ON ALGEBRAICALLY MAXIMAL VALUED FIELDS THAT ARE NOT DEFECTLESS

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ABSTRACT. We use a known example of an algebraically maximal discretely valued field of positive characteristic p which admits purely inseparable extensions of degree p^2 with defect p to construct algebraically maximal valued fields of characteristic p as well as of characteristic 0 and of rank 2 which admit separable extensions of degree p^2 with defect p.

1. Introduction

The notions and notations we will use will be introduced in Section 2.

Françoise Delon gave an example that shows that algebraically maximal valued fields are not necessarily defectless (see [2], Exemple 1.51). A corrected and expanded version was presented in [7, Example 3.25]. We reproduce it in Section 3. A further discussion of this example will be included in Section 5.

For what follows, take a prime p. Example 3.1 proves:

Theorem 1.1. *There are discretely valued algebraically maximal fields* (L_0, v_0) *of characteristic* $p > 0$ *which are not inseparably defectless and admit a purely inseparable extension of degree* p *which is not an algebraically maximal field. In particular, the property "algebraically maximal" does not imply "defectless".*

The question arises whether there are also examples of algebraically maximal fields which admit separable (and hence simple) defect extensions. Using a trick already employed in [7, Example 3.18], we will construct such examples in Section 4, based on which we prove:

Theorem 1.2. *There are algebraically maximal fields* (L, v) *of characteristic* (p, p) *as well as of characteristic* $(0, p)$ *admitting separable extensions of degree* p^2 *with defect* p*, and with intermediate fields of degree* p *over* L *which are not algebraically maximal fields. In particular, the property "algebraically maximal" does not imply "separably defectless" and is not preserved under finite separable extensions.*

The valuations in the examples we give to prove this theorem have rank 2.

Open Problem: Are there algebraically maximal fields of rank 1 which admit separable defect extensions?

Date: 8.1.2026.

2020 *Mathematics Subject Classification.* 12J10, 12J25.

Key words and phrases. valued field extension, defect, defectless valued field, algebraically maximal valued field.

2. Preliminaries

For a valued field (K,v), we denote its value group by vK, its residue field by Kv, and its valuation ring by $\mathcal{O}_K$ with maximal ideal $\mathcal{M}_K$. By $(L|K,v)$ we denote an extension $L|K$ with valuation v on L, where K is endowed with the restriction of v. In this case, there are induced embeddings of vK in vL and of Kv in Lv. The extension $(L|K,v)$ is called **immediate** if these embeddings are onto. A valued field (K,v) is called **algebraically maximal** if it does not admit nontrivial immediate algebraic extensions, and it is called **maximal** if it does not admit any nontrivial immediate extensions.

We say that (K,v) has characteristic (p,p) if $\mathrm{char}\,K = \mathrm{char}\,Kv = p$, and characteristic $(0,p)$ if $\mathrm{char}\,K = 0$ and $\mathrm{char}\,Kv = p$. If $\mathrm{char}\,K = p > 0$ and the extension $K|K^p$ is finite, then there is $k \geq 0$ such that $[K : K^p] = p^k$; we then take the **p-degree of K** (also called **degree of inseparability**) to be k. If $K|K^p$ is infinite, then we take the p-degree to be ∞.

A valued field (K,v) is called **henselian** if each algebraic extension $L|K$ is **unibranched**, that is, the extension of v to L is unique.

If $(L|K,v)$ is a finite unibranched extension, then by the Lemma of Ostrowski ([9, Corollary to Theorem 25, Section G, p. 78]),

$$[L : K] \;=\; \tilde{p}^{\nu} \cdot (vL : vK)[Lv : Kv]\,, \tag{1}$$

where ν is a nonnegative integer and $\tilde{p}$ the **characteristic exponent** of Kv, that is, $\tilde{p} = \mathrm{char}\,Kv$ if it is positive and $\tilde{p} = 1$ otherwise. The factor $d(L|K,v) := \tilde{p}^{\nu}$ is the **defect** of the extension $(L|K,v)$. If $d(L|K,v) = 1$, then the extension $(L|K,v)$ is called **defectless**; otherwise we call it a **defect extension**. A henselian field (K,v) is a **separably defectless field** if every finite unibranched separable extension of (K,v) is defectless, and a **defectless field** if every finite unibranched extension of (K,v) is defectless; note that this is always the case if $\mathrm{char}\,Kv = 0$. An arbitrary valued field is called an **inseparably defectless field** if every finite purely inseparable extension is defectless.

