SOME EXAMPLES OF VALUED FIELDS

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The purpose of this paper is to use generalized power series to provide two classes of examples of valued fields.

In the first set of examples the valued fields are henselian, in general not maximal, with given residue field and value group.

In the second set of examples, the valued fields are all maximal and their families of residue fields and value groups form chains. In both cases, these fields possess additional interesting properties.

To begin, I shall indicate the construction of the generalized power series fields (see [Ri], [El & Ri]). Then, I shall recall the results obtained in [Ri 2], which concern the lifting of algebraic properties from the residue field and the value group to the field of power series.

1. Let R be a commutative field, let S be a torsion-free abelian group. Assume that S is endowed with a compatible partial order $\leq$, that is, if $s, t, u \in S$ and $s \leq t$, then $s + u \leq t + u$.

Let A be the set of all mappings $f : S \to R$, whose support $\mathrm{supp}(f) = \{s \in S | f(s) \neq 0\}$ does not contain any infinite descending chain (the support is *artinian*), nor any infinite subset of pairwise order-incomparable elements (the support is *narrow*). With pointwise addition, A is an abelian group. Let $s \in S$ and $f, g \in A$, then

$$\{(t, u) \in S \times S | t + u = s, f(t) \neq 0, g(u) \neq 0\}$$

is finite. This allows to define the product fg as follows:

$$(fg)(s) = \sum_{t+u=s} f(t)g(u) \text{ for every } s \in S$$

and $fg \in A$. Then A is a commutative ring with unit e ($e(0) = 1$, $e(s) = 0$ for every $s \neq 0$). We also use the notation $A = [[R^{S,\leq}]]$.

The following theorem was proved in [El & Ri]:

(1.1). *A is a field if and only if the order $\leq$ is sub-total, that is, if $s, t \in S$, there exists an integer $k \geq 1$ such that $ks \leq kt$ or $kt \leq ks$.*

In this situation, define $s \leq' t$ when there exists $k \geq 1$ such that $ks \leq kt$. It follows that $\leq'$ is the unique compatible total order on S which is finer than $\leq$.

The natural valuation π' on the field A is defined as follows: $\pi'(0) = \infty$ and $\pi'(f) = \mathrm{Min}_{\leq'}\ \mathrm{supp}(f))$ (for $f \neq 0$).

The value group of π' is S and the residue field is R.

Krull showed ([Kr]) the main implication of:

(1.2). *(A, π') is a maximal valued field if and only if $A = [[R^{S,\leq'}]]$.*

(this means that (A, π') does not have any proper extension as valued field with the same value group and residue field).

Rayner proved ([Ra]) and another simpler proof was given in [Pr-Cr & Ri]:

(1.3). *The valued field (A, π') is henselian.*

2. In this section, I quote some of the numerous results proven in [Ri 2], which are of the following character:

If the field R has a property (P) and the torsion-free subtotally ordered group $(S, \leq)$ has a property (Q), then A has also the property (P).

(2.1). *If R is an algebraically closed field of characteristic 0 and S is a divisible group, then A is an algebraically closed field.*

(2.2). *If R is a real field, then A is a real field.*

(2.3). *If R is a real closed field and S is a divisible group, then A is a real closed field.*

(2.4). *If $R = R^n$, $S = n\dot{S}$, where n is not a multiple of the characteristic of R, then $A = A^n$.*

For the next result, we need the following concept. Let S be a torsion-free group let p be a prime number and assume that S is p-divisible. Let $\leq$ be a compatible order on S. By definition, the set of artinian and narrow subsets of $(S, \leq)$ is p-divisible when, for every artinian and narrow subset X of S, if $Y \leq S$ is such that $pY = X$, then Y is also artinian and narrow.

(2.5). *If R is a perfect field of characteristic p, $S = p\dot{S}$ and the set of artinian and narrow subsets of S is p-divisible, then A is a perfect field.*

(2.6). *If R is a perfect field with no proper solvable extension, if S is a divisible group and if R has characteristic p, the set of artinian and narrow subsets is p-divisible, then A is a perfect field with no proper solvable extension.*

(2.7). *If R is a real pythagorean field, then A is a real pythagorean field.*

(2.8). *If R is a non-real pythagorean field and $S = 2\dot{S}$, then A is a non-real pythagorean field.*

(2.9). *If R is an euclidean field (that is, $R = R^2 \cup (-R^2)$) $S = 2S$ and the set of artinian and narrow subsets qS is 2-divisible, then A is an euclidean field.*

3. In this section, I show how the construction of generalized power series is used to provide interesting examples of valued fields.

