-1} = k(u_2, \zeta_3, \ldots, \zeta_{i-1}) \subset \sigma(\Delta_{\widehat{v}}),$$

where each ζ_k is: either $\{\{u_{kj}\}_{j=1}^{s_k}, u_k\}$ if $\{u_{kj}\}_{j=1}^{s_k}$ are algebraic over Δ_{k-1} and $u_k = \sigma((Z_k/Y_1^{\widehat{v}(Z_k)}) + \mathfrak{m}_{\widehat{v}})$ is a transcendental element over Δ_{k-1} (i.e. the situation of remark 2.5 **5)**) , or $\Delta_{k-1} \subset \Delta_{k-1}(\{u_{kj}\}_{j\geq 1})$ is an algebraic extension (i.e. the

situation of remark 2.5 **6)**). So we have two possible situations concerning variable Y_i:
1) There exists a transformation

$$Z_i = Y_i - \sum_{j=1}^{s_i} u_{ij} Y_1^j,$$

where the elements u_{ij} are algebraic over Δ_{i-1} and $u_i = \sigma((Z_i/Y_1^{\widehat{v}(Z_i)}) + \mathfrak{m}_{\widehat{v}})$ is a transcendental element over Δ_{i-1}. So we have the transcendental extension

$$\Delta_{i-1} \subset \Delta_{i-1}(\{u_{ij}\}_{j=1}^{s_i}, u_i) = \Delta_i.$$

2) All the elements u_{ij} we have constructed are algebraic over Δ_{i-1}, so we have the algebraic extension

$$\Delta_{i-1} \subset \Delta_{i-1}(\{u_{ij}\}_{j\geq 1}) = \Delta_i.$$

In these remarks we have given an explicit construction of elements $\{Y_1, \dots, Y_n\}$ such that they satisfy this important properties:

1. By reordering of elements, we can suppose that the first m elements give us all the transcendental residues over k, i.e. the residue of each Y_i/Y_1 is transcendental over Δ_{i-1} with $i = 2, \dots, m$. So the rest of variables $Y_{m+1}, \dots, Y_n$ are such that we enter in the procedure of remark 2.5 **7)**.
2. With the usual notations, the algebraic extension
$$\Delta_m \subset \Delta_m\left(\{u_{ij}\}_{j\geq 1}\right), \ i = m+1, \dots, n$$
is infinite.

Theorem 2.7. *The residue field of v is*

$$\Delta_n = k\left(u_2, \{u_{3,j}\}_{j=1}^{s_3}, u_3, \dots, \{u_{m,j}\}_{j=1}^{s_m}, u_m\right)\left(\{u_{m+1,j}\}_{j\geq 1}, \dots, \{u_{n,j}\}_{j\geq 1}\right),$$

and the transcendence degree of Δ_n over k is $m-1$.

Proof. In this section we have given a construction by writing the elements Y_i depending on Y_1 and some transcendental and algebraic residues. So we have constructed an application

$$\begin{array}{rcl} \varphi' : L_n((Y_1, \dots, Y_n)) & \longrightarrow & \Delta_n((t)) \\ Y_1 & \longmapsto & t \\ Y_i & \longmapsto & u_i t, \ i = 2, \dots, m \\ Y_k & \longmapsto & \sum_{j\geq 1} u_{k,j} t^j, \ u_{k,1} \neq 0, \ k = m+1, \dots, n. \end{array}$$

This application is not injective in the general case, but we know that $v = \nu_t \circ \varphi'_{|K_n}$. So the residue field of v is equal to the residue field of ν_t, i.e. Δ_n. □

A straightforward consequence of this theorem is the following well-known result

Corollary 2.8. *The usual order function over K_n has dimension $n-1$, i.e. the transcendence degree of its residue field over k is $n-1$.*

Proof. Let ν be the usual order function over K_n. All the residues $X_i/X_1 + \mathfrak{m}_\nu$ are transcendental over $k(X_2/X_1 + \mathfrak{m}_\nu, \dots, X_{i-1}/X_1 + \mathfrak{m}_\nu)$: if this were not the case, there would exist $u_i \in \sigma(\Delta_\nu)$ such that $\nu(X_i - u_i X_1) > 1$ and ν would not be an order function. So $\Delta_\nu = k(X_2/X_1, \dots, X_n/X_1)$. □

We can summarize the constructions of this section in the following theorem wich generalize the results of [2, 3] (Thanks are due to Prof. Mark Spivakovsky for his helpful suggestions concerning the second part of this result).

