ON CERTAIN DEFINABLE COARSENINGS OF VALUATION RINGS AND THEIR APPLICATIONS

FRANZ-VIKTOR KUHLMANN

ABSTRACT. We show how suitable extensions $(L|K,v)$ of prime degree of valued fields give rise to definable coarsenings of the valuation rings of L and K. In the case of Artin-Schreier and Kummer extensions with wild ramification, we can also define the ramification ideal. We demonstrate the use of the coarsenings on L, their maximal ideals, and the ramification ideals for the classification of defects and for the presentation of the Kähler differentials of the extension of the valuation rings of $(L|K,v)$, and their annihilators. Finally, we give a construction that realizes predescribed convex subgroups of suitable value groups as those that are associated with Galois extensions of degree p with independent defect, which in turn give rise to definable coarsenings.

1. Introduction

In this paper, for Galois extensions $(L|K,v)$ of prime degree, as studied in [2, 3], we will discuss definable coarsenings of the valuation rings of L and K, and their applications to the presentation of the Kähler differentials of the extension of the valuaion rings of $(L|K,v)$. As our main interest are these applications, we will only deal with definability in suitable expansions of the language $\mathcal{L}_{\rm val}$ of valued fields, instead of the language of rings.

Moreover, we will be interested in definable coarsenings of both the valuation ring $\mathcal{O}_L$ of v on L and the valuation ring $\mathcal{O}_K$ of v on K; however, it is the former that are important for our applications. Under certain additional assumptions the coarsenings of $\mathcal{O}_K$ have already been shown in [6] to be definable in the ring language.

The notions and notations we will now use will be introduced in Section 2.

1.1. **Coarsenings defined from immediate elements in valued field extensions.**

Take any valued field extension $(L|K,v)$ and an arbitrary element $z\in L\backslash K$. For a nonempty subset $M\subseteq K$ we define

$$v(z-M)\;:=\;\{v(z-c)\,|\,c\in M\}\;\subseteq\;vL\,.$$

If $M=K$, then the set $v(z-K)\cap vK$ is an initial segment of vK. For the properties of the sets $v(z-K)$, see [13, Chapter 2.4]. If $v(z-K)$ has no maximal element, then we call z an **immediate element** of the extension $(L|K,v)$. In this case, $v(z-K)\subseteq vK$.

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In Section 3, we will define from an immediate element z a coarsening $\mathcal{O}_{z-K}$ of the valuation ring $\mathcal{O}_L$ of L in the language $\mathcal{L}_{\mathrm{val},K}$ of valued fields with a predicate for membership in K. This coarsening plays an important role in our study of Galois defect extensions of prime degree. If z does not lie in the completion $\widehat{K}$ of (K,v), then $v(z-K)$ is bounded from above and $-v(z-K)$ is bounded from below in vK. In this case,

$$I_{z-K} \;:=\; \{b \in L \mid \exists c \in K:\; vb \geq -v(z-c)\} \tag{1}$$

is a (possibly fractional) $\mathcal{O}_L$-ideal. When we speak of $\mathcal{O}_L$-ideals, we always include fractional ideals, that is, $\mathcal{O}_L$-modules $I \subset L$ for which there is some $a \in \mathcal{O}_L$ such that $aI \subseteq \mathcal{O}_L$.

For an $\mathcal{O}_L$-ideal I, its invariance valuation ring $\mathcal{O}(I)$ (see the definition in Section 2.4) is the largest of all coarsenings $\mathcal{O}'$ of $\mathcal{O}_L$ such that I is an $\mathcal{O}'$-ideal. It is definable in the ring language augmented by a predicate for membership in I. We define $\mathcal{O}_{z-K}$ to be the invariance valuation ring of I_{z-K}.

1.2. **Galois defect extensions of prime degree.** These extensions have been studied in [16] and [2]. Take a valued field (K,v) with $\mathrm{char}\, Kv = p > 0$, and a Galois defect extension $\mathcal{E} = (L|K,v)$ of prime degree p. For every σ in its Galois group $\mathrm{Gal}\,(L|K)$, with $\sigma \neq \mathrm{id}$, we set

$$\Sigma_\sigma \;:=\; \left\{ v\left(\frac{\sigma b - b}{b}\right) \,\middle|\, b \in L^\times,\, \sigma b \neq b \right\}. \tag{2}$$

This set is a final segment of vK and independent of the choice of σ (see Theorem 4.1); we denote it by $\Sigma_{\mathcal{E}}$. It is shown in [16, Section 2.4] that

$$I_{\mathcal{E}} \;=\; (b \in L \mid vb \in \Sigma_{\mathcal{E}}) \;=\; \{b \in L \mid vb \in \Sigma_{\mathcal{E}} \vee b = 0\} \tag{3}$$

is the unique ramification ideal of $\mathcal{E}$. We set $\mathcal{O}_{\mathcal{E}} := \mathcal{O}(I_{\mathcal{E}})$ and denote its maximal ideal by $\mathcal{M}_{\mathcal{E}}$. We denote by $\mathcal{L}_{\mathrm{val},K}$ the language of valued fields with a predicate for membership in K and prove in Section 4:

Proposition 1.1. *Take a Galois extension $\mathcal{E} = (L|K,v)$ of prime degree p.*

1) The ideals $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$ are $\mathcal{L}_{\mathrm{val},K}$-definable in (L,v).

2) If $\mathrm{char}\, K = 0$*, then assume in addition that K contains a primitive p-th root of unity. Then also the ideal $I_{\mathcal{E}}$ is $\mathcal{L}_{\mathrm{val},K}$-definable in (L,v).*

A main aim of this paper is to describe the role the ideals $I_{\mathcal{E}}$, $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$ play in the description of the structure of Artin-Schreier extensions and Kummer defect extensions of prime degree. This will be done in Section 4.

We say that $\mathcal{E}$ has **independent defect** if

$$I_{\mathcal{E}} \;=\; \mathcal{M}_{\mathcal{E}} \quad \text{and } \mathcal{M}_{\mathcal{E}} \text{ is a nonprincipal } \mathcal{O}_{\mathcal{E}}\text{-ideal,} \tag{4}$$

otherwise we will say that $\mathcal{E}$ has **dependent defect**. We will show in Section 4 that in the case of Artin-Schreier extensions and Kummer extensions of prime degree, this definition is equivalent to the one given in [2].

Let us give an example for the importance of independent defect. A valued field (K,v) is called a **roughly deeply ramified field**, or in short an **rdr field**, if the following conditions hold:

(DRvp) if $\mathrm{char}\, Kv = p > 0$, then vp is not the smallest positive element in the value group vK,

(DRvr) if $\mathrm{char}\, Kv = p > 0$, then $\mathcal{O}_K/p\mathcal{O}_K$ is semiperfect if $\mathrm{char}\, K = 0$, and the completion $\widehat{K}$ of (K, v) is perfect if $\mathrm{char}\, K = p$.

The following is a consequence of [16, Theorem 1.10 1)]:

Theorem 1.2. *Assume that (K, v) is a roughly deeply ramified field. Then every Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree $p = \mathrm{char}\, Kv > 0$ has independent defect.*

1.3. **Deeply ramified fields and Kähler differentials.** We call (K, v) a **deeply ramified field** if it satisfies condition **(DRvr)** together with

(DRvg) whenever $\Gamma_1 \subsetneq \Gamma_2$ are convex subgroups of the value group vK, then Γ_2/Γ_1 is not isomorphic to $\mathbb{Z}$ (that is, no archimedean component of vK is discrete).

Every perfect valued field of positive characteristic p and every perfectoid field is a deeply ramified field with p-divisible value group. Every deeply ramified field is an rdr field.

A theorem of Gabber and Ramero uses Kähler differentials, that is, modules of relative differentials, to characterize deeply ramified fields (cf. [5, Theorem 6.6.12 (vi)] and [3, Theorem 1.2]). When A is a ring and B is an A-algebra, then we denote by $\Omega_{B|A}$ the Kähler differentials of $B|A$ (see Section 2.3). Given a valued field (K, v), we denote by K^{sep} the separable algebraic closure and extend v from K to K^{sep}. The following result does not depend on the choice of the extension of v since all of the possible extensions are conjugate.

Theorem 1.3. *For a valued field (K, v),*

$$\Omega_{\mathcal{O}_{K^{\mathrm{sep}}}|\mathcal{O}_K} = 0 \tag{5}$$

holds if and only if (K, v) is a deeply ramified field.

A main goal of the papers [2, 3] is to compute the Kähler differentials of Galois extensions $\mathcal{E} = (L|K, v)$ of prime degree of valued fields and use this to give an alternative proof of Theorem 1.3. According to [2, Theorem 1.1], these Kähler differentials can be represented in the form

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \simeq U/UV \tag{6}$$

where U and V are certain $\mathcal{O}_L$-ideals. Their computation in the case of defect extensions $\mathcal{E}$ is dealt with in [2] and we will state the results in Section 4.

1.4. **Defectless Galois extensions of prime degree.** The paper [3] is devoted to the case of defectless extensions $\mathcal{E}$; in Section 5 we discuss its results, as well as the $\mathcal{L}_{\mathrm{val},K}$-definition and the role of the valuation ring $\mathcal{O}_{\mathcal{E}}$ and its maximal ideal $\mathcal{M}_{\mathcal{E}}$. The interesting case is the one of Galois extensions $\mathcal{E} = (L|K, v)$ of prime degree $q = (vL : vK)$ (which this time is not necessarily equal to $\mathrm{char}\, Kv$). In order to compute the ideals U and V appearing in (6) we determined in [3] a presentation of $\mathcal{O}_L$ as a union over a chain of simple ring extensions of $\mathcal{O}_K$. It depends on a distinction of three ways in which vK extends to vL, and as a byproduct we obtain definitions of the valuation ideal $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$. We will show in Section 5.2 that the ideal $\mathcal{M}_{\mathcal{E}}$ is necessary for the presentation of U and V, and also for the computation of the annihilator of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$.

In Section 5.1 we will present $\mathcal{L}_{\mathrm{val},K}$-definitions of the ramification ideal $I_{\mathcal{E}}$ for the defectless wildly ramified case.

1.5. **Predescribed associated convex subgroups.** For Galois defect extensions $\mathcal{E} = (L|K,v)$ of prime degree, we have already defined in Section 1.2 the valuation rings $\mathcal{O}_{\mathcal{E}}$. They correspond to convex subgroups of vL via the definition

$$H_{\mathcal{E}} \;:=\; v\mathcal{O}_{\mathcal{E}}^{\times} \;=\; v\mathcal{O}_{\mathcal{E}} \cap -v\mathcal{O}_{\mathcal{E}}\;.$$

Since the extension $\mathcal{E}$ is immediate, $H_{\mathcal{E}}$ is a convex subgroup of both vK and vL.

Using this definition, we can modify the original definition for independent defect given in [16] in the following way: $\mathcal{E}$ has **independent defect** if

$$(7) \qquad \Sigma_{\mathcal{E}} \;=\; \{\alpha \in vK \mid \alpha > H_{\mathcal{E}}\} \;\text{ and } vK/H_{\mathcal{E}} \text{ has no smallest positive element;}$$

otherwise we will say that $\mathcal{E}$ has **dependent defect**. If (K,v) has rank 1 (i.e., its value group is order isomorphic to a subgroup of $\mathbb{R}$), then condition (7) just means that $\Sigma_{\mathcal{E}}$ consists of all positive elements in vK. In the case of independent defect, we will call $H_{\mathcal{E}}$ the **convex subgroup associated with** $\mathcal{E}$. In order not to overload our sentences, we will write "associated convex subgroup" for "convex subgroup associated with a Galois defect extension of prime degree".

For an ordered abelian group Γ, denote by $\mathcal{C}(\Gamma)$ the chain of its proper convex subgroups, and by $\mathcal{C}_{\mathrm{pr}}(\Gamma)$ the chain of its proper principal convex subgroups. If H is a convex subgroup of Γ that is the smallest among all convex subgroups that contain a given element $\gamma \in \Gamma$, then we call it a **principal convex subgroup**, and if it is largest among all convex subgroups that do not contain a given element $\gamma \in \Gamma$, then we call it a **subprincipal convex subgroup**. A subprincipal convex subgroup may or may not be principal. In Section 6 we will prove:

Theorem 1.4. *Let p be a prime and take any totally ordered set I. Then there exists an ordered abelian group Γ with $\mathcal{C}_{\mathrm{pr}}(\Gamma)$ order isomorphic to I such that for any subset $\mathcal{C}^{\mathrm{sp}} \subseteq \mathcal{C}$ containing only subprincipal convex subgroups, the following statements hold.*

1) There exists a perfect henselian valued field of characteristic p with value group Γ for which the associated convex subgroups are exactly the elements of $\mathcal{C}^{\mathrm{sp}}$.