Examples 1. I indicate many examples of henselian valued fields which are not maximal, all fields with fixed arbitrary given residue field and value group. More precisely, to a set of parameters I associate explicitly henselian valued fields, which are not maximal. The set of parameters will be partially ordered and the corresponding valued fields will be related by opposite inclusion.

These fields will also be endowed with additional properties, reflecting properties of the residue field and value group.

Let $(S, \leq)$ be a totally ordered group (not reduced to 0). The set of principal convex subgroups of S is totally ordered by inclusion. Let $(T, \leq)$ be a totally ordered set such that $(C_t)_{t\in T}$ is the family of principal convex subgroups and $t < t'$ if and only if $C_t \subset C_{t'}$. For each C_t there exists the largest convex subgroup properly contained in C_t, which I denote by C_t^-; I stress that C_t^- need not be a principal convex subgroup of S.

For every $t \in T$, C_t/C_t^- is an archimedian totally ordered group, hence it may be identified to a subgroup of $\mathbb{R}$. Let $\varphi_t : C_t \to \mathbb{R}$ be a given ordered group homomorphism, with kernel C_t^-. For each $s \in C_t$, and for the sake of simplicity, I write $\overline{s} = \varphi_t(s) \in \mathbb{R}$. Note that if $0 < \varphi_t(s)$ then $s' < s$ for every $s' \in C_t^-$.

Let $\mathcal{G} = \prod\limits_{t\in T} \varphi_t(C_t)_{\geq 0}$. Thus, if $g \in \mathcal{G}$ then $g = (g_t)_t$ with $0 \leq g_t \in \varphi_t(C_t)$, for every $t \in T$. The *support* of g is $\{t \in T | g_t \neq 0\}$.

I define the following relation $\leq_g$ on S : $\;s \leq_g s'$ when $s = s'$ or, if $s \neq s'$ and $s' - s \in C_t \backslash C_t^-$, then $g_t < \varphi_t(s' - s)$.

It is easy to check that $\leq_g$ is a compatible order relation on S. Moreover, $\leq_g$ is a subtotal order, because $\mathbb{R}$ is archimedian.

If $g \neq g'$ then $\leq_g, \leq_{g'}$ are distinct orders.

Also if $g \leq g'$, that is $g_t \leq g'_t$ for every $t \in T$, then the order $\leq_{g'}$, is coarser than $\leq_g$. Note that if $g = 0$ (i.e. $g_t = 0$ for every $t \in T$) then $\leq_0$ coincides with the given order on S. In particular, if $s <_g s'$ then $s < s'$.

I require the following lemma.

LEMMA 3.1. *With the preceding notations, assume that S is n-divisible (where $n \geq 2$) and let $g \in \mathcal{G}$ have finite support. Then the set of $\leq_g$-artinian and $\leq_g$-narrow subsets of S is n-divisible.*

Proof:

Let X be a $\leq_g$-artinian and $\leq_g$-narrow subset of S, let $Y \subseteq S$ be such that $n \cdot Y = X$. Since X is also $\leq$-artinian (and $\leq$-narrow) by [Ri 1, (1.5)] so Y is $\leq$-artinian and also $\leq_g$-artinian. Assume that Y is not $\leq_g$-narrow, hence there exists an infinite subset Y_1 of Y consisting of pairwise $\leq_g$-incomparable elements. Since

Y_1 is $\leq$-artinian, there exists elements $y_1, y_2, \dots \in Y_1$, such that $y_1 < y_2 < y_3 < \cdots$ Let $y_{i+1} - y_i \in C_{t_i} ackslash C_{t_i}^-$, so $0 < \overline{y_{i+1} - y_i} \leq g_{t_i}$, because y_i, y_{i+1} are $\leq_g$-incomparable. Let $T_1 = \{t_1, t_2, \dots\}$, so $T_1 \subseteq \operatorname{supp}(g)$, hence by hypothesis T_1 is finite.