The defect is multiplicative: if $(L|K,v)$ and $(M|L,v)$ are finite unibranched extensions, then

$$\mathrm{d}(M|K,v) = \mathrm{d}(M|L,v) \cdot \mathrm{d}(L|K,v) \tag{2}$$

(see [7, Equation (4)]).

For a valued field (K,v) and a finite field extension $L|K$, the **Fundamental Inequality** (see (17.5) of [3] or Theorem 19 on p. 55 of [9]) states that there are finitely many extensions of v from K to L, and

$$[L : K] \;\geq\; \sum_{i=1}^{\mathrm{g}} (v_iL : vK)[Lv_i : Kv]\,, \tag{3}$$

where $v_1, \ldots, v_{\mathrm{g}}$ are the distinct extensions.

Lemma 2.1. *If $L|K$ is a separable algebraic extension, then $L^{1/p} = L.K^{1/p}$, and the p-degree of L is equal to that of K. If $L|K$ is an arbitrary finite extension, then again, the p-degree of L is equal to that of K.*

Proof. Let $\mathcal{B}$ be a basis of L over K. Then $L^{1/p} = K^{1/p}(b^{1/p} \mid b \in \mathcal{B})$.

Assume that $L|K$ is separable. Then for every $b \in \mathcal{B}$, we have $K(b) = K(b^p)$ since otherwise, the separable extension $K(b)|K$ would contain a nontrivial purely inseparable extension $K(b)|K(b^p)$, which is impossible. It follows that $L = K(\mathcal{B}) = K(b^p \mid b \in \mathcal{B})$, which gives $L^{1/p} = K^{1/p}(\mathcal{B}) = L.K^{1/p}$. Since L and $K^{1/p}$ are linearly disjoint over K, $L|K$ being separable, it now follows that $[L^{1/p} : L] = [K^{1/p} : K]$.

To prove our second assertion, assume that $L|K$ is an arbitrary finite extension. We have that $[L^{1/p} : K^{1/p}][K^{1/p} : K] = [L^{1/p} : K] = [L^{1/p} : L][L : K]$. The Frobenius endomorphism sends $L^{1/p}$ onto L and $K^{1/p}$ onto K. Thus, $[L^{1/p} : K^{1/p}] = [L : K] < \infty$. If $[K^{1/p} : K]$ is finite, then this yields that $[K^{1/p} : K] = [L^{1/p} : L]$. If $[K^{1/p} : K]$ is infinite, then so is $[L^{1/p} : L]$. □

3. A basic example

Example 3.1. We consider $\mathbb{F}_p((t))$ with its t-adic valuation v_t. Since $\mathbb{F}_p((t))$ has uncountable cardinality, while that of $\mathbb{F}_p(t)$ is countable, the extension $\mathbb{F}_p((t))|\mathbb{F}_p(t)$ has infinite transcendence degree, we can choose elements $x, y \in \mathbb{F}_p((t))$ which are algebraically independent over $\mathbb{F}_p(t)$. We set

$$s := x^p + ty^p \quad \text{and} \quad K := \mathbb{F}_p(t, s) .$$

We note that $K^{1/p} = \mathbb{F}_p(t^{1/p}, s^{1/p}) = K(t^{1/p}, s^{1/p})$. The elements t, s are algebraically independent over $\mathbb{F}_p$. Consequently, the p-degree of K is 2. We define L_0 to be the relative algebraic closure of K in $\mathbb{F}_p((t))$. Then $L_0^{1/p} = L_0.K^{1/p} = L_0(t^{1/p}, s^{1/p})$. Since the elements $1, t^{1/p}, \ldots, t^{(p-1)/p}$ are linearly independent over $\mathbb{F}_p((t))$, the same holds over L_0. Hence, the elements $1, t, \ldots, t^{p-1}$ are linearly independent over L_0^p. Now if L_0 had p-degree 1, then s could be written in a unique way as an L_0^p-linear combination of $1, t, \ldots, t^{p-1}$. But this is not possible since $s = x^p + ty^p$ and x, y are transcendental over L_0. Hence, the p-degree of L_0 is still 2 (as it cannot increase through algebraic extensions); more precisely,

$$L_0^{1/p} = L_0(t^{1/p}, s^{1/p}) \quad \text{with} \quad [L_0^{1/p} : L_0] = p^2 .$$

On the other hand, since $s^{1/p} \in \mathbb{F}_p(t^{1/p}, x, y) \subset \mathbb{F}_p((t^{1/p}))$, we have

$$(v_t L_0^{1/p} : v_t L_0) = p \quad \text{and} \quad [L_0^{1/p} v_t : L_0 v_t] = 1$$

(where we extend v_t to the algebraic closure of L_0). As a relatively algebraically closed subfield of the henselian field $(\mathbb{F}_p((t)), v_t)$, also (L_0, v_t) is henselian. Thus the extension $(L_0^{1/p}|L_0, v_t)$ is unibranched and consequently has defect p.