2) Assume in addition that Γ has a largest proper convex subgroup. Then there exists a henselian deeply ramified field of characteristic 0 and residue characteristic p with value group Γ for which the associated convex subgroups are exactly the elements of $\mathcal{C}^{\mathrm{sp}}$.

In an ordered abelian group with only finitely many proper convex subgroups, each of them is subprincipal. Therefore, the next result follows immediately from our theorem:

Corollary 1.5. *Let p be a prime and take any finite totally ordered set I. Then there exists an ordered abelian group Γ with $\mathcal{C}_{\mathrm{pr}}(\Gamma)$ order isomorphic to I such that for any set $\mathcal{H}$ of proper convex subgroups of Γ, there exists a perfect henselian valued field of characteristic p as well as a henselian deeply ramified field of characteristic*

0 and residue characteristic p with value group Γ for which the associated convex subgroups are exactly the elements of $\mathcal{H}$.

2. Preliminaries

2.1. **Notation.** For a valued field (K,v), we denote the value group by vK, the residue field by Kv, the valuation ring by $\mathcal{O}_K$, and its maximal ideal by $\mathcal{M}_K$. We set $vK^{>0} := \{\alpha \in vK \mid \alpha > 0\}$ and $vK^{<0} := \{\alpha \in vK \mid \alpha < 0\}$. Throughout, we will use the convention that $v0 = \infty > \alpha$ for all $\alpha \in vK$.

By $(L|K,v)$ we denote a field extension $L|K$ with a valuation v on L, where K is endowed with the restriction of v. In this case, there are induced embeddings of vK in vL and of Kv in Lv. The extension $(L|K,v)$ is called **immediate** if these embeddings are onto. In this case, if $z \in L \setminus K$, then $v(z-K)$ has no maximal element, and therefore z is an immediate element of $(L|K,v)$; this follows from [13, Lemma 2.9 2)] and the fact that each subextension of an immediate extension is immediate.

We call $(L|K,v)$ **unibranched** if the valuation v has only one extension from K to L. A valued field is **henselian** if and only if all of its algebraic extensions are unibranched.

If $(L|K,v)$ is a finite unibranched extension, then by the Lemma of Ostrowski ([18, Corollary to Theorem 25, Section G, p. 78]),

$$[L:K] \;=\; \tilde{p}^{\nu}\cdot(vL:vK)[Lv:Kv]\,, \tag{8}$$

where ν is a non-negative integer and $\tilde{p}$ the **characteristic exponent** of Kv, that is, $\tilde{p} = \operatorname{char} Kv$ if it is positive and $\tilde{p} = 1$ otherwise. The factor $d(L|K,v) := \tilde{p}^{\nu}$ is the **defect** of the extension $(L|K,v)$. If $d(L|K,v) = 1$, then the extension $(L|K,v)$ is called **defectless**; otherwise we call it a **defect extension**. A henselian field (K,v) is a **defectless field** if every finite unibranched extension of (K,v) is defectless; note that this is always the case if $\operatorname{char} Kv = 0$.

2.2. **Ramification ideals.** If $L|K$ is Galois, then we denote its Galois group by $\operatorname{Gal} L|K$. In this case, a nontrivial $\mathcal{O}_L$-ideal contained in $\mathcal{M}_L$ is called a **ramification ideal** of $(L|K,v)$ if it is of the form

$$\left(\frac{\sigma b - b}{b} \;\middle|\; \sigma \in H\,,\; b \in L^{\times}\right) \tag{9}$$

for some subgroup H of $\operatorname{Gal} L|K$. For more information on ramification ideals, see [14].

2.3. **Kähler differentials.** Assume that A is a ring and B is an A-algebra. Then $\Omega_{B|A}$ denotes the module of relative differentials (Kähler differentials), that is, the B-module for which there is a universal derivation

$$d:\; B \;\to\; \Omega_{B|A}$$

such that for every B-module M and derivation $\delta : B \to M$ there is a unique B-module homomorphism

$$\phi:\; \Omega_{B|A} \;\to\; M$$

such that $\delta = \phi \circ d$.

2.4. **Invariance group and invariance valuation ring.**
Take any valued field (L,v) and $\mathcal{O}_L$-ideal I. We set

$$\mathcal{O}(I) \;:=\; \{b\in L \mid bI\subseteq I\} \qquad\text{and}\qquad \mathcal{M}(I) \;=\; \{b\in L \mid bI\subsetneq I\}\,. \tag{10}$$

We call $\mathcal{O}(I)$ the **invariance valuation ring** of I. The following is part of [17, Theorem 3.6]:

Proposition 2.1. *For every $\mathcal{O}_L$-ideal I, $\mathcal{O}(I)$ is a valuation ring of L containing $\mathcal{O}_L$, with maximal ideal $\mathcal{M}(I)$, which is a prime $\mathcal{O}_L$-ideal. It is the largest of all valuation rings $\mathcal{O}$ of L containing $\mathcal{O}_L$ for which I is an $\mathcal{O}$-ideal.*

If the ideal I is definable in an expansion $\mathcal{L}$ of $\mathcal{L}_{\mathrm{val}}$, then also $\mathcal{O}(I)$ and $\mathcal{M}(I)$ are $\mathcal{L}$-definable:

$$\mathcal{O}(I) \;:=\; \{b\in L \mid \forall c\in I:\ bc\in I\}, \tag{11}$$

$$\mathcal{M}(I) \;:=\; \{b\in \mathcal{O}(I) \mid \exists a\in I\,\forall c\in I:\ bc\neq a\}\,. \tag{12}$$

For a subset M of an ordered abelian group Γ, we define its **invariance group** to be

$$\mathcal{G}(M) \;:=\; \{\gamma\in\Gamma \mid M+\gamma = M\}\,.$$

This is a subgroup of Γ, and it is a convex subgroup if M is convex (which in particular is the case if M is an initial or a final segment of Γ). If S is a final segment of Γ and $\gamma\in\Gamma$, then $\gamma+S:=\{\gamma+\alpha\mid\alpha\in S\}$ and $-S:=\{-\alpha\mid\alpha\in S\}$ are again final segments of Γ with

$$\mathcal{G}(\gamma+S) \;=\; \mathcal{G}(S) \;=\; \mathcal{G}(-S)\,. \tag{13}$$

For these facts and more information on invariance groups, see [17, Section 2.4] and [12].

For every coarsening $\mathcal{O}$ of $\mathcal{O}_L$, we set

$$H(\mathcal{O}) \;:=\; v\mathcal{O}\cap -v\mathcal{O} \;=\; v\mathcal{O}^\times\,.$$

This is a convex subgroup of the value group vL of (L,v). If $\mathcal{M}$ is the maximal ideal of $\mathcal{O}$, then

$$v\mathcal{M} \;=\; \{\alpha\in vL \mid \alpha > v\mathcal{O}^\times\} \;=\; \{\alpha\in vL\mid \alpha > H(\mathcal{O})\}\,. \tag{14}$$

The valuation w associated with $\mathcal{O}$ is (up to equivalence) given by

$$wa \;=\; va/H(\mathcal{O}) \tag{15}$$

for every $a\in K$, the value group of w is canonically isomorphic to $vK/H(\mathcal{O})$, and the value group of the valuation induced by v on the residue field Kw is canonically isomorphic to $H(\mathcal{O})$ (cf. [18]). The function $\mathcal{O}\mapsto H(\mathcal{O})$ sends every coarsening $\mathcal{O}$ of $\mathcal{O}_L$ to a convex subgroup of vL. Its inverse is given by sending a convex subgroup H of vL to

$$\mathcal{O}(H) \;:=\; \{b\in K \mid \exists\alpha\in H:\ \alpha\le vb\}\,. \tag{16}$$

We call this the **coarsening of $\mathcal{O}_L$ associated with H.**

Further, for every $\mathcal{O}_v$-ideal I we define

$$H(I) \;:=\; H(\mathcal{O}(I))\,.$$

By [17, Theorem 3.6 3)],

$$H(I) \;=\; H(\mathcal{O}(I)) \;=\; \mathcal{G}(vI) \tag{17}$$

and

$$\mathcal{O}(I) \;=\; \mathcal{O}(\mathcal{G}(vI)) \;=\; \mathcal{O}(H(I))\,. \tag{18}$$

The next result is part of [17, Lemma 3.5].

Lemma 2.2. *For every coarsening $\mathcal{O}$ of $\mathcal{O}_L$ with maximal ideal $\mathcal{M}$,*

$$H(\mathcal{O}) \;=\; \mathcal{G}(v\mathcal{O}) \;=\; \mathcal{G}(v\mathcal{M})\,. \tag{19}$$

We leave the straightforward proof of the following result to the reader.

Lemma 2.3. *If I is an $\mathcal{O}_L$-ideal and $J = aI$ with $0 \neq a \in L$, then $\mathcal{O}(J) = \mathcal{O}(I)$, $\mathcal{M}(J) = \mathcal{M}(I)$ and $H(J) = H(I)$.*

3. Immediate elements in arbitrary valued field extensions

Take any valued field extension $(L|K,v)$ and $z \in L \setminus K$ an immediate element in $(L|K,v)$, that is, the set $v(z-K)$ has no maximal element and is an initial segment of vK. We define

$$I_{z-K;K} \;:=\; (b \in K \mid vb \in -v(z-K)\} \tag{20}$$

and

$$I_{z-K} \;:=\; (b \in L \mid \exists c \in K :\; vb \geq -v(z-c)\}\,. \tag{21}$$

If $v(z-K) = vK$, then $I_{z-K;K} = K$. If $v(z-K)$ is bounded from above, then $-v(z-K)$ is bounded from below and therefore, $I_{z-K;K}$ is a fractional $\mathcal{O}_K$-ideal and I_{z-K} is a fractional $\mathcal{O}_K$-ideal. We set $\mathcal{O}_{z-K;K} := \mathcal{O}(I_{z-K;K})$ (taken in (K,v)), and denote its maximal ideal by $\mathcal{M}_{z-K;K}$. Likewise, we set $\mathcal{O}_{z-K} := \mathcal{O}(I_{z-K})$ (taken in (L,v)), and denote its maximal ideal by $\mathcal{M}_{z-K}$. We see that by (21), I_{z-K} is definable in L in the language $\mathcal{L}_{\mathrm{val},K}$ with parameter z. Hence by (11) and (12), also the invariance valuation ring $\mathcal{O}_{z-K}$ and its maximal ideal $\mathcal{M}_{z-K}$ are $\mathcal{L}_{\mathrm{val},K}$-definable in L with parameter z.

Since z is not a parameter in K, we may in general not have an elementary definition of $\mathcal{O}_{z-K;K}$ and $I_{z-K;K}$ in (K,v). For example, we have not even excluded the case that z is transcendental over K. On the other hand, if z is algebraic over K with a suitable minimal polynomial, then the situation may change, as we will see in Sections 4.1 and 4.2.

Now we define

$$H_{z-K;K} \;:=\; H(\mathcal{O}_{z-K;K}) \;=\; H(\mathcal{O}(I_{z-K;K})) \tag{22}$$

and

$$H_{z-K} \;:=\; H(\mathcal{O}_{z-K}) \;=\; H(\mathcal{O}(I_{z-K}))\,. \tag{23}$$

By (17) and (13),

$$H_{z-K;K} \;=\; \mathcal{G}(vI_{z-K;K}) \;=\; \mathcal{G}(-v(z-K)) \;=\; \mathcal{G}(v(z-K))\,.$$

We observe that $H_{z-K;K}$ is a proper convex subgroup of vK if and only if $v(z-K)$ is bounded from above, and that this holds if and only if z does not lie in the completion of (K,v). If $H_{z-K;K}$ is not a proper convex subgroup, that is, $H_{z-K;K} = vK$, then $\mathcal{O}_{z-K;K} = K$, i.e., the corresponding valuation is trivial. Otherwise, this coarsening of $\mathcal{O}_K$ is nontrivial.

Now assume in addition that the extension $(L|K,v)$ is immediate. Then as mentioned in Section 2.1, every $z \in L\setminus K$ is an immediate element in $(L|K,v)$, and $v(z-K)$ is an initial segment of $vL = vK$. In this case, $H_{z-K;K}$ is also a convex subgroup in vL, and moreover,

$$vI_{z-K} \;=\; \{\alpha \in vL \mid \exists c \in K:\; \alpha \geq -v(z-c)\} \;=\; -v(z-K) \;=\; vI_{z-K;K}\,.$$

Using this together with (17), we obtain:

$$H_{z-K} \;=\; H(\mathcal{O}(I_{z-K})) \;=\; \mathcal{G}(vI_{z-K}) \;=\; \mathcal{G}(vI_{z-K;K}) \;=\; H(\mathcal{O}(I_{z-K;K})) \;=\; H_{z-K;K}\,.$$

4. Defect extensions of prime degree

Take a Galois defect extension $\mathcal{E} = (L|K,v)$ of prime degree p. We set

$$H_{\mathcal{E}} \;:=\; H(\mathcal{O}_{\mathcal{E}}) \;=\; H(\mathcal{O}(I_{\mathcal{E}}))\,.$$

By Lemma 2.2, $H_{\mathcal{E}}$ is the invariance group of $v\mathcal{O}_{\mathcal{E}}$ and of $v\mathcal{M}_{\mathcal{E}}$. By (17), Lemma 2.2 and the definition of $I_{\mathcal{E}}$,

$$H_{\mathcal{E}} \;=\; \mathcal{G}(vI_{\mathcal{E}}) \;=\; \mathcal{G}(\Sigma_{\mathcal{E}}) \;=\; \mathcal{G}(v\mathcal{O}_{\mathcal{E}}) \;=\; \mathcal{G}(v\mathcal{M}_{\mathcal{E}})\,. \tag{24}$$

Theorem 4.1. *For every Galois defect extension $\mathcal{E} = (L|K,v)$ of prime degree p, the following statements hold.*

1) The set Σ_σ is a final segment of $vK^{>0}$ and independent of the choice of a generator σ of Gal $L|K$.