Denote by u the largest element of T_1 such that there exists an infinite sequence $i_1 < i_2 < \dots$ with $y_{i_k+1} - y_{i_k} \in C_u \backslash C_u^-$. By deleting finitely many initial terms, I may also assume that $y_{h+1} - y_h \in C_u^-$ when $h \neq i_1, i_2, \dots$ Then $y_{i_{k+1}} - y_{i_k} = (y_{i_{k+1}} - y_{i_{k+1}-1}) + (y_{i_{k+1}-1} - y_{i_{k+1}-2}) + \cdots + (y_{i_k+1} - y_{i_k}) \in C_u \backslash C_u^-$. I consider only this subsequence, and by simplicity I renumber it as $y_1 < y_2 < \cdots$ with $y_{j+1} - y_j \in C_u \backslash C_u^-$ and $0 < \overline{y_{j+1} - y_j} \leq g_u$. So $0 < \overline{x_{j+1} - x_j} \leq n g_u$, hence there exist infinitely many indices j and some k, $0 \leq k \leq n-1$, such that $k g_u < \overline{x_{j+1} - x_j} \leq (k+1) g_u$. This implies that the subset $\{x_1, x_2, \dots\}$ of X is not $\leq_g$-narrow, which is a contradiction.

$\blacksquare$

I define the following relation in $\mathcal{G} : g \sim g'$ whenever the following conditions are satisfied:

i) if $\varphi_t(C_t)$ is a dense subgroup of $\mathbb{R}$ then $g_t = 0$ if and only if $g'_t = 0$

ii) $\{t \in T \mid g_t \neq g'_t\}$ is finite.

This is an equivalence relation. Let $\mathcal{R} = \mathcal{G}/\sim$ be the set of equivalence classes. I denote by $\widetilde{g}$ the equivalence class of g.

The relation $\leq$ on $\mathcal{R}$ is defined as follows:

$\widetilde{g} \leq \widetilde{h}$ whenever

i) if $t \in T$ is such that $\varphi_t(C_t)$ is a dense subgroup of $\mathbb{R}$ and $h_t = 0$ then $g_t = 0$;

ii) $\{t \in T \mid h_t < g_t\}$ is finite.

It is easy to check that if $g \sim g'$, $h \sim h'$, then g', h' satisfy also the above conditions (i), (ii), hence the relation $\leq$ on $\mathcal{R}$ is well-defined.

It is also easy to verify that $\leq$ is an order relation.

Let R be any field, for each $g \in \mathcal{G}$ let $A_g = [[R^{S, \leq_g}]]$, so by (1.1), A_g is a field, and by [Ri 1, (2.3)] it is a subfield of $A = [[R^{S,\leq}]]$. Endowed with its natural valuation π, (A, π) is a maximal valued field by (1.2). For each g, let π_g be the restriction of

π to A_g; then (A_g, π_g) is a henselian valued field, having residue field R and value group S, as indicated in (1.3).

Now I prove:

(3.1). *$A_g \subseteq A_{g'}$ if and only if $\widetilde{g'} \leq \widetilde{g}$.*

Proof:

1°) First I assume that $\widetilde{g'} \not\leq \widetilde{g}$. Then either

i) there exists $t \in T$ such that $\varphi_t(C_t)$ is a dense subgroup of $\mathbb{R}$ and $g_t = 0,\ g'_t \neq 0$;

or

ii) the set $T_1 = \{t \in T \mid g_t < g'_t\}$ is infinite.

First assume (i). Then there exists an infinite sequence of real numbers $0 < r_1 < r_2 < \cdots < g'_t$, with each $r_i \in \varphi_t(C_t)$. Let $s_i \in C_t \backslash C_t^-$ be such that $\varphi_t(s_i) = r_i$, so $0 < s_1 < s_2 < \cdots$. Thus $0 <_g s_1 <_g s_2 \cdots$, hence the set $\{s_1, s_2, \dots\}$ is $\leq_g$-artinian and $\leq_g$-narrow, but it is not $\leq_{g'}$-narrow. Let $f : S \to R$ be such that $\mathrm{supp}(f) = \{s_1, s_2, \dots\}$ and $f(s_i) = 1$ for every $i = 1, 2, \dots$. Hence $f \in A_g$, but $f \notin A_{g'}$, which is a contradiction.