Theorem 2.9. *Let v be a discrete valuation of $K_n|k$, then*

1. *If the dimension of v is $n-1$, we can embed $k[\![X_1,\dots,X_n]\!]$ into a ring $L[\![Y_1,\dots,Y_n]\!]$, where $L\subset\sigma(\Delta_{\widehat{v}})$ and the extended valuation of v over the field $L((Y_1,\dots,Y_n))$ is the usual order function.*
2. *If the dimension of v is $m-1<n-1$, we can embed $k[\![X_1,\dots,X_n]\!]$ into a ring $L[\![Y_1,\dots,Y_n]\!]$, where $L\subset\sigma(\Delta_{\widehat{v}})$ and the restriction into $L((Y_1,\dots,Y_m))$ of the "extended valuation" of v over $L((Y_1,\dots,Y_n))$ is the usual order function.*

Proof. We have the following map:

$$\begin{array}{rcl}\varphi' : L_n((Y_1,\dots,Y_n)) & \longrightarrow & \Delta_n((t))\\ Y_1 & \longmapsto & t\\ Y_i & \longmapsto & u_i t,\ i=2,\dots,m\\ Y_k & \longmapsto & \sum_{j\geq 1} u_{k,j}t^j,\ u_{k,1}\neq 0,\ k=m+1,\dots,n,\end{array}$$

where $m-1$ is de dimension of v. Let us prove the theorem:

1. In the case $m=n$, $\varphi'(Y_i)=u_i t$ for all $i=2,\dots,n$. Let ν_t be the usual order funtion over $\Delta_n((t))$. The homomorphism φ' is injective and the valuation $v'=\nu_t\circ\varphi'$ of $L((Y_1,\dots,Y_n))$ is the usual order fuction over this field. Obviously v' extends v.
2. If $m<n$ we can consider the elements $W_k=Y_k-\sum_{j\geq 1}u_{kj}Y_1^j$. Hence we have $L((Y_1,\dots,Y_n))=L((Y_1,\dots,Y_m,W_{m+1},\dots,W_n))$. We define the discrete valuation of rank $n-m+1$ over $L((Y_1,\dots,Y_n))$:
$$v'(Y_1)=\ldots=v'(Y_m)=(0,\dots,0,1),$$
$$v'(W_{m+1})=(0,\dots,1,0),\dots,v'(W_n)=(1,0,\dots,0).$$
The restriction of this valuation to K_n is a rank one discrete valuation, because the value of any element is in $0\times\cdots\times 0\times\mathbb{Z}$. In fact $v'(f)=(0,\dots,0,v(f))$ for all $f\in K_n$, so v' "extends" v in this sense. Obviously $v'_{|L((Y_1,\dots,Y_m))}$ is the usual order function.

□

For the case of valuations of dimension $n-1$, we can combine corollary 2.8 and assertion 1 of the previous theorem:

Corollary 2.10. *Let v be a discrete valuation of $K_n|k$. The following conditions are equivalent:*

1) The transcendence degree of $\Delta_{\widehat{v}}$ over k is $n-1$.

2) There exists a finite sequence of monoidal transformations and coordinates changes which take v into an order function.

Example 2.11. Let us consider the embedding

$$\begin{array}{rcl}\Psi : \mathbb{C}[\![X_1,X_2,X_3,X_4]\!] & \longrightarrow & \mathbb{C}[\![t,T_2,T_3,T_4]\!]\\ X_1 & \longmapsto & t\\ X_2 & \longmapsto & T_2 t\\ X_3 & \longmapsto & T_2^2 t+T_2t^2+T_3t^3\\ X_4 & \longmapsto & T_2^3t+T_2^2t^2+T_3t^3+T_4t^4,\end{array}$$

with t, T_2, T_3 and T_4 variables over $\mathbb{C}$. We are going to denote its extension to the quotient fields by Ψ. The composition of this injective homomorphism with the order function in t gives a discrete valuation of $\mathbb{C}((X_1, X_2, X_3, X_4))|\mathbb{C}$, $v = \nu_t \circ \Psi$. The residues of X_i/X_1 are not in $\mathbb{C}$ for $i = 2, 3, 4$.

Let us put $u_2 = \sigma(X_2/X_1 + \mathfrak{m}_v)$, a transcendental element over $\mathbb{C}$. By remarks 2.4 we know that we can construct σ step by step, so let take us $u_2 = X_2/X_1$ and $\Delta_2 = \mathbb{C}(u_2)$.

The residue $X_3/X_1 + \mathfrak{m}_v$ is algebraic over $\mathbb{C}(u_2)$, in fact

$$\frac{X_3}{X_1} + \mathfrak{m}_v = \frac{X_2^2}{X_1^2} + \mathfrak{m}_v.$$

So we can take $u_{31} = \sigma((X_3/X_1) + \mathfrak{m}_v) = u_2^2$. The value of $X_3 - u_{31}X_1$ is 2, therefore we have to see if the residue

$$\frac{X_3 - u_{31}X_1}{X_1^2} + \mathfrak{m}_v$$

is algebraic over $\mathbb{C}(u_2)$. We have that

$$\frac{X_3 - u_{31}X_1}{X_1^2} + \mathfrak{m}_v = \frac{X_2}{X_1} + \mathfrak{m}_v,$$