2) For every $a \in L\setminus K$ and every generator σ of Gal $L|K$,

$$\Sigma_{\mathcal{E}} \;=\; -v(a-K) + v(\sigma a - a) \tag{25}$$

and

$$I_{\mathcal{E}} \;=\; (\sigma a - a)I_{a-K},\;\; H_{\mathcal{E}} \;=\; H_{a-K},\;\; \mathcal{O}_{\mathcal{E}} \;=\; \mathcal{O}_{a-K},\;\; \text{and}\;\; \mathcal{M}_{\mathcal{E}} \;=\; \mathcal{M}_{a-K}\,. \tag{26}$$

3) For every $a \in L\setminus K$,

$$H_{\mathcal{E}} \;=\; \mathcal{G}(v(a-K)) \;=\; \mathcal{G}(-v(a-K))\,. \tag{27}$$

Proof. 1): By [16, Theorem 3.5], Σ_σ is independent of the choice of a generator σ of Gal $L|K$; so we denote it by $\Sigma_{\mathcal{E}}$. By [16, Theorem 3.4], $\Sigma_{\mathcal{E}}$ is a final segment of $vK^{>0}$. Note that $vK^{>0} = vL^{>0}$ since $vK = vL$, as the extension $(L|K,v)$ is immediate.

2): Equation (25) is part of [16, Theorem 3.4]. It implies Equation (26) by way of the definitions of $I_{\mathcal{E}}$ and I_{a-K}, and Lemma 2.3.

3): From (24) we know that $H_{\mathcal{E}}$ is equal to $\mathcal{G}(\Sigma_{\mathcal{E}})$, and by (25) this is equal to $\mathcal{G}(-v(a-K)+v(\sigma a - a))$. Since $-v(a-K)+v(\sigma a - a)$ is a final segment of vK by part 1) of our theorem and $\alpha := v(\sigma a - a) \in vL$, we can infer from equation (13) that $\mathcal{G}(-v(a-K)+v(\sigma a - a)) = \mathcal{G}(-v(a-K))$. Finally, the equality $\mathcal{G}(v(a-K)) = \mathcal{G}(-v(a-K))$ follows from [17, Lemma 2.12 3)]. □

Based on part 2) of this theorem, we can now give the

Proof of part 1) of Proposition 1.1:
We have $\mathcal{O}_{\mathcal{E}} = \mathcal{O}(I_{a-K}) = \{b \in L \mid bI_{a-K} \subseteq I_{a-K}\} = \{b \in L \mid \forall c \in K\ \exists c' \in K : vb(a-c) = v(a-c')\} \cup \{0\}$, where we use that $v(a-K)$ is a final segment of vL. Since $\mathcal{O}(I_{a-K})$ does not depend on the choice of $a \in L \setminus K$, $\mathcal{O}_{\mathcal{E}}$ has the following parameter free definitions in the language $\mathcal{L}_{\mathrm{val},K}$:

$$\mathcal{O}_{\mathcal{E}} \;=\; \{b \in L \mid \forall x \in L \setminus K\ \forall c \in K\ \exists c' \in K :\ vb(x-c) = v(x-c')\} \cup \{0\}\,, \tag{28}$$

and the quantifier "$\forall x \in L \setminus K$" can also be replaced by "$\exists x \in L \setminus K$". Further, $\mathcal{M}_{\mathcal{E}} = \{b \in L \mid bI_{a-K} \subsetneq I_{a-K}\}$ has the following parameter free definition in the language $\mathcal{L}_{\mathrm{val},K}$:

$$\mathcal{M}_{\mathcal{E}} \;=\; \{b \in L \mid b \in \mathcal{O}_{\mathcal{E}} \wedge \exists c \in K\ \forall c' \in K :\ vb(x-c) \neq v(x-c')\}\,. \tag{29}$$

Let us show that our definitions (4) and (7) of independent defect are equivalent. By definition of $I_{\mathcal{E}}$ we have $\Sigma_{\mathcal{E}} = vI_{\mathcal{E}}$. By (14), $v\mathcal{M}_{\mathcal{E}} = \{\alpha \in vK \mid \alpha > H(\mathcal{O}_{\mathcal{E}})\} = \{\alpha \in vK \mid \alpha > H_{\mathcal{E}}\}$. Hence (7) reads as $vI_{\mathcal{E}} = v\mathcal{M}_{\mathcal{E}}$. Since the function $M \mapsto vM := \{va \mid a \in M\}$ that sends every $\mathcal{O}_L$-module $M \subseteq L$ to a corresponding final segment in vL is bijective, the latter equality is equivalent to the equality $I_{\mathcal{E}} = \mathcal{M}_{\mathcal{E}}$. Further, as $vK/H_{\mathcal{E}}$ is the value group of $\mathcal{O}_{\mathcal{E}}$, $vK/H_{\mathcal{E}}$ having no smallest positive element is equivalent to $\mathcal{M}_{\mathcal{E}}$ being a nonprincipal $\mathcal{O}_{\mathcal{E}}$-module.

Finally, we show that (7) is equivalent to the condition (6) in the original definition of independent defect in [16]. It is obvious that (7) implies the latter. For the converse, assume that $\Sigma_{\mathcal{E}} = \{\alpha \in vK \mid \alpha > H\}$ for some proper convex subgroup H of vL such that vL/H has no smallest positive element. Then by [17, Lemma 2.13 5)], H is the invariance group of $\Sigma_{\mathcal{E}}$, hence by (24), it is equal to $H_{\mathcal{E}}$.

While we have given elementary definitions of $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$, the problem with doing the same for $I_{\mathcal{E}}$ is that we may not have enough information on the factor $\sigma a - a$. We will now show that this changes when we know that the extension is an Artin-Schreier or a Kummer extension of prime degree. We will thereby prove part 2) of Proposition 1.1.

4.1. **The equal characteristic case.** Let us first discuss the case where (K,v) is of equal positive characteristic, that is, $\mathrm{char}\,K = \mathrm{char}\,Kv = p > 0$. Then every Galois defect extension $\mathcal{E} = (L|K,v)$ of prime degree p is an Artin-Schreier extension, that is, generated by an **Artin-Schreier generator** $\vartheta \in L \setminus K$ with $\vartheta^p - \vartheta \in K$. By [16, Theorem 3.5],

$$\Sigma_{\mathcal{E}} \;=\; -v(\vartheta - K)\,. \tag{30}$$

for every such ϑ. Further, $v(\sigma\vartheta - \vartheta) = 0$, hence

$$I_{\mathcal{E}} \;=\; I_{\vartheta-K} \;=\; \{b \in L \mid \exists c \in K :\ vb \geq -v(\vartheta - c)\}$$

in this case. Equation (30) shows that the set $v(\vartheta - K)$ does not depend on the choice of the Artin-Schreier generator of $L|K$, hence $I_{\mathcal{E}}$ has the following parameter free definitions in the language $\mathcal{L}_{\mathrm{val},K}$:

$$I_{\mathcal{E}} \;=\; \{b \in L \mid \exists x \in L \setminus K\ \exists c \in K :\ x^p - x \in K\ \wedge\ vb \geq -v(\vartheta - c)\} \tag{31}$$

and

$$I_{\mathcal{E}} \;=\; \{b\in L \mid \forall x\in L\setminus K\;\exists c\in K:\; x^p-x\in K \;\rightarrow\; vb\geq -v(\vartheta-c)\}\;. \tag{32}$$

Now assume that $\mathcal{E}$ has independent defect with associated convex subgroup $H_{\mathcal{E}}$. By [2, Theorem 1.7], this holds if and only if

$$v(\vartheta^p-\vartheta-\wp(K)) \;=\; \{\alpha\in pvK \mid \alpha<H_{\mathcal{E}}\}\;. \tag{33}$$

Since $vL/H_{\mathcal{E}}$ has no smallest positive element, equation (33) is equivalent to

$$H_{\mathcal{E}} \;=\; \{\beta\in vK \mid \beta>v(\vartheta^p-\vartheta-\wp(K)) \text{ and } -\beta>v(\vartheta^p-\vartheta-\wp(K))\}\;. \tag{34}$$

In other words,

$$H_{\mathcal{E}} \;=\; \{\pm\beta\in vK \mid \forall c\in K: v(\vartheta^p-\vartheta-c^p+c)<\beta\leq 0\}\;. \tag{35}$$

The convex subgroup $H_{\mathcal{E}}$ gives rise to an $\mathcal{L}_{\rm val}$-definition of the coarsening $\mathcal{O}_{\mathcal{E};K} = \mathcal{O}(H_{\mathcal{E}}) = \{b\in K \mid \exists\alpha\in H_{\mathcal{E}}:\; \alpha\leq vb\}$ (taken in K) of the valuation ring $\mathcal{O}_K$, namely

$$\mathcal{O}_{\mathcal{E};K} \;=\; \{b\in K \mid \forall c\in K: v(\vartheta^p-\vartheta-c^p+c)<vb\}\;, \tag{36}$$

whose value group is $vK/H_{\mathcal{E}}$.

For our applications, we are more interested in the coarsening of $\mathcal{O}_L$ corresponding to $H_{\mathcal{E}}$. By (16),

$$\mathcal{O}_{\mathcal{E}} \;=\; \mathcal{O}(H_{\mathcal{E}}) \;=\; \{b\in L \mid \exists\alpha\in H_{\mathcal{E}}:\; \alpha\leq vb\}\;. \tag{37}$$

By the definition of independent defect combined with Equation (30), if $\mathcal{E}$ has independent defect with associated convex subgroup $H_{\mathcal{E}}$, then

$$v(\vartheta-K) \;=\; \{\alpha\in vK \mid \alpha<H_{\mathcal{E}}\}\;. \tag{38}$$

Since this does not depend on the choice of the Artin-Schreier generator of $L|K$, $\mathcal{O}_{\mathcal{E}}$ has the following parameter free definitions in the language $\mathcal{L}_{{\rm val},K}$:

$$\mathcal{O}_{\mathcal{E}} \;=\; \{b\in L \mid \forall x\in L\setminus K\;\; \forall c\in K:\; x^p-x\in K \;\rightarrow\; v(x-c)<vb\} \tag{39}$$

and

$$\mathcal{O}_{\mathcal{E}} \;=\; \{b\in L \mid \exists x\in L\setminus K\;\; \forall c\in K:\; x^p-x\in K \;\wedge\; v(x-c)<vb\}\;. \tag{40}$$

Also the maximal ideal $\mathcal{M}_{\mathcal{E}}$ of $\mathcal{O}_{\mathcal{E}}$ has a parameter free definition in the language $\mathcal{L}_{{\rm val},K}$:

$$\mathcal{M}_{\mathcal{E}} \;=\; \{b\in L \mid \exists x\in L\setminus K\;\exists c\in K:\; x^p-x\in K \;\wedge\; -v(x-c)\leq vb\}\;. \tag{41}$$

4.2. **The mixed characteristic case.** Now we discuss the case where (K,v) is of mixed characteristic, that is, $\mathrm{char}\,K = 0$ and $\mathrm{char}\,Kv = p > 0$. We assume in addition that K contains a primitive p-th root of unity ζ_p. Then every Galois defect extension $\mathcal{E} = (L|K,v)$ of prime degree p is a **Kummer extension**, that is, generated by a **Kummer generator** $\eta \in L \setminus K$ with $\eta^p \in K$. Then by [16, Theorem 3.5] and [16, Lemma 2.5],

$$\Sigma_{\mathcal{E}} \;=\; v(\zeta_p - 1) - v(\eta - K) \;=\; \frac{1}{p-1} vp - v(\eta - K) \tag{42}$$

for every such η. Further, $\sigma\eta - \eta = \zeta_p - 1$ for suitable ζ_p, hence

$$I_{\mathcal{E}} \;=\; (\zeta_p - 1) I_{\vartheta - K} \;=\; \{b \in L \mid \exists c \in K :\, vb \geq \frac{1}{p-1} vp - v(\vartheta - c)\}$$

in this case. Similarly as in the equal characteristic case, $I_{\mathcal{E}}$ has the following parameter free definitions in the language $\mathcal{L}_{\mathrm{val},K}$:

$$I_{\mathcal{E}} \;=\; \{b \in L \mid \exists x \in L \setminus K\ \exists c \in K :\, x^p - x \in K \,\wedge\, vb \geq \frac{1}{p-1} vp - v(x - c)\} \tag{43}$$

and

$$I_{\mathcal{E}} \;=\; \{b \in L \mid \forall x \in L \setminus K\ \exists c \in K :\, x^p - x \in K \,\rightarrow\, vb \geq \frac{1}{p-1} vp - v(x - c)\}\,. \tag{44}$$