Now assume (ii). Since T_1 is infinite and totally ordered, either there exists an increasing sequence $t_1 < t_2 < \cdots$ in T_1, or there exists a decreasing sequence $t_1 > t_2 > \cdots$ in T_1.

I shall define an infinite sequence $s_1, s_2, \dots$ of elements of S, such that $s_1 <_g s_2 <_g \cdots$ and $s_1, s_2, \cdots$ are pairwise incomparable with respect to $\leq_{g'}$.

If $t_1 < t_2 < \cdots$ take $s_i \in C_{t_i} \backslash C_{t_i}^-$ such that $\overline{s_i} = g'_{t_i}$, for every $i \geq 1$. Then the set $\{s_1, s_2, \dots\}$ satisfies the first required condition.

Moreover, if $i < j$ then s_i, s_j are incomparable with respect to the order $\leq_{g'}$. Indeed, $t_i < t_j$ so $C_{t_i} \subseteq C_{t_j}^- \subset C_{t_j}$ and $s_j - s_i \in C_{t_j} \backslash C_{t_j}^-$; since $g_{t_j} < \overline{s_j - s_i} = g'_{t_j}$ then $s_i \not\leq_{g'} s_j$; also, if $s_j \leq_{g'} s_i$ then $s_j < s_i \in C_{t_i} \subseteq C_{t_j}^-$, which is a contradiction.

If $t_1 > t_2 > \cdots$ I define $s'_i \in C_{t_i} \backslash C_{t_i}^-$ such that $\overline{s'_i} = g'_{t_i}$; let $s_i = -s'_i$, then again the set $\{s_1, s_2, \dots\}$ satisfies the required conditions and consists of pairwise $\leq_{g'}$-incomparable elements.

If $f : S \to R$, with $\mathrm{supp}(f) = \{s_1, s_2, \dots\}$, and $f(s_i) = 1$ for $i = 1, 2, \dots$, then

$f \in A_g$ but $f \notin A_{g'}$, which is a contradiction.

2°) Now I assume that $A_g \not\subseteq A_{g'}$, so there exists a subset X of S which is $\leq_g$-artinian and $\leq_g$-narrow, but not simultaneously $\leq_{g'}$-artinian and $\leq_{g'}$-narrow.

For every $t \in T$, let $g''_t = \min\{g_t, g'_t\}$, so the order $\leq_{g''}$ is finer than the orders $\leq_g$ and $\leq_{g'}$. By [Ri 1, (1.5)] , X is also $\leq_{g''}$-artinian and $\leq_{g''}$-narrow. It follows that X is $\leq_{g'}$-artinian.

I conclude that X is not $\leq_{g'}$-narrow. Hence X is infinite and there exists an infinite subset $\{s_1, s_2, \dots\}$ of X which consists of pairwise incomparable elements (with respect to $\leq_{g'}$). By [Ri 1, (1.2)], there is a subsequence which is strictly increasing with respect to $\leq_g$; considering only this subsequence, I may assume that $s_1 <_g s_2 <_g \cdots$ but $s_1, s_2, \dots$ are pairwise $\leq_{g'}$-incomparable. By adding $-s_1$ to each element of the above set, and changing notation, if necessary, I may further assume without loss of generality that $0 <_g s_1$. Note that $0 < s_1 < s_2 < \dots$.

For every $i \geq 1$, let $t_i \in T$ be such that $s_{i+1} - s_i \in C_{t_i} \backslash C^-_{t_i}$.

If the set $Z = \{t_1, t_2, \dots\}$ is infinite, from $s_i <_g s_{i+1}$ and $s_i \not\leq_{g'} s_{i+1}$, it follows that $g_{t_i} < \overline{s_{i+1} - s_i} \leq g'_{t_i}$ and therefore $\widetilde{g'} \leq \widetilde{g}$ is not true.