On the other hand, $\mathbb{F}_p((t))$ is the completion of (L_0, v_t) since it is already the completion of $\mathbb{F}_p(t) \subseteq L_0$. This shows that $\mathbb{F}_p((t))$ is the unique maximal immediate extension of L_0 (up to valuation preserving isomorphism over L_0). If L_0 would admit a proper immediate algebraic extension L_1, then a maximal immediate extension of L_1 would also be a maximal immediate extension of L_0 and would thus be isomorphic over L_0 to $\mathbb{F}_p((t))$. But we have chosen L_0 to be relatively algebraically closed in $\mathbb{F}_p((t))$. This proves that (L_0, v) must be algebraically maximal. Hence

$$(L_0(s^{1/p}|L_0, v_t) \quad \text{and} \quad (L_0(t^{1/p}|L_0, v_t) ,$$

being of prime degree, cannot be immediate and are therefore defectless. Thus the defect of $L_0^{1/p}|L_0$ by multiplicativity (2) implies that both

$$(L_0^{1/p}|L_0(s^{1/p}, v_t) \quad \text{and} \quad (L_0^{1/p}|L_0(t^{1/p}, v_t)$$

must have defect p. Consequently, $(L_0(s^{1/p}, v_t)$ and $(L_0(t^{1/p}, v_t)$ are not algebraically maximal. ♢

We summarize the properties of this example, thereby adjusting the notation for later use.

Proposition 3.2. *There exists a discretely valued algebraically maximal field* (L_0, v_0) *of characteristic* $p > 0$ *and purely inseparable defectless extensions* $(L_0(a_0)|L_0, v_0)$ *and* $(L_0(b_0)|L_0, v_0)$ *of degree p such that the unibranched extension* $(L_0(a_0, b_0)|L_0, v_0)$ *of degree* p^2 *has defect* p*, as* $(v_0L_0(a_0, b_0) : v_0L_0) = p$ *and* $[L_0(a_0, b_0)v_0 : L_0v_0] = 1$*, and neither* $(L_0(a_0), v_0)$ *nor* $(L_0(b_0), v_0)$ *is an algebraically maximal field.* □

4. Examples with composite valuations

Lemma 4.1. *Take any field* L_0 *of positive characteristic. There exist henselian defectless discretely valued fields* (L, w) *with residue field* L_0*. They can be chosen such that either* $\mathrm{char}\, L = 0$*, or* $\mathrm{char}\, L = \mathrm{char}\, L_0$*.*

Proof. For $\mathrm{char}\, L = 0$: Take an extension of $(\mathbb{Q}, v_p)$, where v_p denotes the p-adic valuation, with value group equal to $v_p\mathbb{Q}$ and residue field L_0. For the construction of such extensions, see [5, Theorem 2.14]. Let (L, v_p) be the henselization of this field. Since (L, v_p) is henselian discretely valued of characteristic 0, it is a defectless field by [4, Theorem 8.32]. Alternatively, one can also take the completion in place of the henselization; as the valuation is still discrete, this field is maximal and therefore a henselian defectless field (see the discussion at the beginning of Section 4 in [1]).

For $\mathrm{char}\, L = \mathrm{char}\, L_0$: Take an element z transcendental over L_0, the z-adic valuation v_z on $L_0(z)$, and (L, v_z) to be the henselization of $(L_0(z), v_z)$. Then (L, v_z) is henselian discretely valued, and by [6, Theorem 1.1], it is a defectless field. □

Lemma 4.2. *Take* (L_0, v_0)*,* a_0 *and* b_0 *as in Lemma 3.2, and* (L, w) *as in the previous lemma. Set* $v := w \circ v_0$*. Then* (L, v) *is algebraically maximal.*