Now assume that $\mathcal{E}$ has independent defect with associated convex subgroup $H_{\mathcal{E}}$. By [2, Theorem 1.7], this holds if and only if

$$v(\eta^p - K^p) \;=\; v(\zeta_p - 1)^p + \{\alpha \in pvK \mid \alpha < H_{\mathcal{E}}\}\,. \tag{45}$$

Similarly as in the equal characteristic case, we obtain that

$$H_{\mathcal{E}} \;=\; \{\pm\beta \in vK \mid \forall c \in K : v(\eta^p - c^p) - \frac{p}{p-1} vp < \beta \leq 0\}\,, \tag{46}$$

and we have the $\mathcal{L}_{\mathrm{val}}$-definition

$$\mathcal{O}_{\mathcal{E};K} \;:=\; \{b \in K \mid \forall c \in K : v(\eta^p - c^p) - \frac{p}{p-1} vp < vb\}\,. \tag{47}$$

Note that for this definition and definition (36) it is not needed that (K,v) be henselian, and that in fact, they will be applied to deeply ramified fields, which are not required to be henselian. For the case of henselian fields (K,v), these definitions are used in [6, Theorem 4.11] to define corresponding henselian valuations on K that are definable in the language of rings.

By the definition of independent defect combined with Equation (42), if $\mathcal{E}$ has independent defect with associated convex subgroup $H_{\mathcal{E}}$, then

$$v(\vartheta - K) - v(\zeta_p - 1) \;=\; \{\alpha \in vK \mid \alpha < H_{\mathcal{E}}\}\,. \tag{48}$$

Hence in this case, using again (37) together with the fact that equation (48) is independent of the choice of $\eta \in L \setminus K$ satisfying $\eta^p \in K$), we can give the following parameter free $\mathcal{L}_{\mathrm{val},K}$-definitions of $\mathcal{O}_{\mathcal{E}}$:

$$\begin{aligned}\mathcal{O}_{\mathcal{E}} \;=\; \{b \in L \;\mid\; & \forall x \in L \setminus K\ \ \forall c \in K :\, (x^p \in K \,\wedge\, v(x-1) > 0) \\ & \rightarrow\, v(x - c) - \frac{1}{p-1} vp < vb\}\end{aligned}$$

and

$$\mathcal{O}_{\mathcal{E}} \;=\; \{b\in L \;\mid\; \exists x\in L\setminus K : x^p\in K \,\wedge\, v(x-1)>0 \\ \wedge\, \forall c\in K : v(x-c)-\frac{1}{p-1}vp < vb\}\,.$$

Also the maximal ideal of $\mathcal{O}_{\mathcal{E}}$ admits a parameterfree $\mathcal{L}_{\mathrm{val},K}$-definition:

$$\mathcal{M}_{\mathcal{E}} \;=\; \{b\in L \;\mid\; \exists x\in L\setminus K : x^p\in K \,\wedge\, v(x-1)>0 \\ \wedge\, \exists c\in K : -v(x-c)+\frac{1}{p-1}vp \le vb\}\,.$$

4.3. **Properties and applications of $I_{\mathcal{E}}$, $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$.** We keep our assumption that $\mathcal{E}=(L|K,v)$ is a Galois defect extension of prime degree p.

Equations (31), (32), (43) and (44) prove part 2) of Theorem 1.1.

The following facts are proven in [2]. Part 1) follows directly from our definition $\mathcal{O}_{\mathcal{E}}=\mathcal{O}(I_{\mathcal{E}})$ in the introduction together with Proposition 2.1 which implies the assertion. However, in [2], under the additional assumption that K contains a primitive p-th root of unity if $\operatorname{char} K=0$, $\mathcal{O}_{\mathcal{E}}$ is defined in a different way, and our assertion is part of [2, Theorem 1.4], as is part 2).

Proposition 4.2. *Take a Galois extension $\mathcal{E}=(L|K,v)$ of prime degree p with independent defect.*

1) The ideal $\mathcal{M}_{\mathcal{E}}$ is equal to the ramification ideal $I_{\mathcal{E}}$, and $\mathcal{O}_{\mathcal{E}}$ is the largest of all coarsenings $\mathcal{O}'$ of $\mathcal{O}_L$ such that $I_{\mathcal{E}}$ is an $\mathcal{O}'$-ideal.

2) If $\operatorname{char} K=0$, then assume in addition that K contains a primitive p-th root of unity. Then the trace $\operatorname{Tr}_{L|K}(\mathcal{M}_L)$ is equal to $\mathcal{M}_{\mathcal{E}}\cap K$.

The valuation ring $\mathcal{O}_{\mathcal{E}}$ is of interest for the computation of the annihilator of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$. The annihilator of an $\mathcal{O}_L$-module M is the largest among all $\mathcal{O}_L$-ideals J for which $JM=\{0\}$; we denote it by $\operatorname{ann} M$. From [2, Theorem 1.4] we know that

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;\cong\; I_{\mathcal{E}}/I_{\mathcal{E}}^p\,,$$

which is zero if and only if $\mathcal{E}$ has independent defect; in this case, $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K}=\mathcal{O}_L$. For the case of dependent defect, we infer from [2, Proposition 4.7 2)], denoting by $v_{\mathcal{E}}$ the valuation on L having valuation ring $\mathcal{O}_{\mathcal{E}}$:

Proposition 4.3. *If there is $a\in K$ such that $v_{\mathcal{E}}I_{\mathcal{E}}^{p-1}$ has infimum $v_{\mathcal{E}}a$ in $v_{\mathcal{E}}L$ but does not contain this infimum, then*

$$\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;=\; a\,\mathcal{O}(I_{\mathcal{E}})\,, \tag{49}$$

which properly contains $I_{\mathcal{E}}^{p-1}$. In all other cases, $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K}=I_{\mathcal{E}}^{p-1}$.

5. Defectless extensions

We take an extension $\mathcal{E} = (L|K,v)$ of prime degree q, not necessarily equal to $\mathrm{char}\, Kv$. Then either $[L:K] = (vL:vK)$ or $[L:K] = [Lv:Kv]$. *We will discuss the more interesting case of* $[L:K] = (vL:vK)$*, which we will assume throughout.*

We define $H_\mathcal{E}$ to be the largest convex subgroup of vL which is also a convex subgroup of vK; it exists since unions over arbitrary collections of convex subgroups are again convex subgroups. We take $\mathcal{O}_\mathcal{E}$ to be the coarsening $\mathcal{O}(H_\mathcal{E})$ of $\mathcal{O}_L$ associated with $H_\mathcal{E}$ so that its value group is $vL/H_\mathcal{E}$, and denote its maximal ideal by $\mathcal{M}_\mathcal{E}$.

The subgroup $H_\mathcal{E}$ defined here has important similarities with the convex subgroup $H_\mathcal{E}$ defined in the defect case.

We distinguish three mutually exclusive cases describing how vK extends to vL; for convenience, we use the notation of [3]:

(DL2a): there is no smallest convex subgroup of vL that properly contains $H_\mathcal{E}$;

(DL2b): there is a smallest convex subgroup $\tilde{H}_\mathcal{E}$ of vL that properly contains $H_\mathcal{E}$, and the archimedean quotient $\tilde{H}_\mathcal{E}/H_\mathcal{E}$ is dense;

(DL2c): there is a smallest convex subgroup $\tilde{H}_\mathcal{E}$ of vL that properly contains $H_\mathcal{E}$, and the archimedean quotient $\tilde{H}_\mathcal{E}/H_\mathcal{E}$ is discrete.

Our goal is to find an element $x \in L$ with $vx \notin vK$ such that

$$\mathcal{O}_L \;=\; \bigcup_{c\in K \text{ with } vcx>0} \mathcal{O}_K[cx]\,. \tag{50}$$

If $c,c' \in K$ with $vc \geq vc'$, then $cx = \frac{c}{c'}c'x \in \mathcal{O}_K[c'x]$, hence $\mathcal{O}_K[cx] \subseteq \mathcal{O}_K[c'x]$.

Theorem 5.1. [3, Theorem 3.3] *Take an extension $\mathcal{E} = (L|K,v)$ of prime degree $q = (vL:vK)$, with $x_0 \in L$ such that $vx_0 \notin vK$. Then $qvx_0 \in vK$, and the following assertions hold.*

1) If $\mathcal{E}$ is of type (DL2a) or (DL2b), then (50) holds for $x = x_0$.

2) If $\mathcal{E}$ is of type (DL2c), then (50) holds for $x = x_0^j$ with suitable $j \in \{1,\ldots,q-1\}$. If in addition $H_\mathcal{E} = \{0\}$, then $\mathcal{O}_L = \mathcal{O}_K[cx]$ for suitable $c \in K$.

The assumption of part 1) holds in particular when every archimedean component of vK is dense, and this in turn holds for every deeply ramified field (K,v).

With x as in this theorem, we have:

Proposition 5.2. [3, Proposition 3.4] *The $\mathcal{O}_L$-ideal $\mathcal{M}_\mathcal{E}$ is equal to the $\mathcal{O}_L$-ideal*

$$I_x \;:=\; (cx \mid c \in K \text{ with } vcx > 0)\,. \tag{51}$$

Corollary 5.3. *The set $\{vcx \mid c \in K \text{ with } vcx > 0\}$ is coinitial in $vK^{>0} \setminus H_\mathcal{E}$.*

Lemma 5.4. *For every x_0 with $vx_0 \notin vK$ we have $I_{x_0} \subseteq I_x$.*

Proof. Take $c_0 \in K$ such that $vc_0x_0 > 0$, so that $c_0x_0 \in \mathcal{O}_L$. If x is as in Theorem 5.1, then there is $c \in K$ such that $c_0x_0 \in \mathcal{O}_K[cx]$. Consequently, $c_0x_0 \in I_x$. This proves that $I_{x_0} \subseteq I_x$. $\square$

From this lemma together with Proposition 5.2 we obtain the following parameter free $\mathcal{L}_{\mathrm{val},K}$-definition of $\mathcal{M}_\mathcal{E}$:

$$\mathcal{M}_\mathcal{E} = \{b \in L \mid \exists x \in L\setminus K : (\forall y \in K : vx \neq vy) \wedge \exists c \in K : \ va \geq vcx > 0\}. \tag{52}$$

From this, we can define $\mathcal{O}_\mathcal{E}$ by including the units of $\mathcal{O}_\mathcal{E}$:

$$\mathcal{O}_\mathcal{E} \;=\; \{b \in L \mid \forall x \in \mathcal{M}_\mathcal{E} : \ -vx < vb\}\,.$$

5.1. **The ramification ideal.** Take a unibranched defectless Galois extension $\mathcal{E} = (L|K,v)$ of prime degree $p = (vL : vK) = \mathrm{char}\, Kv$. We denote by $I_\mathcal{E}$ the ramification ideal of $\mathcal{E}$. From [14, Theorem 3.15] we obtain:

Theorem 5.5. *1) If $\mathcal{E}$ is an Artin-Schreier extension, then it admits an Artin-Schreier generator ϑ of value $v\vartheta \leq 0$ such that $v\vartheta \notin vK$. For every such ϑ,*

$$I_\mathcal{E} \;=\; \left(\frac{1}{\vartheta}\right)\,. \tag{53}$$

2) Let $\mathcal{E}$ be a Kummer extension. Then there are two cases:

a) $\mathcal{E}$ admits a Kummer generator η such that $0 < v\eta \notin vK$. For every such η,

$$I_\mathcal{E} \;=\; (\zeta_p - 1)\,. \tag{54}$$

b) $\mathcal{E}$ admits a Kummer generator η such that η is a 1-unit with $v(\zeta_p - 1) \geq v(\eta - 1) \notin vK$. For every such η,

$$I_\mathcal{E} \;=\; \left(\frac{\zeta_p - 1}{\eta - 1}\right)\,. \tag{55}$$

Let us show that under the assumptions of the theorem, $I_\mathcal{E}$ always has a parameter free $\mathcal{L}_{\mathrm{val},K}$-definition. If $\mathcal{E}$ is an Artin-Schreier extension, then we can define

$$\begin{aligned} I_\mathcal{E} \;:=\; \{\, b \in L \;\mid\; & \exists x \in L : \ x^p - x \in K \;\wedge\; vx \leq 0 \\ & \wedge (\forall y \in K : \ vx \neq vy) \;\wedge\; vb \geq vx \,\}\,. \end{aligned}$$

If $\mathcal{E}$ is a Kummer extension, then in case 2)a) of the theorem, we have $v\eta > 0$ and therefore, $v(\eta - 1) = 0$. Thus, we can also in this case use (55) for the definition of $I_\mathcal{E}$:

$$\begin{aligned} I_\mathcal{E} \;:=\; \{\, b \in L \;\mid\; & \exists x \in L : \ x^p \in K \;\wedge\; vp \geq (p-1)v(x-1) \\ & \wedge (\forall y \in K : \ 0 < vx \neq vy \;\vee\; 0 < v(x-1) \neq vy) \\ & \wedge (p-1)vb \geq vp - (p-1)v(x-1) \,\}\,. \end{aligned}$$

5.2. **Importance of the ideals $I_{\mathcal{E}}$, $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$.** We will now summarize the results for defectless Galois extensions $\mathcal{E} = (L|K,v)$ which will demonstrate the importance of the ideals $I_{\mathcal{E}}$, $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$. If $[L:K] = q \neq \mathrm{char}\, K$, then we will assume that K contains a q-th root of unity.