If $Z = \{t_1, t_2, \dots\}$ is finite, there exists the largest $t \in Z$ for which there exists an infinite subsequence $i_1 < i_2 < \dots$ such that $s_{i_j+1} - s_{i_j} \in C_t \backslash C^-_t$ (for every $j \geq 1$).

By deleting finitely many initial terms of the sequence $\{s_1, s_2, \dots\}$, I may assume without loss of generality that, for any $i \neq i_j$, $s_{i+1} - s_i \in C_u \backslash C^-_u$ where $u \in Z$, $u < t$; hence $s_{i+1} - s_i \in C_u \subseteq C^-_t$. Thus

$$s_{i_{j+1}} - s_{i_j} = \left(s_{i_{j+1}} - s_{i_{j+1}-1}\right) + \cdots + \left(s_{i_j+1} - s_{i_j}\right) \in C_t \backslash C^-_t .$$

So, restricting to the subsequence $\left(s_{i_j}\right)_{j\geq 1}$ and changing again notation, I may assume without loss of generality that $s_1 < s_2 < \cdots$ with $s_{i+1} - s_i \in C_t \backslash C^-_t$ (for every $i \geq 1$).

Note now that for every $m \geq 1$, $\sum_{j=1}^{m} \overline{s_{j+1} - s_j} = \overline{s_{m+1} - s_1}$. Since $0 < s_{i+1} - s_i \notin C^-_t$ then $0 < \overline{s_{i+1} - s_i}$, so $0 < \overline{s_2 - s_1} < \overline{s_3 - s_1} < \cdots < \overline{s_{m+1} - s_1} < \cdots < g'_t$. Hence $\varphi_t(C_t)$ is a dense subgroup of $\mathbb{R}$. Since $g_t < \overline{s_{i+1} - s_i}$ for $i \geq 1$, then $mg_t < \overline{s_{m+1} - s_1} \leq g'_t$ for every $m \geq 1$.

By the archimedian property, $g_t = 0,\ g'_t \neq 0$. Since $\varphi_t(C_t)$ is a dense subgroup of $\mathbb{R}$, then $\widetilde{g'} \leq \widetilde{g}$ is not true, concluding the proof.

$\blacksquare$

In particular, $A_g = A_{g'}$ if and only if $\widetilde{g} = \widetilde{g'}$.

Let $U = \{t \in T \mid \varphi_t(C_t) \text{ is a dense subgroup of } \mathbb{R}\}$.

If T is finite, then $\widetilde{g} = \widetilde{g'}$ if and only if $\{u \in U \mid g_u = 0\} = \{u \in U \mid g'_u = 0\}$. In this situation, there is a bijection between $\mathcal{R}$ and the set $\mathcal{P}(U)$ of all subsets of U, and this bijection is order-reversing: $\widetilde{g} \leq \widetilde{g'}$ if and only if $\{u \in U \mid g'_u = 0\} \subseteq \{u \in U \mid g_u = 0\}$.

Thus, if T is finite and $U = \varnothing$, then $\mathcal{R}$ consists of only one element, so $A = [[R^{S,\leq}]]$ is the only field which may be obtained by the above method. For example, this is the case when $S = \mathbb{Z} \times \cdots \times \mathbb{Z}$ (k copies of $\mathbb{Z}$), endowed with the lexicographic order.

More generally, I can also describe $\mathcal{R}$ when $U = \varnothing$ and T is not finite.

For this purpose, I consider partitions $\mathcal{Z} = (Z_n)_{n \geq 0}$ of T into pairwise disjoint subsets $Z_n : T = \bigcup_{n=0}^{\infty} Z_n$. I define the following equivalence relation: $\mathcal{Z} \sim \mathcal{Z}'$ when:

i) there exists n_0 such that $Z_n = Z'_n$ for every $n \geq n_0$;

ii) for every n, the symmetric difference $Z_n \triangle Z'_n = \{t \in T \mid t \in Z_n, t \notin Z'_n\} \cup \{t \in T \mid t \in Z'_n, t \notin Z_n\}$ is finite.

iii) $Z_0 \cap U = Z'_0 \cap U$.

