Positive cones and gauges on algebras with involution

(Joint work with Vincent Astier)

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Sums of Squares
Real Algebraic Geometry and its Applications
Innsbruck, 21.08.2017

Real theory of central simple algebras with involution

Aim: to develop real algebra for (finite dimensional) central simple algebras with involution

Examples:
$(\mathbb{M}_n(\mathbb{R}), t)$, $(\mathbb{M}_n(\mathbb{C}), -^t)$, $(\mathbb{M}_n(\mathbb{H}), -^t)$, $\big(\mathbb{M}_n(\mathbb{Q}(\sqrt{2})), \mathrm{Int}(a) \circ t\big)$

In general:

Theorem (≈ Wedderburn's Theorem)

$$(A, \sigma) \cong (\mathbb{M}_n(D), \mathrm{Int}(u) \circ \vartheta^t)$$

Real theory of central simple algebras with involution

Fix notation (for the next few slides):

- F: (formally) real field
- X_F: space of orderings of F
- F_P: real closure of F at $P \in X_F$
- $a \geq_P b \overset{\text{def}}{\iff} a - b \in P$
- A: simple F-algebra with[1] $Z(A) = F$ and $[A : F] < \infty$

[1] Our results are also valid for unitary involutions, but in order to simplify the presentation, I ignore them in this talk.

First step: signatures, positivity

Witt group: $W(A,\sigma) \approx$ hermitian forms over (A,σ)

Signature at $P \in X_F$: $\mathrm{sign}_P^\eta : W(A,\sigma) \longrightarrow \mathbb{Z}$ (Nice properties !)

Positivity: Fix η. Let $P \in X_F$ and $a \in \mathrm{Sym}(A,\sigma)$. Then

$$a >_P 0 \overset{\text{def}}{\iff} \mathrm{sign}_P^\eta \langle a \rangle_\sigma \text{ is maximal}$$

$$\iff \mathrm{sign}_P^\eta \langle a \rangle_\sigma = n_P := \text{ matrix size of } A \otimes F_P \quad \text{(= Theorem !)}$$

Application: answer[3] to Procesi-Schacher question[13]

(*When are totally positive elements in (A,σ) sums of hermitian squares ?* – cf. Hilbert's 17th Problem)

Question: Can positivity be defined intrinsically ?

Positive cones on (A, σ)

Prepositive cone on (A, σ): $\mathscr{P} \subseteq \mathrm{Sym}(A, \sigma)$ with

(1) $\mathscr{P} + \mathscr{P} \subseteq \mathscr{P}$

(2) $\sigma(x) \cdot \mathscr{P} \cdot x \subseteq \mathscr{P}, \quad \forall x \in A$

(3) $\mathscr{P} \cap -\mathscr{P} = \{0\}$

(4) $\mathscr{P}_F := \{\alpha \in F \mid \alpha \mathscr{P} \subseteq \mathscr{P}\} \in X_F$

Positive cone := a maximal prepositive cone

Partial ordering $\succcurlyeq_{\mathscr{P}}$ on $\mathrm{Sym}(A, \sigma)$: $a \succcurlyeq_{\mathscr{P}} b \overset{\text{def}}{\iff} a - b \in \mathscr{P}$

Theorem (A.-U. 2017[4])

$X_{(A,\sigma)} := \{$*positive cones on* $(A, \sigma)\}$ *is a spectral space w.r.t. the Harrison topology with basis*

$$H_\sigma(a_1, \ldots, a_k) := \{\mathscr{P} \in X_{(A,\sigma)} \mid a_1 \succcurlyeq_{\mathscr{P}} 0, \ldots, a_k \succcurlyeq_{\mathscr{P}} 0\}.$$

Positive cones on (A,σ)

Also[4]:

"Artin" Theorem: describes totally positive elements

"Artin-Schreier" Theorem: describes "formally real" algebras

Examples

- $(A,\sigma) = (F, \mathrm{id}_F) \implies X_{(A,\sigma)} = \{P, -P \mid P \in X_F\}$
- $(A,\sigma) = (\mathbb{M}_n(\mathbb{R}), t) \implies X_{(A,\sigma)} = \{\mathrm{PSD}, -\mathrm{PSD}\}$
- (A,σ) with A division algebra

$$\implies X_{(A,\sigma)} = \{\mathscr{M}_P, -\mathscr{M}_P \mid P \in X_F, \mathrm{sign}_P^\eta \not\equiv 0\}$$

with $\mathscr{M}_P := \{a \in \mathrm{Sym}(A,\sigma)^\times \mid \mathrm{sign}_P^\eta \langle a \rangle_\sigma \text{ is maximal}\} \cup \{0\}$

Positive cones on (A, σ)

Theorem ("Sylvester Inertia", A.-U. 2017[4])

$\exists t = t(A,\sigma) \in \mathbb{N}$ such that $\forall \mathscr{P} \in X_{(A,\sigma)}$, and $\forall h \in W(A,\sigma)$, $\exists u_1, \ldots, u_t \in P := \mathscr{P}_F \in X_F$, $a_i \in \mathscr{P}$ and $b_j \in -\mathscr{P}$ such that

$$n_P^2 \times \langle u_1, \ldots, u_t \rangle \otimes h \simeq \langle a_1, \ldots, a_r \rangle_\sigma \perp \langle b_1, \ldots, b_s \rangle_\sigma .$$

Furthermore, r and s only depend on $\mathscr{P}$ and rank *h.*

Signature at $\mathscr{P}$:

$$\operatorname{sign}_{\mathscr{P}} h := \frac{r - s}{n_P t} \in \mathbb{Z}$$

$$= \varepsilon_P \operatorname{sign}_P^\eta h, \qquad \varepsilon_P \in \{-1, 1\} \quad \text{(= Theorem !)}$$

Valuation rings / valuations on real fields

Let $P \in X_F$. Then

$$R_P := \{x \in F \mid \exists r \in \mathbb{Q} \quad |x|_P \leq_P r\}$$

is a valuation ring in F with unique maximal ideal

$$I_P := \{x \in F \mid \forall r \in \mathbb{Q}_{>0} \quad |x|_P \leq_P r\}.$$

$R_P \longleftrightarrow$ valuation v_P on F, and $1 + I_P \subset P$ (v_P is compatible with P).

Valuation rings / valuations on real fields

v: valuation on F with residue field F_v and compatible with $P \in X_F$. Then P induces an ordering $\overline{P}$ on F_v.

Conversely, the Baer-Krull theorem describes the orderings on F, compatible with v, that induce a given ordering on F_v.

Question:
Similar connection between $\mathscr{P} \in X_{(A,\sigma)}$ and noncommutative valuations / valuation rings ?

Noncommutative valuation rings / valuations

1950s: Valuations on division rings and invariant valuation rings

1980s: Dubrovin valuation rings of simple Artinian rings

1990s: Morandi value functions on simple Artinian rings

2010s: Tignol-Wadsworth gauges on semisimple rings

Conventions for rest of talk

To simplify presentation :

- D: division algebra with centre a field F, $[D : F] < \infty$
- A: central simple algebra with centre a field F, $[A : F] < \infty$
- Γ: divisible totally ordered abelian group

  (large enough to contain the values of all valuations and the degrees of all gradings that occur)

Valuations on division rings / invariant valuation rings

Valuation defined in the usual way: $v : D \longrightarrow \Gamma \cup \{\infty\}$ such that

(1) $v(x) = \infty \iff x = 0$

(2) $v(x+y) \geq \min\{v(x), v(y)\}$

(3) $v(xy) = v(x) + v(y)$

$\Gamma_v := v(D^\times)$: value group (subgroup of Γ)

$R_v := \{d \in D \mid v(d) \geq 0\}$ is an invariant valuation ring of D, i.e. $\forall\, d \in D^\times$,

(a) $d \in R_v$ or $d^{-1} \in R_v$

(b) $dR_vd^{-1} = R_v$

valuations $\longleftrightarrow$ invariant valuation rings

Valuations on division rings / invariant valuation rings

Very useful ! E.g. Amitsur (1972): construction of non-crossed product algebras.