Theorem 5.6. [3, Theorem 4.6] *Take an Artin-Schreier extension $\mathcal{E} = (L|K,v)$ of degree $p = (vL : vK)$. Then*

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}}/(I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^p \tag{56}$$

as $\mathcal{O}_L$-modules; in particular, $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$.

The following is a reformulation of [3, Theorem 4.6].

Theorem 5.7. *Let $\mathcal{E} = (L|K,v)$ be a Kummer extension of prime degree q with $\mathrm{e}(L|K) = q$.*

If $q \neq \mathrm{char}\, Kv$, then

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong \mathcal{M}_{\mathcal{E}}/\mathcal{M}_{\mathcal{E}}^q \tag{57}$$

as $\mathcal{O}_L$-modules.

If $q = \mathrm{char}\, Kv$, then

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}}/(I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^q \tag{58}$$

as $\mathcal{O}_L$-modules.

In case 2)a) of Theorem 5.5, we have that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ if and only if $q \notin \mathcal{M}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$ is a nonprincipal $\mathcal{O}_{\mathcal{E}}$-ideal. The condition $q \notin \mathcal{M}_{\mathcal{E}}$ always holds when $q \neq \mathrm{char}\, Kv$.

In case 2)b) of Theorem 5.5, we always have that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$.

Let us compute the annihilators of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ in the above cases whenever it is nonzero. The following is Poposition 3.21 of [17], adapted to our current notation.

Proposition 5.8. *Take $n \geq 2$, $a \in \mathcal{O}_L$ and $\mathcal{O}$ a valuation ring containing $\mathcal{O}_L$ with maximal ideal $\mathcal{M}$. Assume that $(a\mathcal{M})^n \neq a\mathcal{M}$. Then the following statements hold.*

1) We have that

$$\mathrm{ann}\, a\mathcal{M}/(a\mathcal{M})^n = \begin{cases} (a\mathcal{M})^{n-1} & \text{if } \mathcal{M} \text{ is a principal } \mathcal{O}\text{-ideal,} \\ (a\mathcal{O})^{n-1} = a^{n-1}\mathcal{O} & \text{if } \mathcal{M} \text{ is a nonprincipal } \mathcal{O}\text{-ideal.} \end{cases}$$

2) The annihilator is equal to $\mathcal{M}_L$ if and only if $n = 2$, $a \notin \mathcal{M}_L = \mathcal{M}$ and $\mathcal{M}_L$ is a principal $\mathcal{O}_L$-ideal.

Since $I_{\mathcal{E}}$ is a principal $\mathcal{O}_L$-ideal, we can choose $a \in \mathcal{O}_L$ such that $I_{\mathcal{E}} = a\mathcal{O}_L$ to obtain that

$$I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}} = a\mathcal{M}_{\mathcal{E}}\,.$$

Now we apply Proposition 5.8.

Proposition 5.9. *Let $\mathcal{E}$ be an Artin-Schreier extension or a Kummer extension of degree $p = \mathrm{char}\, Kv$. Assume that $[L:K] = (vL : vK)$ and that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$. Then*

$$\mathrm{ann}\, \Omega_{\mathcal{O}_L|\mathcal{O}_K} = \begin{cases} (a\mathcal{M}_{\mathcal{E}})^{p-1} & \text{if } \mathcal{M}_{\mathcal{E}} \text{ is a principal } \mathcal{O}_{\mathcal{E}}\text{-ideal,} \\ (a\mathcal{O}_{\mathcal{E}})^{p-1} = a^{p-1}\mathcal{O}_{\mathcal{E}} & \text{if } \mathcal{M}_{\mathcal{E}} \text{ is a nonprincipal } \mathcal{O}_{\mathcal{E}}\text{-ideal.} \end{cases}$$

Further, $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{M}_L$ *if and only if* $p = 2$, $a \notin \mathcal{M}_L = \mathcal{M}_{\mathcal{E}}$ *and* $\mathcal{M}_L$ *is a principal* $\mathcal{O}_L$*-ideal.*

Let us note that if (DRvp) holds (and in particular, if (K, v) is a deeply ramified field), then the maximal ideal of any coarsening of $\mathcal{O}_L$ is never principal. In this case, $\mathcal{M}_L$ is never the annihilator of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$.

In the case of a Kummer extension of prime degree $q = (vL : vK) \neq \operatorname{char} Kv$, (57) holds, and we set $a = 1$. Then we obtain from Theorem 5.7 and Proposition 5.8:

Proposition 5.10. *Let* $\mathcal{E}$ *be a Kummer extension of degree* $q = (vL : vK) \neq \operatorname{char} Kv$. *Assume that* $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ *is nonzero. Then* $\mathcal{M}_{\mathcal{E}}$ *is a principal* $\mathcal{O}_{\mathcal{E}}$*-ideal, and*

$$\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;=\; \mathcal{M}_{\mathcal{E}}^{q-1}\,.$$

Further, $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{M}_L$ *if and only if* $q = 2$ *and* $\mathcal{M}_{\mathcal{E}} = \mathcal{M}_L$.

6. Deeply ramified fields in equal characteristic with prescribed associated convex subgroups

6.1. **Preliminaries from ramification theory.** An algebraic extension $(L|K, v)$ of a henselian valued field (K, v) is called **tame** if every finite subextension $K'|K$ satisfies the following conditions:

(T1) the ramification index $(vK' : vK)$ is not divisible by $\operatorname{char} Kv$,

(T2) the residue field extension $K'v|Kv$ is separable,

(T3) the extension $(K'|K, v)$ is defectless.

A henselian valued field (K, v) is called a **tame field** if the algebraic closure K^{ac} of K with the unique extension of v is a tame extension of (K, v). It follows from conditions (T1)–(T3) that all tame fields are perfect defectless fields. For the algebra and model theory of tame fields, see [11].

The **ramification field** of a Galois extension $(L|K, v)$ with Galois group $G = \operatorname{Gal}(L|K)$ is the fixed field in L of the **ramification group**

$$G^r \;:=\; \left\{\sigma \in G \;\middle|\; \frac{\sigma b - b}{b} \in \mathcal{M}_L \;\text{ for all } b \in L^\times \right\}. \tag{59}$$

When dealing with a valued field (K, v), we will tacitly assume v extended to its algebraic closure. Then the **absolute ramification field of** (K, v) (with respect to the chosen extension of v), denoted by (K^r, v), is the ramification field of the Galois extension $(K^{\mathrm{sep}}|K, v)$. If $(K(a)|K, v)$ is finite and a defect extension, then $(K^r(a)|K^r, v)$ is a defect extension with the same defect (see [16, Proposition 2.13]). On the other hand, $K^{\mathrm{sep}}|K^r$ is a p-extension (see [4, Theorem (20.18)]), so every finite extension of K^r is a tower of purely inseparable extensions and Galois extensions of degree p. If (K, v) is henselian, then (K^r, v) is its unique maximal tame extension (see [15, Proposition 4.1]). Hence the next fact follows from [16, Proposition 2.13]:

Lemma 6.1. *If* (K, v) *is henselian,* $(K(a)|K, v)$ *is finite and a defect extension, and* $(L|K, v)$ *is a tame extension, then* $d(L(a)|L, v) = d(K(a)|K, v)$.

An extension (L,v) of a henselian field (K,v) is called **purely wild** if every finite subextension $(L_0|K,v)$ satisfies:

(a) $(vL_0 : vK)$ is a power of the characteristic exponent of Kv,

(b) $L_0v|Kv$ is purely inseparable.

The extension (L,v) of (K,v) is purely wild if and only if it is linearly disjoint from K^r over K (see [15, Lemma 4.2]).

Lemma 6.2. *Every maximal purely wild extension of a henselian field is a tame field.*

Proof. By [15, Theorem 4.3], every maximal purely wild extension W of a henselian field (K,v) is a K-complement of K^r, that is, $W\cap K^r=K$ and $W.K^r=K^{\mathrm{ac}}$. By [15, Lemma 2.1 (i)], there is also a W-complement W' of W^r. Again by [15, Theorem 4.3], W' is a maximal purely wild extension of W. By the maximality of W, we must have $W'=W$. Hence $K^{\mathrm{ac}}=W'.W^r=W^r$, which shows that W is a tame field. □

6.2. **Technical preliminaries.** For the following result, see [1, Lemma 4.1] (cf. also [8, Lemma 2.21]):

Lemma 6.3. *Assume that $(K(a)|K,v)$ is a unibranched extension of prime degree such that $v(a-K)$ has no maximal element. Then the extension $(K(a)|K,v)$ is immediate and hence a defect extension.*

Lemma 6.4. *1) Let (K_0,v) be a valued field of characteristic $p>0$ whose value group is not p-divisible. Take $a\in K_0$ such that $va<0$ is not divisible by p. Let ϑ be a root of the Artin–Schreier polynomial X^p-X-a. Then $(K_0^{1/p^\infty}(\vartheta)|K_0^{1/p^\infty},v)$ is a defect extension with independent defect, and $v(\vartheta-K_0^{1/p^\infty})\subseteq(vK_0^{1/p^\infty})^{<0}$.*

2) Take a perfect field k of characteristic $p>0$ and K_0 to be $k(t)$, $k(t)^h$ or $k((t))$, equipped with the t-adic valuation $v=v_t$. Let ϑ be a root of the Artin–Schreier polynomial X^p-X-1/t. Then the assertion of part 1) holds, and

$$v(\vartheta-K_0^{1/p^\infty})\;=\;(vK_0^{1/p^\infty})^{<0}\;.$$

Proof. 1): We have that $v\vartheta=va/p$ and $[K_0(\vartheta):K_0]=p=(vK_0(\vartheta):vK_0)$. The Fundamental Inequality (cf. (17.5) of [4] or Theorem 19 on p. 55 of [18] shows that $K_0(\vartheta)v=K_0v$ and that the extension $(K_0(\vartheta)|K_0,v)$ is unibranched. The further extension of v to the perfect hull

$$K_0(\vartheta)^{1/p^\infty}=K_0^{1/p^\infty}(\vartheta)$$

is unique, as the extension is purely inseparable. It follows that also the extension $(K_0^{1/p^\infty}(\vartheta)|K_0^{1/p^\infty},v)$ is unibranched. On the other hand, $[K_0^{1/p^\infty}(\vartheta):K_0^{1/p^\infty}]=p$ since the separable extension $K_0(\vartheta)|K_0$ is linearly disjoint from $K_0^{1/p^\infty}|K_0$. The value group $vK_0^{1/p^\infty}(\vartheta)=vK_0(\vartheta)^{1/p^\infty}$ is the p-divisible hull of $vK_0(\vartheta)=vK_0+\mathbb{Z}v\vartheta$. Since $pv\vartheta\in vK$, this is the same as the p-divisible hull of vK_0, which in turn is equal to vK_0^{1/p^∞}. The residue field of $K_0^{1/p^\infty}(\vartheta)$ is the perfect hull of $K_0(\vartheta)v=K_0v$. Hence it is equal to the residue field of K_0^{1/p^∞}. It follows that the extension $(K_0^{1/p^\infty}(\vartheta)|K_0^{1/p^\infty},v)$ is immediate and that its defect is p, equal to its degree. Since K_0^{1/p^∞} is perfect, it is deeply ramified and hence according to

[16, part (1) of Theorem 1.10] the extension must have independent defect. The inclusion $v(\vartheta - K_0^{1/p^\infty}) \subseteq (vK)^{<0}$ follows from [8, Corollary 2.30].