If $g \in \mathcal{G}$, let $Z_{g,n} = \{t \in T \mid g_t = n\}$ for every $n \geq 0$; let $\mathcal{Z}_g = (Z_{g,n})_{n \geq 0}$ be the partition of T thus obtained. If $g' \sim g$ then $\mathcal{Z}_{g'} \sim \mathcal{Z}_g$ as easily checked. To $\widetilde{g} \in \mathcal{R}$ I associate the equivalence class $\mathcal{Z}_{\widetilde{g}}$ of $\mathcal{Z}_g$. It is now immediate to verify that this establishes a bijection between $\mathcal{R}$ and the set of equivalence classes of partitions of T.

Now I shall obtain valued fields A_g enjoying an additional property. The method is simply the following. Assume:

i) R satisfies property (P) and $(S, \leq_g)$ satisfies property (Q).

ii) There is a theorem which states that if (i) holds, then $A = [[R^{S,\leq}]]$ satisfies property (P).

Applying this method, I obtain:

(3.2). *For each of the cases described below, the valued fields (A_g, π_g), with residue field R and value group $(S, \leq_g)$ satisfy property (P), when R satisfies property (P) and $(S, \leq_g)$ satisfies property (Q):*

a) *R is an algebraically closed field of characteristic 0, S is a divisible group and $g \in \mathcal{G}$ is arbitrary.*

b) *R is a real field, S and $g \in \mathcal{G}$ are arbitrary.*

c) *R is a real closed field, S is a divisible group and $g \in \mathcal{G}$ is arbitrary.*

d) *$n \geq 2$, the characteristic of R does not divide n, $R = R^n$, $S = n \cdot S$ and $g \in \mathcal{G}$ is arbitrary.*

e) *R is a perfect field of characteristic p, $S = p \cdot S$, the set of artinian and narrow subsets of S is p-divisible and $g \in \mathcal{G}$ has finite support.*

f) *R is a perfect field with no proper solvable extension, S is a divisible group and if R has characteristic p, then the set of artinian and narrow subsets of S is p-divisible and $g \in \mathcal{G}$ has finite support.*

g) *R is a real pythagorean field, S and $g \in \mathcal{G}$ are arbitrary.*

h) *R is a non-real pythagorean field, $S = 2 \cdot S$ and $g \in \mathcal{G}$ is arbitrary.*

i) *R is an euclidean field, $S = 2 \cdot S$ and if R has characteristic 2, then $g \in \mathcal{G}$ has finite support.*

Proof:

This follows at once from (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), (2.8), (2.9) and lemma 3.1.

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Examples 2. Now I give many examples of maximal valued fields, for which the collections of residue fields, respectively value groups, satisfy prescribed properties.

The following terminology will be used. Let $(T, \leq)$ be a totally ordered set, let $(X_t)_{t\in T}$ be a family of sets, resp. $(X_t, \leq_t)_{t\in T}$ be a family of totally ordered abelian groups, resp. $(X_t)_{t\in T}$ be a family of fields, resp. $(X_t, \pi_t)_{t\in T}$ be a family of fields X_t

endowed with a valuation π_t. It is called an *only lower jumping chain of sets* (resp. *totally ordered abelian groups*, resp. *fields*, resp. *valued fields*) when the following condi tions are satisfied:

i) If $t < t'$ then $X_t \subset X_{t'}$, resp. X_t is a proper subgroup of $X_{t'}$ and $\leq_t$ is the restriction of $\leq_{t'}$, resp. X_t is a proper subfield of $X_{t'}$, resp. X_t is a proper subfield of $X_{t'}$ and π_t is the restriction of $\pi_{t'}$.

ii) $X_t \neq \bigcup\limits_{u<t} X_u$.

iii) If $U \subseteq T$, $u = \inf U$, then $X_u = \bigcap\limits_{t \in U} X_t$.

In similar way, I define *no lower jumping chains of sets*, resp. *totally ordered abelian groups*, resp. *fields with valuations*, requiring that condition (i) be satisfied, as well as:

ii′) If $U \subseteq T$, $u = \sup U$, then $X_u = \bigcup\limits_{t \in U} X_t$.

I shall show the existence of such chains of totally ordered abelian groups, fields and valued fields, which moreover possess additional interesting properties.