Proof. Suppose that $(L'|L, v)$ is a nontrivial immediate algebraic extension. Then $vL' = vL$, which implies that $wL' = wL$ and that $v_0(L'w) = v_0(Lw)$. Since (L, w) is a henselian defectless field, we have $[L'w : Lw] = [L' : L]$. Since $Lw = L_0$ is algebraically maximal under its valuation v_0, it follows that $(v_0(L'w) : v_0(Lw)) > 1$, which implies $(vL' : vL) > 1$, or $[(L'w)v_0 : (Lw)v_0] > 1$, which implies $[L'v : Lv] > 1$. This contradicts our assumption that $(L'|L, v)$ is immediate. □

Lemma 4.3. *Let* (L, v) *be as in the previous lemma. Then there are elements* a, b *in the separable-algebraic closure of* L *such that* $[L(a) : L] = [L(b) : L] = p$*,* $L(a)w = L_0(a_0)$*, and* $L(b)w = L_0(b_0)$*.*

Proof. Take $c, d \in L$ such that $cw = a_0^p \in L_0$ and $dc = b_0^p \in L_0$. If $\operatorname{char} L = 0$, then take a to be a p-th root of c and b to be a p-th root of d. If $\operatorname{char} L = \operatorname{char} L_0 = p$, then take a to be a root of the polynomial $X^p - rX - c$ and b to be a root of the polynomial $X^p - rX - d$ for some $r \in L$ with $wr > 0$. Then in both cases, $aw = a_0$ and $bw = b_0$. It follows that

$$p \;\geq\; [L(a) : L] \;\geq\; [L(a)w : Lw] \;\geq\; [L_0(a_0) : L_0] \;=\; p\,.$$

Hence equality holds everywhere, which proves that $[L(a) : L] = p$ and $L(a)w = L_0(a_0)$. The proof for b in place of a is similar. □

Now we are ready for the

Proof of Theorem 1.2: We shall prove that the valued field (L, v) of the previous lemma has the properties stated in Theorem 1.2. As the extensions $L(a)|L$ and $L(b)|L$ are separable, so is the extension $L(a, b)|L$. Since $a_0, b_0 \in L(a, b)w$, we have

$$p^2 \;\geq\; [L(a, b) : L] \;\geq\; [L(a, b)w : Lw] \;\geq\; [L_0(a_0, b_0) : L_0] \;=\; p^2\,,$$

hence equality holds everywhere, showing that $[L(a, b) : L] = p^2$ and $L(a, b)w = L_0(a_0, b_0)$, so that $L(a, b)v = L_0(a_0, b_0)v_0$. On the other hand, $wL(a, b) = wL$ by the Fundamental Equality (3) since $[L(a, b) : L] = [L(a, b)w : Lw]$. Further, by Proposition 3.2, $(v_0L_0(a_0, b_0) : v_0L_0) = p$ and $[L_0(a_0, b_0)v_0 : L_0v_0] = 1$. So

$$(vL(a, b) : vL) \;=\; (v_0(L(a, b)w : v_0(Lw)) \;=\; (v_0L_0(a_0, b_0) : v_0L_0) = p$$

and

$$[L(a, b)v : Lv] = [L_0(a_0, b_0)v_0 : L_0v_0] \;=\; 1\,,$$

hence the extension $(L(a, b)|L, v)$ has defect p.

Finally, $[L(a) : L] = p$ and $(vL(a)v : vL) = (v_0L_0(a_0) : v_0L_0) = p$, showing that the extension $(L(a)|L, v)$ is defectless. Since the defect is multiplicative, it follows that $(L(a, b)|L(a), v)$ has defect p, which shows that $(L(a), v)$ is not algebraically maximal. The same proof works for b in place of a. □

5. Appendix

Remark 5.1. In [7, Example 3.25] it is stated that the relative algebraic closure L_0 of (K, v_t) in $\mathbb{F}_p((t))$ is a separable extension of K and therefore is the henselization of K. However, the proof contains a gap, so the question about the separability of the extension remains open.

In what follows we will state results from [7, Example 3.25] that are not affected by the gap.

We set

$$F \;:=\; \mathbb{F}_p(t, x, y)$$

and note that $t^{1/p} \notin F$. Hence F and $K(t^{1/p})$ are linearly disjoint over K and $F \cap K(t^{1/p}) = K$.

Lemma 5.2. *The following assertions hold:*

1) K is relatively algebraically closed in F.

2) F and $K^{1/p} = K(t^{1/p}, s^{1/p})$ are not linearly disjoint over K.

3) While $K(t^{1/p})$ is linearly disjoint from F over K, it is not relatively algebraically closed in $F(t^{1/p}) = F.K(t^{1/p})$ since $s^{1/p} \in F(t^{1/p}) \setminus K(t^{1/p})$.