2): In all three cases we have that $K_0^{1/p^\infty} = K_0(t^{1/p^k} \mid k \in \mathbb{N})$. For the partial sums

$$b_k \;:=\; \sum_{i=1}^{k} t^{-1/p^i} \;\in\; K_0^{1/p^\infty} \tag{60}$$

we have

$$(\vartheta - b_k)^p - (\vartheta - b_k) \;=\; \vartheta^p - \vartheta - b_k^p + b_k \;=\; \frac{1}{t} - \sum_{i=0}^{k-1} t^{-1/p^i} + \sum_{i=1}^{k} t^{-1/p^i} \;=\; t^{-1/p^k}\;,$$

so

$$v(\vartheta - b_k) \;=\; -\frac{1}{p^{k+1}} \;<\; 0\;.$$

Suppose that there is $c \in K_0^{1/p^\infty}$ such that $v(\vartheta - c) > -1/p^k$ for all k. Then $v(c-b_k) = \min\{v(\vartheta - c), v(\vartheta - b_k)\} = -1/p^{k+1}$ for all k. On the other hand, there is some k such that $c \in K_0(t^{-1/p}, \ldots, t^{-1/p^k}) = K_0(t^{-1/p^k})$. But this contradicts the fact that $v(c - t^{-1/p} - \ldots - t^{-1/p^k}) = v(c - b_k) = -1/p^{k+1} \notin vK_0(t^{-1/p^k})$. As $v(\vartheta - K_0^{1/p^\infty}) \subseteq (vK_0^{1/p^\infty})^{<0}$ by part 1), this proves that the values $-1/p^k$ are cofinal in $v(\vartheta - K_0^{1/p^\infty})$. Since vK_0^{1/p^∞} is a subgroup of the rationals, this shows that the least upper bound of $v(\vartheta - K_0^{1/p^\infty})$ in vK_0^{1/p^∞} is the element 0. As $v(\vartheta - K_0^{1/p^\infty})$ is an initial segment of vK_0^{1/p^∞} by [8, Lemma 2.19], we conclude that $v(\vartheta - K_0^{1/p^\infty}) = (vK_0^{1/p^\infty})^{<0}$. □

When we take $K_0 = \mathbb{F}_p((t))$ in part 1) of this lemma, where $\mathbb{F}_p$ is the field with p elements, and $a = 1/t$, we obtain "Abhyankar's Example", see [9, Example 3.12].

Lemma 6.5. *Take a valued field (K, v) of characteristic $p > 0$, a decomposition $v = w \circ \bar{w}$, and an Artin-Schreier extension of K with Artin-Schreier generator ϑ. Then the following assertions hold.*

1) $(K(\vartheta)|K, v)$ is a defect extension with $v(\vartheta - K) = \{\alpha \in vK(\vartheta) \mid \alpha < \bar{w}(K(\vartheta)w)\}$ if and only if $(K(\vartheta)|K, w)$ is a defect extension with $w(\vartheta - K) = (wK(\vartheta))^{<0}$.

2) If $w\vartheta = 0$ and $\bar{w}(\vartheta w - Kw) = (\bar{w}(K(\vartheta)w))^{<0}$, then $(K(\vartheta)|K, v)$ is a defect extension with $v(\vartheta - K) = (vK(\vartheta))^{<0}$.

Proof. 1): We can write $wK(\vartheta) = vK(\vartheta)/\bar{w}(K(\vartheta)w)$ and $wa = va + \bar{w}(K(\vartheta)w)$ for each $a \in K(\vartheta)$. This implies that $v(\vartheta - K) = \{\alpha \in vK \mid \alpha < \bar{w}(K(\vartheta)w)$ if and only if $w(\vartheta - K) = (wK(\vartheta))^{<0}$. By Lemma 6.3, $(K(\vartheta)|K, v)$ is a defect extension if $v(\vartheta - K)$ is a subset of $vK(\vartheta)$ without maximal element, and similarly for w in place of v. This fact together with the equivalence we have already shown proves the assertion of our lemma.

2): Our assumption implies that $v(\vartheta - K) \subseteq (vK(\vartheta))^{<0}$ since if there is $c \in K$ such that $v(\vartheta - c) \geq 0$, then $vc = v\vartheta$, so $wc = w\vartheta = 0$, and $\bar{w}(\vartheta w - cw) \geq 0$. On the other hand, $\bar{w}(Kw)$ is a convex subgroup of vK, so $(\bar{w}(Kw))^{<0}$ and thus also $\bar{w}(\vartheta w - Kw)$ is cofinal in $vK^{<0}$. Since for every $b \in Kw$ there is $c \in K$ with $cw = b$ and $v(\vartheta - c) = \bar{w}(\vartheta w - cw)$, it follows that $v(\vartheta - K) = (vK(\vartheta))^{<0}$. Again by Lemma 6.3, $(K(\vartheta)|K, v)$ is a defect extension. □

Proposition 6.6. *Take any perfect field K of characteristic $p > 0$ and a field L_1 of characteristic 0 carrying a p-adic valuation v_p such that $v_pL_1 = \mathbb{Z}v_pp$ and $L_1v_p = K$. Set $a_0 := p$ and by induction, choose elements a_i in the algebraic closure L_1^{ac} of L_1 such that $a_i^p = a_{i-1}$ for $i \in \mathbb{N}$, and set*

$$L_2 := L_1(a_i \mid i \in \mathbb{N}) .$$

Further, take $a \in \mathbb{Q}^{\mathrm{ac}} \subseteq L_2^{\mathrm{ac}}$ such that

$$a^p - a = \frac{1}{p} . \tag{61}$$

Then the following assertions hold.

1) There is a unique extension of the valuation v_p to L_2.

2) (L_2, v_p) is a deeply ramified field with value group $v_pL_2 = \frac{1}{p^\infty}\mathbb{Z}v_pp$ and residue field $L_2v_p = K$.

3) $(L_2(a)|L_2, v_p)$ is a defect extension of degree p.

4) Assume that $\mathbb{F}_p^{\mathrm{ac}} \subseteq K$ and there is $L_0 \subseteq L_1$ such that (L_0, v_p) is henselian with $L_0v_p = \mathbb{F}_p^{\mathrm{ac}}$. Then there is a finite extension $(L|L_2, v_p)$ such that $Lv_p = L_2v_p = K$, (L, v_p) is a deeply ramified field, and $(L(a)|L, v_p)$ is a Galois defect extension of degree p with independent defect and associated convex subgroup $\{0\}$.

Proof. By our choice of the a_i, $\frac{v_pp}{p^i} = v_pa_i \in v_pL_1(a_i)$. Therefore,

$$p^i \le (v_pL_1(a_i) : v_pL_1) \le (v_pL_1(a_i) : v_pL_1)[L_1(a_i)v_p : L_1v_p] \le [L_1(a_i) : L_1] \le p^i .$$

Hence equality holds everywhere, and $[L_1(a_i)v_p : L_1v_p] = 1$. We thus obtain that $v_pL_1(a_i) = \frac{1}{p^i}v_pL_1$ and $L_1(a_i)v_p = L_1v_p$. Consequently,

$$v_pL_2 = \bigcup_{i\in\mathbb{N}} v_pL_1(a_i) = \frac{1}{p^\infty}\mathbb{Z} \quad \text{and} \quad L_2v_p = L_1v_p = K ,$$

and the extension $(L_2|L_1, v_p)$ is unibranched, which proves assertion 1). We see that v_pL_2 is p-divisible, so (L_2, v_p) satisfies (DRvg). In order to show that (L_2, v_p) is a deeply ramified field it remains to show that it satisfies (DRvr).

Take $b \in \mathcal{O}_{L_2}$. Then $b \in L_1(a_i)$ for some $i \in \mathbb{N}$ and we can write

$$b = \sum_{j=0}^{p^i-1} c_j a_i^j$$

with $c_j \in L_1$. Since the values $v_pa_i^j$, $0 \le j \le p^i-1$ lie in distinct cosets modulo v_pL_1 (hence the elements a_i^j, $0 \le j \le p^i - 1$ form a valuation basis of $(L_1(a_i)|L_1, v_p)$), we have that $v_pb = \min_{0\le j\le p^i-1} v_pc_ja_i^j$. As $b \in \mathcal{O}_{L_2}$, it follows that $c_ja_i^j \in \mathcal{O}_{L_2}$ for all j. We observe that $v_pa_i^j \le \frac{p^i-1}{p^i}v_pp < v_pp$, so $v_pc_j \in \mathbb{Z}v_pp$ cannot be negative. This shows that $c_j \in \mathcal{O}_{L_1}$ for all j.

For every $c \in \mathcal{O}_{L_1}$ there is $d \in \mathcal{O}_{L_1}$ such that $c \equiv d^p \mod p\mathcal{O}_{L_1}$; indeed, as K is perfect, there is $\xi \in K$ such that $cv_p = \xi^p$, so we can choose $d \in \mathcal{O}_{L_1}$ such that $dv_p = \xi$. Then we obtain $v_p(c - d) \ge v_pp$.

For each j we now choose $d_j \in L_1$ such that $c_j \equiv d_j^p \mod p\mathcal{O}_{L_1}$. Then

$$\left(\sum_{j=0}^{p^i-1} d_j a_{i+1}^j\right)^p \equiv \sum_{j=0}^{p^i-1} d_j^p (a_{i+1}^p)^j \equiv \sum_{j=0}^{p^i-1} c_j a_i^j = b \mod p\mathcal{O}_{L_1(a_i)} .$$

In view of [16, Lemma 4.1 (2)], this shows that (K, v_p) satisfies (DRvr) and is therefore a deeply ramified field, which proves assertion 2).

Our next aim is to show that the extension $(L_2(a)|L_2, v_p)$ is nontrivial and immediate. For each $i \in \mathbb{N}$, we set

$$b_i = \sum_{j=1}^{i} \frac{1}{a_j} \in L_1(a_i)$$

and compute, using [16, Lemma 2.17 (2)]:

$$\begin{aligned}(a-b_i)^p - (a-b_i) &\equiv a^p - \sum_{j=1}^{i} \frac{1}{a_j^p} - a + \sum_{j=1}^{i} \frac{1}{a_j} \\ &= \frac{1}{p} - \frac{1}{p} - \sum_{j=1}^{i-1} \frac{1}{a_j} + \sum_{j=1}^{i} \frac{1}{a_j} = \frac{1}{a_i} \mod \mathcal{O}_{L_1(a_i)} .\end{aligned}$$

It follows that $v_p(a-b_i) < 0$ and

$$-\frac{v_p p}{p^i} = v_p \frac{1}{a_i} = \min\{p v_p(a-b_i), v_p(a-b_i)\} = p v_p(a-b_i) ,$$

whence

$$v_p(a-b_i) = -\frac{v_p p}{p^{i+1}} . \tag{62}$$

We have that

$$\begin{aligned} p &\leq (v_p L_1(a_i,a) : v_p L_1(a_i)) \leq (v_p L_1(a_i,a) : v_p L_1(a_i))[L_1(a_i,a)v_p : L_1(a_i)v_p] \\ &\leq [L_1(a_i,a) : L_1] \leq p .\end{aligned}$$

Thus equality holds everywhere and we have that $(v_p L_1(a_i,a) : v_p L_1(a_i)) = p$, the extension is unibranched, $L_1(a_i,a)v_p = L_1(a_i)v_p = L_1 v_p$, and for all $i \in \mathbb{N}$, $a \notin L_1(a_i)$. Hence $a \notin L_2$, and we have:

$$v_p L_2(a) = \bigcup_{i\in\mathbb{N}} v_p L_1(a_i,a) = \frac{1}{p^\infty}\mathbb{Z} = v_p L_2 \quad \text{and} \quad L_2(a)v_p = L_1 v_p = L_2 v_p .$$

This shows that $(L_2(a)|L_2, v_p)$ is nontrivial and immediate, as asserted. The extension is also unibranched since each extension $(L_1(a_i,a)|L_1(a_i), v_p)$ is unibranched. Therefore, it is a defect extension of degree p, which proves assertion 3).

Since $[L_0(a) : L_0] = p$, there is an element $\zeta' \in L_0^{\mathrm{ac}}$ such that $[L_0(\zeta') : L_0]$ divides $(p-1)!$ and $L_0(a,\zeta')|L_0(\zeta')$ is Galois. As (L_0, v_p) is henselian, p does not divide $[L_0(\zeta') : L_0]$, and $L_0 v_p$ is algebraically closed, the Lemma of Ostrowski shows that $[L_0(\zeta') : L_0] = (v_p L_0(\zeta') : v_p L_0)$. It follows that also $[L_1(\zeta') : L_1] = (v_p L_1(\zeta') : v_p L_1)$, and that $(L_1(\zeta')|L_1, v_p)$ is a tame extension. Hence by Lemma 6.1, also $(L_1(a,\zeta')|L_1(\zeta'), v_p)$ is a defect extension. By [16, Theorem 1.5], the algebraic extension $(L_1(\zeta'), v_p)$ of (L_1, v_p) is again a deeply ramified field and hence an rdr field.