so it is algebraic and we can take $u_{32} = u_2$. Now $v(X_3 - u_{31}X_1 - u_{32}X_1^2) = 3$ and we have to check if

$$\frac{X_3 - u_{31}X_1 - u_{32}X_1^2}{X_1^3} + \mathfrak{m}_v$$

is algebraic over Δ_2. In this case, as

$$\Psi\left(\frac{X_3 - u_{31}X_1 - u_{32}X_1^2}{X_1^3} + \mathfrak{m}_v\right) = T_3,$$

this residue is transcendental. So we take

$$u_3 = \sigma\left(\frac{X_3 - u_{31}X_1 - u_{32}X_1^2}{X_1^3} + \mathfrak{m}_v\right) = \frac{X_1X_3 - X_2^2 - X_1^2X_2}{X_1^4}.$$

Let us take $\Delta_3 = \mathbb{C}(u_2, u_3)$.

Now we have to apply this procedure to X_4. The residue $X_4/X_1 + \mathfrak{m}_v$ is algebraic over Δ_3 because

$$\frac{X_4}{X_1} + \mathfrak{m}_v = \frac{X_2^3}{X_1^3} + \mathfrak{m}_v,$$

so we can take $u_{41} = \sigma((X_4/X_1) + \mathfrak{m}_v) = u_2^3 \in \Delta_3$. $v(X_4 - u_{41}X_1) = 2$, so we have to check what happens with the residue

$$\frac{X_4 - u_{41}X_1}{X_1^2} + \mathfrak{m}_v.$$

Now

$$\frac{X_4 - u_{41}X_1}{X_1^2} + \mathfrak{m}_v = \frac{X_1^2}{X_2^2} + \mathfrak{m}_v,$$

so it holds

$$u_{42} = \sigma\left(\frac{X_4 - u_{41}X_1}{X_1^2} + \mathfrak{m}_v\right) = u_2^2.$$

As $v(X_4 - u_{41}X_1 - u_{42}X_1^2) = 3$ and

$$\frac{X_4 - u_{41}X_1 - u_{42}X_1^2}{X_1^3} + \mathfrak{m}_v = \frac{X_1X_3 - X_2^2 - X_1^2X_2}{X_1^4} + \mathfrak{m}_v,$$

we have

$$u_{43} = \sigma\left(\frac{X_4 - u_{41}X_1 - u_{42}X_1^2}{X_1^3} + \mathfrak{m}_v\right) = u_3.$$

The following residue is transcendental because $v(X_4 - u_{41}X_1 - u_{42}X_1^2 - u_{43}X_1^3) = 4$ and

$$\Psi\left(\frac{X_4 - u_{41}X_1 - u_{42}X_1^2 - u_{43}X_1^3}{X_1^4}\right) = T_4.$$

Then we can take

$$u_4 = \sigma\left(\frac{X_4 - u_{41}X_1 - u_{42}X_1^2 - u_{43}X_1^3}{X_1^4} + \mathfrak{m}_v\right) =$$

$$= \frac{X_1^2X_4 - X_3^2 - X_1^2X_2^2 - X_1^2X_3 - X_1X_2^2 - X_1^2X_2}{X_1^6}.$$

So the residue field of v is

$$\Delta_v = \mathbb{C}\left(\frac{X_2}{X_1} + \mathfrak{m}_v, \frac{X_1X_3 - X_2^2 - X_1^2X_2}{X_1^4} + \mathfrak{m}_v,\right.$$
$$\left.\frac{X_1^2X_4 - X_3^2 - X_1^2X_2^2 - X_1^2X_3 - X_1X_2^2 - X_1^2X_2}{X_1^6} + \mathfrak{m}_v\right).$$

Hence, by the transformation

$$\begin{array}{lcl} X_1 & \longrightarrow & Y_1 \\ X_2 & \longrightarrow & Y_2 \\ X_3 & \longrightarrow & Y_1^2Y_3 + u_{3,1}Y_1 + u_{3,2}Y_1^2 \\ X_4 & \longrightarrow & Y_1^3Y_4 + u_{4,1}Y_1 + u_{4,2}Y_1^2 + u_{4,3}Y_1^3, \end{array}$$

we have extended the valuation v to the usual order function over the field $\mathbb{C}((Y_1, Y_2, Y_3, Y_4))|\mathbb{C}$.

Remark 2.12. There exists discrete valuations of $K_n|k$ of dimension strictly smaller than $n-1$. In [1] we have an example of a discrete valuation over $K_3|k$ of dimension 1, and [4] proves that there exist discrete valuations of $K_n|k$ of arbitrary dimension between 1 and $n-1$. This proof is constructive. Therefore the proof of assertion 2 of theorem 2.9 gives us a method to construct valuations of any dimension.

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