4) Since $F|K$ is not linearly disjoint from $K^{1/p}|K$, it is not a separable extension. Although being finitely generated, it is consequently not separably generated; in particular, it is not a rational function field.

5) If $a \in K \setminus K^p$ (and in particular, if $a = s$), then F and $K(a^{1/p})$ are linearly disjoint over K, $F(a^{1/p}) = F.K^{1/p}$ and $K^{1/p}$ is a nontrivial purely inseparable algebraic extension of $K(a^{1/p})$ in $F(a^{1/p})$.

Proof. 1): Take $b \in F$ algebraic over K. The element b^p is algebraic over K and lies in $F^p = \mathbb{F}_p(t^p, x^p, y^p)$ and thus also in $K(x) = \mathbb{F}_p(t, x, y^p)$. Since $\operatorname{trdeg} K|\mathbb{F}_p = 2$ while $\operatorname{trdeg} \mathbb{F}_p(t, x, y^p)|\mathbb{F}_p = 3$, we see that x is transcendental over K. Therefore, K is relatively algebraically closed in $K(x)$ and thus, $b^p \in K$. Consequently, $b \in K^{1/p} = \mathbb{F}_p(t^{1/p}, s^{1/p})$. Write

$$b \;=\; r_0 + r_1 s^{\frac{1}{p}} + \ldots + r_{p-1} s^{\frac{p-1}{p}} \quad \text{ with } \; r_i \in \mathbb{F}_p(t^{1/p}, s) \;=\; K(t^{1/p})\,.$$

Since $s^{1/p} = x + t^{1/p} y$, we have that

$$b \;=\; r_0 \;+\; r_1 x + \ldots + r_{p-1} x^{p-1} \;+ \ldots +\; t^{1/p} r_1 y + \ldots + t^{(p-1)/p} r_{p-1} y^{p-1}$$

(in the middle, we have omitted the summands in which both x and y appear). Since x, y are algebraically independent over $\mathbb{F}_p$, the p-degree of $\mathbb{F}_p(x, y)$ is 2, and the elements $x^i y^j$, $0 \le i < p$, $0 \le j < p$, form a basis of $\mathbb{F}_p(x, y)|\mathbb{F}_p(x^p, y^p)$. Since $t^{1/p}$ is transcendental over $\mathbb{F}_p(x^p, y^p)$, we know that $\mathbb{F}_p(x, y)$ is linearly disjoint from $\mathbb{F}_p(t^{1/p}, x^p, y^p)$ and hence also from $\mathbb{F}_p(t, x^p, y^p)$ over $\mathbb{F}_p(x^p, y^p)$. This shows that the elements $x^i y^j$ also form a basis of $F|\mathbb{F}_p(t, x^p, y^p)$ and are still $\mathbb{F}_p(t^{1/p}, x^p, y^p)$-linearly independent. Hence, b can also be written as a linear combination of these elements with coefficients in $\mathbb{F}_p(t, x^p, y^p)$, and this must coincide with the above $\mathbb{F}_p(t^{1/p}, x^p, y^p)$-linear combination which represents b. That is, all coefficients r_i and $t^{i/p} r_i$, $1 \le i < p$, are in $\mathbb{F}_p(t, x^p, y^p)$. Since $t^{i/p} \notin \mathbb{F}_p(t, x^p, y^p)$, this is impossible unless they are zero. It follows that $b = r_0 \in K(t^{1/p})$, whence $b \in F \cap K(t^{1/p}) = K$. This proves that K is relatively algebraically closed in F.

2): We have $s^{1/p} = x + t^{1/p} y \in \mathbb{F}_p(t^{1/p}, x, y) = F(t^{1/p})$, which implies

$$[F.K^{1/p} : F] \;=\; [F.K(t^{1/p}, s^{1/p}) : F] \;=\; [F(t^{1/p}) : F] \;=\; p \;<\; p^2 \;=\; [K^{1/p} : K]\,.$$

3) and 4) need no further proof.

5): We know that $a^{1/p} \notin F$ since K is relatively algebraically closed in F. Hence, F and $K(a^{1/p})$ are linearly disjoint over K and $[F(a^{1/p}) : F] = p = [F.K^{1/p} : F]$. This shows that $F(a^{1/p}) = F.K^{1/p}$ and that $K(a^{1/p})$ admits the nontrivial purely inseparable algebraic extension $K^{1/p}$ in $F(a^{1/p})$. □

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