Thus it follows from Theorem 1.2 that the Galois extension $(L_1(a,\zeta')|L_1(\zeta'),v_p)$ has independent defect. Since (L_2,v_p) has rank 1, the convex subgroup associated with the extension $(L_2(\zeta',a)|L_2(\zeta'),v_p)$ is $\{0\}$. With $L:=L_2(\zeta')$, we have now proved assertion 4). □

6.3. **The case of equal characteristic: some examples.**

Example 6.7. Take $K_0=\mathbb{F}_p((t))$. Then $K:=K_0^{1/p^\infty}=\mathbb{F}_p((t))^{1/p^\infty}$ is a perfect field, so under the canonical t-adic valuation v_t it is a deeply ramified field. Moreover, (K,v_t) is henselian and of rank 1. Now set $L_0:=K((x))$ and $L:=L_0^{1/p^\infty}$, and equip L with the canonical x-adic valuation v_x. Both v_x and v_t are henselian, hence so is their composition $v:=v_x\circ v_t$ on L. As L is perfect, (L,v) is a deeply ramified field. Its value group vL has rank 2, i.e., it has two proper convex subgroups. Let ϑ_t be a root of $X^p-X-\frac{1}{t}$, and ϑ_x a root of $X^p-X-\frac{1}{x}$. We note that $K=Lv_x$.

By part 1) of Lemma 6.4, both extensions $(L(\vartheta_x)|L,v)$ and $(L(\vartheta_x)|L,v_x)$ are defect extensions with independent defect. By part 2) of Lemma 6.4, $v_x(\vartheta_x-L)=v_xL^{<0}$, which by part 1) of Lemma 6.5 implies that $v(\vartheta_x-L)=\{\alpha\in vL\mid \alpha<v_tK\}$. Hence for $\mathcal{E}_x=(L(\vartheta_x)|L,v)$, its associated convex subgroup $H_{\mathcal{E}_x}$ is the convex subgroup v_tK of vL.

Again by part 1) of Lemma 6.4, the extension $(K(\vartheta_t)|K,v_t)$ has independent defect, and by part 2) of Lemma 6.4, $v_t(\vartheta_t-K)=v_tK^{<0}$. By part 2) of Lemma 6.5, $(L(\vartheta_t)|L,v)$ is a defect extension with $v(\vartheta_t-L)=(vL(\vartheta_t)^{<0}$. Therefore, $\{0\}$ is the convex subgroup associated with the defect extension $(L(\vartheta_t)|L,v)$.

We have shown that both convex subgroups of vL, v_tK and $\{0\}$, appear as the convex subgroups associated with Galois defect extensions of (L,v). ♢

Let us present a modification of this example.

Example 6.8. In the previous example, we replace K by some algebraically closed (or just henselian defectless) field with an arbitrary nontrivial valuation v_t. Then (K,v_t) has no defect extensions, and $H=v_tK$ will be the only convex subgroup of vL associated with Galois defect extensions. This is seen as follows. As in Example 6.7 we have that $(L(\vartheta_x)|L,v)$ is a defect extension with $v(\vartheta_x-L)=\{\alpha\in vL\mid\alpha<v_tK\}$. On the other hand, suppose that there is a defect extension $(L(\vartheta)|L,v)$ with $v(\vartheta-L)=vL^{<0}$. Then there is $b\in L$ such that $v(\vartheta-b)\in v_tK$. Set $\bar\vartheta:=(\vartheta-b)v_x$. Since $\vartheta-b$ is a root of an Artin-Schreier polynomial over L, $\bar\vartheta$ is a root of an Artin-Schreier polynomial over K. By construction, $(K(\bar\vartheta)|K,v_t)$ cannot be a defect extension, so there is $\bar c\in K$ such that $v_t(\bar\vartheta-\bar c)$ is the maximum of $v_t(\bar\vartheta-K)$. Choose $c\in L$ such that $cv_x=\bar c$. Then $v(\vartheta-b-c)$ is the maximum of $v(\vartheta-L)$, contradicting our assumption that $(L(\vartheta)|L,v)$ is a defect extension.

At the other extreme, we can keep (K,v_t) and (L_0,v_x) as in the previous example, but now take (L,v_x) to be a maximal purely wild extension of (L_0,v_x). As $L_0v_x=K$ is perfect, we have $Lv_x=L_0v_x=K$. By Lemma 6.2, (L,v_x) is a tame field and thus has no defect extensions. Therefore v_tK cannot appear as a convex subgroup associated with any Galois defect extension; indeed, if $v(\vartheta-L)=\{\alpha\in vL\mid\alpha< v_tK\}$, then by part 1) of Lemma 6.5, $(L(\vartheta)|L,v_x)$ would be a defect extension. However, as in the previous example one shows that $\mathcal{E}_t=(L(\vartheta_t)|L,v)$ is a defect extension with $H_{\mathcal{E}_t}=\{0\}$. ♢

6.4. **The case of equal characteristic: a general construction.** We will now present a much more general construction. Given any ordered index set I and for every $i\in I$ an arbitrary ordered abelian group C_i, we can form the **Hahn sum** $\coprod_{i\in I} C_i$. As an abelian group, this is the direct sum of the groups C_i, represented as the set of all tuples $(\alpha_i)_{i\in I}$ with only finitely many of the $\alpha_i\in C_i$ nonzero. An ordering on $\coprod_{i\in I} C_i$ is introduced as follows. For $(\alpha_i)_{i\in I}\in\coprod_{i\in I} C_i$, set $i_{\min}:=\min\{i\in I\mid \alpha_i\ne 0\}$. Then define $(\alpha_i)_{i\in I}>0$ if $\alpha_{i_{\min}}>0$. If all C_i are archimedean ordered, then the principal convex subgroups are exactly the subsets of the form $\{(\alpha_i)_{i\in I}\in\coprod_{i\in I} C_i\mid \alpha_i=0 \text{ for all } i<i_0\}$ for some $i_0\in I$; this subgroup is generated by any $(\alpha_i)_{i\in I}\in\coprod_{i\in I} C_i$ with $\alpha_{i_0}\ne 0$; likewise, the subprincipal convex subgroups are exactly the subsets of the form $\{(\alpha_i)_{i\in I}\in\coprod_{i\in I} C_i\mid \alpha_i=0 \text{ for all } i\le i_0\}$ for some $i_0\in I$.

Now take any ordered index set I. Set $C_i=\mathbb{Z}$ for all I and let Γ_0 be the Hahn sum $\coprod_{i\in I} C_i$. For each $\ell\in I$ let 1_ℓ denote the element $(\alpha_i)_{i\in I}$ with $\alpha_i=1\in\mathbb{Z}$ if $i=\ell$ and $\alpha_i=0$ otherwise. Now the elements 1_ℓ generate all principal convex subgroups of Γ_0. Note that if $\ell<\ell'$, then $1_\ell\gg 1_{\ell'}$, that is, $1_\ell>n1_{\ell'}$ for all $n\in\mathbb{N}$.

Take a perfect field k of characteristic $p>0$ and a set $\{t_i\mid i\in I\}$ of elements algebraically independent over k and define a valuation v on the field $k(t_i\mid i\in I)$ by setting $vt_i=1_i$ for each $i\in I$. Let (K_0,v) be the henselization of $(k(t_i\mid i\in I),v)$.

For each $i\in I$ there are:

- a decomposition $v=v_i\circ\bar v_i$, where v_i is the finest coarsening of v on K_0 that is trivial on t_i and $\bar v_i$ is the valuation induced by v on the residue field K_0v_i, which can be identified with $k(t_j\mid i\le j\in I)$, and
- a decomposition $\bar v_i=w_i\circ\bar w_i$, where w_i is the t_i-adic valuation on K_0v_i and $\bar w_i$ is the valuation induced by $\bar v_i$ on the residue field $K_0v_iw_i$, which can be identified with $k(t_j\mid i<j\in I)$.

Note that v_j is strictly coarser than v_i if $j<i$.

We take K_1 to be the perfect hull of K_0, that is, $K_1=k(t_i^{1/p^n}\mid i\in I,\, n\in\mathbb{N})$. The valuations v and v_i, $i\in I$, have unique extensions to K_1, and vK_1 is the p-divisible hull $\frac{1}{p^\infty}\Gamma$ of vK_0. Further, K_1v_i is the perfect hull $k(t_j^{1/p^n}\mid i\le j\in I,\, n\in\mathbb{N})$ of $k(t_j\mid i\le j\in I)$, so the valuations $\bar v_i$ and w_i have unique extensions to K_1v_i and the decompositions $v=v_i\circ\bar v_i$ again hold on K_1. Likewise, $K_1v_iw_i$ is the perfect hull $k(t_j^{1/p^n}\mid i<j\in I,\, n\in\mathbb{N})$ of $k(t_j\mid i<j\in I)$, so also the valuations $\bar v_i$ have unique extensions to $K_1v_iw_i$ and the decompositions $v_i=w_i\circ\bar w_i$ again hold on K_1.

We set $\Gamma:=vK_1=\frac{1}{p^\infty}\Gamma_0$ and define H_i to be the largest convex subgroup of Γ that does not contain 1_i, that is, $H_i=\bar w_i(K_1v_iw_i)$. Consequently, the H_i are exactly all subprincipal convex subgroups of Γ. The principal convex subgroups of Γ are exactly all smallest convex subgroups that contain 1_i for some $i\in I$; they are of the form $\bar v_i(K_1v_i)$.

The next theorem proves part 1) of Theorem 1.4.

Theorem 6.9. *Take any subset $J\subseteq I$. Then there exists an algebraic extension (K_2,v) of (K_1,v) which is a henselian deeply ramified field and such that the convex subgroups associated with Galois defect extensions of prime degree of (K_2,v) are exactly the convex subgroups H_j with $j\in J$.*

Proof. Since (K_1, v) is henselian and perfect, each algebraic extension (K, v) of (K_1, v) is a henselian deeply ramified field.

For each $i \in I$ we let ϑ_i be a root of $X^p - X - \frac{1}{t_i}$. Since v_i is trivial on t_i, we can identify ϑ_i with $\vartheta_i v_i$. By part 1) of Lemma 6.4, $(K_1(\vartheta_i)|K_1, v)$, $(K_1 v_i(\vartheta_i)|K_1 v_i, \bar{v}_i)$ and $(K_1 v_i(\vartheta_i)|K_1 v_i, w_i)$ are defect extensions with independent defect. By construction, $K_1 v_i = K_1 v_i w_i(t_i^{1/p^n} \mid n \in \mathbb{N}) = K_1 v_i w_i(t_i)^{1/p^\infty}$, where $K_1 v_i w_i$ is a perfect field. Hence by part 2) of Lemma 6.4, $w_i(\vartheta_i - K_1 v_i) = (w_i(K_1 v_i))^{<0}$, which by part 1) of Lemma 6.5 implies that $\bar{v}_i(\vartheta_i - K_1 v_i) = \{\alpha \in \bar{v}_i(K_1 v_i) \mid \alpha < \bar{v}_i(K_1 v_i w_i)\} = \{\alpha \in \bar{v}_i(K_1 v_i) \mid \alpha < H_i\}$. We claim that this implies that $v(\vartheta_i - K_1) = \{\alpha \in vK_1 \mid \alpha < H_i\}$, that is, H_i is the convex subgroup associated with the defect extension $(K_1(\vartheta_i)|K_1, v)$. For the proof of the claim, observe that $K_1 v_i \subset K_1$ and $v|_{K_1 v_i} = \bar{v}_1$; so $\bar{v}_i(\vartheta_i - K_1 v_i) \subseteq v(\vartheta_i - K_1)$. We show that the former is cofinal in the latter, which will prove our claim. Take any $c \in K_1$. If $vc < v\vartheta_i$, then $\bar{v}_i \vartheta_i = v\vartheta_i > v(\vartheta_i - c)$. If $vc \geq v\vartheta_i$, then we can write $c = cv_i + c'$ with $cv_i \in K_1 v_i$ and $c' \in K_1$ with $v_i c' > 0$. It follows that $vc' > \bar{v}_i(K_1 v_i)$ and consequently, $vc' > \bar{v}_i(\vartheta_i - cv_i)$ and $v(\vartheta_i - c) = v_i(\vartheta_i - cv_i) \in \bar{v}_i(\vartheta_i - K_1 v_i)$. Hence by our construction, all H_i for $i \in I$ appear as the convex subgroups associated with Galois defect extensions of (K_1, v). We now have to find an algebraic extension of (K_1, v) which will admit exactly all H_i for $i \in I \setminus J$.