LEMMA 3.2. *Let $(T, \leq)$ be any totally ordered set. Then there exists an only lower jumping chain $(S_t, \leq_t)_{t \in T}$ of totally ordered abelian groups, resp. totally ordered divisible abelian group, resp. totally ordered n-divisible abelian groups (where $n \geq 2$).*

Proof:

For every $t \in T$, let L_t be the lower class of T generated by $t : L_t = \{t' \in T \mid t' \leq t\}$. It is easy to see:

a) If $t < t'$ then $L_t \subset L_{t'}$, because $t' \in L_{t'} \backslash L_t$.

b) If $U \subseteq T$ and $t = \inf U$, then $L_t = \bigcap\limits_{u \in U} L_u$.

Let $(G, \leq)$ be a totally ordered group, $G \neq \{0\}$. Let S be the set of all mappings $s : T \to G$, such that $\mathrm{supp}(s) = \{t \in T \mid s(t) \neq 0\}$ is a well-ordered subset of $(T, \leq)$. With pointwise addition, S is a group. Define on S the lexicographic order, still denoted by $\leq$: $s \leq s'$ when $s = s'$ or $s \neq s'$ and $s(t) < s'(t)$, where t is the smallest element in $\mathrm{supp}(s - s')$. Then $(S, \leq)$ is a totally ordered group.

For every $t \in T$, let $S_t = \{s \in S \mid \mathrm{supp}(s) \subseteq L_t\}$.

S_t is a subgroup of S; let $\leq_t$ be the restriction to S_t of the total order $\leq$ of S.

I show now:

i) This is trivial.

ii) If $t \in T$, if $g \in G, g \neq 0$, let $s \in S$ be such that $s(t) = g \in G$, $s(u) = 0$, for every $u \in T$, $u \neq t$; then $s \in S_t$, $s \notin \bigcup\limits_{u<t} S_u$.

iii) If $U \subseteq T$ and $t = \inf U$, then $S_t = \bigcap\limits_{u \in U} S_u$. Indeed, if $s \in \bigcap\limits_{u \in U} S_u$, then $\mathrm{supp}(s) \subseteq \bigcap\limits_{u \in U} L_u = L_t$.

To conclude, note that if G is a divisible, resp. n-divisible group, then each group S_t is divisible, resp. n-divisible.

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LEMMA 3.3. *Let $(T, \leq)$ be any totally ordered set. Then there exists a no lower jumping chain $(S'_t, \leq_t)_{t \in T}$ of totally ordered abelian groups, resp. divisible totally ordered abelian groups, resp. n-divisible totally ordered abelian groups (where $n \geq 2$).*

Proof:

I keep the notations of the preceding lemma.

For every $t \in T$, define $S'_t = \bigcup\limits_{u<t} S_u$. S'_t is endowed with a compatible total order $\leq'_t$, which is unique inducing the order $\leq_u$ on each group S_u. I prove:

i) If $t < t'$ then $S'_t \subset S'_{t'}$. Indeed, let $g \in G, g \neq 0$, let $s \in S$ be such that $s(t) = g$, $s(u) = 0$ for every $u \in T$, $u \neq t$. Then $s \in S_t \subseteq S'_{t'}$, but $s \notin S'_t$. Clearly $\leq'_t$ is the restriction of $\leq'_{t'}$.

ii$'$) If $U \subseteq T$ and $t = \sup U$, then $S'_t = \bigcup\limits_{u \in U} S'_u$. Indeed, let $s \in S'_t$, so there exists $v < t$, such that $s \in S_v$; there exists also $u \in U$, such that $v < u \leq t$. Then $s \in S'_u$, proving the statement.

To conclude, note that if G is divisible, resp. n-divisible, so are the groups S'_t.

■

The next lemmas concern fields.