Let (K_2, v) be a maximal algebraic extension of (K_1, v) for which $vK_2 = vK_1$, the above decompositions carry over to K_2 for suitable extensions of the valuations v_i, $\bar{v}_i$, w_i, $\bar{v}_i$, and for all $j \in J$, $K_2 v_j = K_2 v_j w_j(t_j)^{1/p^\infty}$. As $K_2 v_j w_j$ is perfect, being an algebraic extension of the perfect field $K_1 v_j w_j$, part 2) of Lemma 6.4 shows that $(K_2 v_j(\vartheta_j)|K_2 v_j, w_j)$ is (still) a defect extension with $w_j(\vartheta_j - K_2 v_j) = (w_j(K_2 v_j))^{<0}$. By part 2) of Lemma 6.5, $(K_2(\vartheta_j)|K_2, v_j \circ w_j)$ is a defect extension with $v_j \circ w_j(\vartheta_j - K_2) = (v_j \circ w_j(K_2))^{<0}$. Now by part 1) of Lemma 6.5, $(K_2(\vartheta_j)|K_2, v)$ is a defect extension with $v(\vartheta_j - K_2) = \{\alpha \in vK_2 \mid \alpha < \bar{v}_j(K_2 v_j w_j)\}$, that is, $H_j = \bar{v}_j(K_1 v_j w_j) = \bar{v}_j(K_2 v_j w_j)$ is its associated convex subgroup.

Suppose that there is some $i \in I \setminus J$ such that H_i is also the convex subgroup associated with some Galois defect extension of (K_2, v). In this case we take (L, w_i) to be a maximal purely wild extension (L, w_i) of $(K_2 v_i, w_i)$. By Lemma 6.2, (L, w_i) is a tame field and thus does not have any nontrivial defect extensions. As $K_2 v_i$ is perfect, being an algebraic extension of $K_1 v_i$, we have that (L, w_i) is an immediate extension of $(K_2 v_i, w_i)$, that is, $w_i L = w_i(K_2 v_i)$ and $L w_i = K_2 v_i w_i$. We take (K_3, v_i) to be an algebraic extension of (K_2, v_i) such that $v_i K_3 = v_i K_2$ and $K_3 v_i = L$, and that $[K_2' : K_2] = [K_2' v_i : K_2 v_i]$ holds for every finite subextension $K_2'|K_2$ of $K_3|K_2$; for the construction of such extensions, see [7, Section 2.3]. We set $v = v_i \circ w_i \circ \bar{v}_i$ on K_3; since $v_i K_3 = v_i K_2$ and $(K_3 v_i, w_i) = (L, w_i)$ is an immediate extension of $(K_2 v_i, w_i)$, also (K_3, v) is an immediate extension of (K_2, v).

Take any $j \in J$; we will show that we still have $K_3 v_j = K_3 v_j w_j(t_i)^{1/p^\infty}$. Since $K_3 v_i w_i$ is perfect, being an algebraic extension of the perfect field $K_1 v_i w_i$, it will then follow as in the beginning of this proof that $(K_3(\vartheta_j)|K_3, v)$ is still a defect extension with associated convex subgroup H_j.

First assume that $j > i$. Then $K_3 v_j = K_2 v_j = K_2 v_j w_j(t_j)^{1/p^\infty} = K_3 v_j w_j(t_j)^{1/p^\infty}$ since $K_3 v_i w_i = L w_i = K_2 v_i w_i$ and $K_3 v_j$ and $K_3 v_j w_j$ are equal to or residue fields of $K_3 v_i w_i$.

Now assume that $j < i$. Suppose that K_3v_j properly contains $K_3v_jw_j(t_j)^{1/p^\infty}$. Then there is a finite subextension $K_2'|K_2$ of $K_3|K_2$ such that $K_2'v_j$ properly contains $K_2'v_jw_j(t_j)^{1/p^\infty}$. Using that $K_2v_j = K_2v_jw_j(t_j)^{1/p^\infty}$ and that $K_2'v_i$ is equal to or a residue field of $K_2'v_jw_j$ and K_2v_i is equal to or a residue field of $K_2v_jw_j$, we compute:

$$\begin{aligned}[K_2' : K_2] &\geq [K_2'v_j : K_2v_j] > [K_2'v_jw_j(t_j)^{1/p^\infty} : K_2v_jw_j(t_j)^{1/p^\infty}] \\ &= [K_2'v_jw_j : K_2v_jw_j] \geq [K_2'v_i : K_2v_i] = [K_2' : K_2] .\end{aligned}$$

This contradiction proves that $K_3v_j = K_3v_jw_j(t_i)^{1/p^\infty}$ also holds in this case.

Finally, we show that H_i cannot appear as the convex subgroup associated with any Galois defect extension of (K_3, v). This will contradict the maximality of (K_2, v) and show that it satisfies the statement of our theorem. Suppose the contrary, and let $(K_3(\vartheta)|K_3, v)$ be an Artin-Schreier defect extension with H_i as its associated convex subgroup. Since $\bar{v}_i(K_3v_i)$ properly contains $H_i = \bar{v}_i(K_1v_iw_i)$, it follows that there is some $b \in K_3$ such that $v(\vartheta - b) \in \bar{v}_i(K_3v_i)$. With $\vartheta' := \vartheta - b$ we obtain that $\bar{v}_i(\vartheta' - K_3v_i) = \{\alpha \in \bar{v}_i(K_3v_i) \mid \alpha < \bar{v}_i(K_1v_iw_i)\}$. Hence Lemma 6.3 shows that $(K_3v_i(\vartheta v_i)|K_3v_i, \bar{v}_i)$ is a nontrivial defect extension. Thus by part 1) of Lemma 6.5, also $(K_3v_i(\vartheta v_i)|K_3v_i, w_i)$ is a nontrivial defect extension. However, this contradicts the fact that by our construction, (K_3v_i, w_i) is a tame and thus defectless field with respect to w_i. □

6.5. **The case of mixed characteristic.** We choose a perfect field K of characteristic $p > 0$ containing $\mathbb{F}_p^{\mathrm{ac}}$. We denote the p-adic valuation on $\mathbb{Q}$ by v_p and take an algebraic extension (L_0, v_p) such that (L_0, v_p) is henselian, $v_pL_0 = v_p\mathbb{Q}$ and $L_0v_p = \mathbb{F}_p^{\mathrm{ac}}$. Then we construct an extension (L_1, v_p) of (L_0, v_p) such that $v_pL_1 = v_p\mathbb{Q}$ and $L_1v_p = K$. See [7, Section 2.3] for information on the construction of such extensions. By Proposition 6.6 there is an algebraic extension (L, v_p) of (L_0, v_p) such that $Lv_p = K$ and (L, v_p) is a deeply ramified field admitting a Galois defect extension $(L(a)|L, v_p)$ of degree p with independent defect and associated convex subgroup $\{0\}$.

Example 6.10. Now take any nontrivial valuation $\bar{v}$ on K. If we choose K to be algebraically closed, then it does not admit any Galois defect extension. Still, $(L(a)|L, v_p \circ \bar{v})$ is a Galois defect extension, and as the convex subgroup associated with the extension $(L(a)|L, v_p)$ is $\{0\}$, by part 1) of Lemma 6.5 the convex subgroup associated with the extension $(L(a)|L, v_p \circ \bar{v})$ is $\bar{v}K$. This is the only convex subgroup of $(v_p \circ \bar{v})L$ that appears as convex subgroup associated with some Galois defect extension of $(L, v_p \circ \bar{v})$. ♢

The situation changes when $(K, \bar{v})$ itself admits Galois defect extensions. Then these can be lifted to Galois defect extensions of $(L, v_p \circ \bar{v})$, and the convex subgroups associated with Galois defect extensions of $(K, \bar{v})$ appear as convex subgroups associated with Galois defect extensions of $(L, v_p \circ \bar{v})$ that are properly contained in $\bar{v}(Lv_p)$. This will be exploited in the

Proof of part 2) of Theorem 1.4. Let Δ denote the largest proper convex subgroup of Γ. Denote by $\mathcal{C}_\Delta^{\mathrm{sp}}$ the set of all proper convex subgroups of Δ in $\mathcal{C}^{\mathrm{sp}}$. By part 1) of Theorem 1.4 we can choose a perfect henselian valued field $(K, \bar{v})$ of characteristic

p for which the associated convex subgroups are exactly the elements of $\mathcal{C}^{\rm sp}_\Delta$. We take (L, v_p) as described at the beginning of this section and consider $(L, v_p\circ\bar v)$ which is a deeply ramified field since (L, v_p) is and K is perfect.

Now $\bar vK$ is the largest proper convex subgroup of $(v_p\circ\bar v)L$ and it is shown as in the proof of part 1) of Theorem 1.4 that a convex subgroup of $\bar vK$ is an associated convex subgroup for $(K,\bar v)$ if and only if it is an associated convex subgroup for $(L, v_p\circ\bar v)$.

It remains to deal with the convex subgroup $\bar vK$ of $v_p\circ\bar vL$. If it is an element of $\mathcal{C}^{\rm sp}$, then we are done because $(L(a)|L, v_p)$ is a Galois defect extension of degree p with independent defect, and it follows that also $(L(a)|L, v_p\circ\bar v)$ is a Galois defect extension of degree p with independent defect.

Finally, assume that $\bar vK$ is not an element of $\mathcal{C}^{\rm sp}$. Then we replace (L, v_p) by a maximal purely wild extension, which does not change the residue field K because it is perfect, and is a tame field by Lemma 6.2. After this, (L, v_p) does not admit any defect extension and $\bar vK$ cannot be an associated convex subgroup for (L, v_p). It is then shown as in the proof of part 1) of Theorem 1.4 that it also cannot be an associated convex subgroup for $(L, v_p\circ\bar v)$. This completes the proof of part 2) of Theorem 1.4. □

References

[1] Anna Blaszczok: *Distances of elements in valued field extensions*, Manuscripta Mathematica **159** (2019), 397–429

[2] Steven Dale Cutkosky – Franz-Viktor Kuhlmann – Anna Rzepka: *Characterizations of Galois extensions with independent defect*,

[3] Steven Dale Cutkosky – Franz-Viktor Kuhlmann: *Kähler differentials of extensions of valuation rings and deeply ramified fields*, submitted; https://arxiv.org/abs/2306.04967

[4] Otto Endler: *Valuation theory*, Berlin (1972)

[5] Ofer Gabber – Lorenzo Ramero: *Almost ring theory*, Lecture Notes in Mathematics **1800**, Springer-Verlag, Berlin, 2003

[6] Margarete Ketelsen – Simone Ramello – Piotr Szewczyk: *Definable henselian valuations in positive residue characteristic*, J. Symb. Logic, DOI: https://doi.org/10.1017/jsl.2024.55

[7] Franz-Viktor Kuhlmann: *Value groups, residue fields and bad places of rational function fields*, Trans. Amer. Math. Soc. **356** (2004), 4559–4600

[8] Franz-Viktor Kuhlmann: *A classification of Artin–Schreier defect extensions and a characterization of defectless fields*, Illinois J. Math. **54** (2010), 397–448

[9] Franz-Viktor Kuhlmann: *Defect*, in: Commutative Algebra - Noetherian and non-Noetherian perspectives, Fontana, M., Kabbaj, S.-E., Olberding, B., Swanson, I. (Eds.), 277–318, Springer-Verlag, New York, 2011

[10] Franz-Viktor Kuhlmann: *Approximation of elements in henselizations*, Manuscripta Math. **136** (2011), 461–474

[11] Franz-Viktor Kuhlmann: *The algebra and model theory of tame valued fields*, J. reine angew. Math. **719** (2016), 1–43

[12] Franz-Viktor Kuhlmann: *Selected methods for the classification of cuts and their applications*, Proceedings of the ALANT 5 conference 2018, Banach Center Publications **121** (2020), 85–106

[13] Franz-Viktor Kuhlmann: *Approximation types describing extensions of valuations to rational function fields*, Mathematische Zeitschrift **301** (2022), 2509–2546

[14] Franz-Viktor Kuhlmann: *Topics in higher ramification theory, I: ramification ideals*, in preparation; available at https://fvkuhlmann.de/topIJun3-2025.pdf

[15] Franz-Viktor Kuhlmann – Matthias Pank – Peter Roquette: *Immediate and purely wild extensions of valued fields*, Manuscripta math. **55** (1986), 39–67

[16] Franz-Viktor Kuhlmann – Anna Rzepka: *The valuation theory of deeply ramified fields and its connection with defect extensions*, Transactions Amer. Math. Soc. **376** (2023), 2693–2738

[17] Franz-Viktor Kuhlmann – Katarzyna Kuhlmann: *Arithmetic of cuts in ordered abelian groups and of ideals over valuation rings*, to appear in Pacific J. Math.; arXiv:2406.10545

[18] Oscar Zariski – Pierre Samuel: *Commutative Algebra*, Vol. II, New York–Heidelberg–Berlin, 1960

Institute of Mathematics, University of Szczecin, ul. Wielkopolska 15, 70-451 Szczecin, Poland

Email address: fvk@usz.edu.pl