LEMMA 3.4. *Let $(T, \leq)$ be any totally ordered set. Then there exists an only lower jumping chain $(R_t)_{t\in T}$ of fields, each one satisfying any one of the following properties*

a) *algebraically closed field of characteristic* 0.
b) *algebraically closed field of characteristic* p.
c) *real closed field.*
d) *pythagorean and real, but not a real closed field.*
e) *real, but not a pythagorean field.*
f) *pythagorean, but not a real field.*
g) *perfect of characteristic p, but not an algebraically closed field.*
h) *perfect of characteristic p with no proper solvable extension, but not an algebraically closed field.*
i) *euclidean, but not a pythagorean field.*
j) *the characteristic does not divide $n \geq 2$, the field is closed under taking n^{th} roots, but not an algebraically closed field.*

Proof:

Let R be a field and let a $(S_t, \leq_t)_{t\in T}$ be a no lower jumping chain of totally ordered abelian groups, as given in lemma 3.2. Let $R_t = [[R^{S_t, \leq_t}]]$, for each $t \in T$. Then (with natural identifications), $(R_t)_{t\in T}$ is a no lower jumping chain of fields.

By choosing the groups S_t to be divisible, resp. n-divisible (for some $n \geq 2$) and by choosing the field R appropriately, it follows from (2.1) to (2.9) that I obtain fields R_t having any one of the required properties (a) to (j).

■

LEMMA 3.5. *Let $(T, \leq)$ be a totally ordered set. Then there exists a no lower jumping chain $(R'_t)_{t\in T}$ of fields, each one satisfying any one of the properties (a) to (j) of lemma 3.4.*

Proof:

I keep the notations of the preceding lemma. For every $t \in T$, let $R'_t = \bigcup\limits_{u<t} R_u$, so R'_t is a field.

Now I prove properties (i) and (ii$'$).

i) If $t < t'$ then R'_t is a subfield of $R'_{t'}$. Moreover by (ii$'$) of lemma 3.3, $R'_t = \bigcup_{u<t} R_u \subset R_t \subseteq R'_{t'}$.

ii$'$) Let $t = \sup U$. I show that $R'_t = \bigcup_{u\in U} R'_u$. If $f \in R'_t = \bigcup_{v<t} R_v$, there exists $v < t$ such that $f \in R_v$. Let $u \in U$ be such that $v < u \leq t$, then $f \in R'_u$.

To conclude the proof, it suffices to note that taking each one of the fields R_t with one of the properties (a) to (j), then the fields $R'_t = \bigcup_{u<t} R_u$ satisfy the same property.

$\blacksquare$

The preceding lemmas are applied to obtain the following examples:

(3.3). *Let $(T, \leq)$, $(V, \leq)$ be totally ordered sets. Then there exists a family $(A_{(t,v)}, \pi_{(t,v)})_{(t,v)\in T\times V}$ of valued fields which are maximal and with the following properties:*

1) *if $t, t' \in T$ and $v \in V$ then the value groups of $A_{(t,v)}$ and of $A_{(t',v)}$ are equal and denoted by $(S_v, \leq_v)$.*
2) *if $t \in T$, $v, v' \in V$, then the residue fields of $A_{(t,v)}$ and of $A_{(t,v')}$ are equal and denoted by R_t.*
3) *$(S_v, \leq_v)_{v\in V}$ is an only lower jumping chain (respectively, a no lower jumping chain) of ordered abelian groups.*
4) *$(R_t)_{t\in T}$ is an only lower jumping chain (respectively, a no lower jumping chain) of fields. Moreover, if each R_t satisfies any one of the properties (a) - (j) of lemma 3.4, then so does each field $A_{(t,v)}$, for every $v \in V$.*

Proof:

By lemma 3.2 (resp. lemma 33) there exists an only lower jumping chain (resp. a no lower jumping chain) of totally ordered abelian groups $(S_v, \leq_v)_{v\in V}$. By lemma 3.4 (resp. lemma 3.5), there exists an only lower jumping chain (resp. a no lower jumping chain) of fields $(R_t)_{t\in T}$. For each $(t, v) \in T \times V$, let $A_{(t,v)} = [[R_t^{S_v,\leq_v}]]$, endowed with its natural valuation $\pi_{(t,v)}$. The value group is $(S_v, \leq_v)$, the residue field is R_t. Thus, properties (1), (2), (3) and (4) are satisfield. Moreover, by choosing app S_v with appropriate property and all R_t with any one of the properties (a) to (j), it follows that each field $A_{(t,v)}$ satisfies the same property.

■

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