Problem: valuations on F need not extend to D !

Theorem (Ershov 1988[9], Morandi 1989[11])
v extends from F to $D \iff D \otimes F^h$ is division (F^h: Henselization). The extension is unique.

Problem: tensor products, scalar extensions may give matrices: v does not extend (zero divisors !)

Dubrovin valuation rings

A subring B of A with Jacobson radical $J = J(B)$ is a Dubrovin valuation ring of A if

(1) B/J is a simple ring

(2) $\forall a \in A \setminus B$, $\exists b, b' \in B$ such that $ab, b'a \in B \setminus J$

Examples: (matrices over) invariant valuation rings, Azumaya algebras over commutative valuation rings

Going down:

- $B \cap F = Z(B)$
- $Z(B)$ is a valuation ring of F with maximal ideal $J \cap F$

Dubrovin valuation rings

Going up:

Theorem (Dubrovin 1984[8])

V: valuation ring of $F \implies \exists$ Dubrovin valuation ring B of A with $B \cap F = V$

Theorem (Wadsworth 1989[17])

B is unique up to isomorphism

Problem: in general there is no "valuation" on A associated to B

Morandi value functions

$w : A \longrightarrow \Gamma \cup \{\infty\}$ is a Morandi value function if

(1) $w(x) = \infty \iff x = 0$

(2) $w(x + y) \geq \min\{w(x), w(y)\}$

(3) $w(xy) \geq w(x) + w(y)$

(4) $w(-1) = 0$

(5) $\Gamma_w := w(A^\times) = w\big(\{x \in A^\times \mid w(x^{-1}) = -w(x)\}\big)$

Let

$B_w := \{x \in A \mid w(x) \geq 0\}$ (the "valuation ring" of w)

$J_w := \{x \in A \mid w(x) > 0\}$ (two-sided ideal)

$A_w := B_w / J_w$ (the "residue ring" of w)

Morandi value functions

Theorem (Morandi 1989[12])

Let B be a Dubrovin valuation ring of A. Then B is integral over $V := Z(B) = B \cap F \iff \exists$ a Morandi value function w on A with $B = B_w$ and $J(B) = J_w$.

In this case:

(1) $v := w|_F$ *is a valuation on F corresponding to V*

(2) $w(x) \leq \frac{1}{n} v(\mathrm{Nrd}(x))$ *in general, where* $n = \sqrt{[A : F]}$

(3) w *is uniquely determined on A by v and B*

Morandi value functions

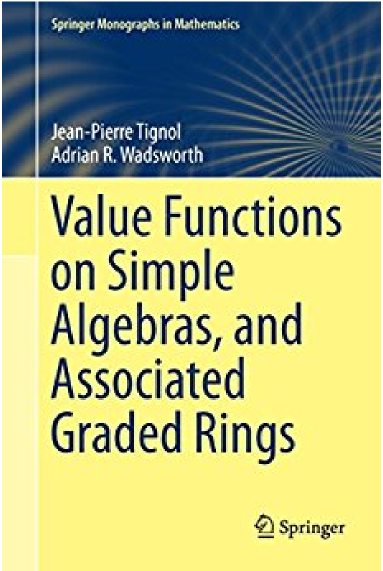
Theorem (Wadsworth 1989 cf. [12])

Let w be a Morandi value function on A. Then A_w simple $\implies$ B_w is a Dubrovin valuation ring of A with $J(B_w) = J_w$.

In this case, the defect of w is the integer

$$\delta(w) := \frac{[A : F]}{[A_w : F_v]\,|\Gamma_w : \Gamma_v|}$$

Tignol-Wadsworth gauges



Tignol-Wadsworth gauges

Let $v : F \longrightarrow \Gamma \cup \{\infty\}$ be a valuation.

Then $g : A \longrightarrow \Gamma \cup \{\infty\}$ is a v-value function if $\forall x, y \in A$, $\forall \lambda \in F$,

(1) $g(x) = \infty \iff x = 0$

(2) $g(x + y) \geq \min\{g(x), g(y)\}$

(3) $g(\lambda x) = v(\lambda) + g(x)$

g is surmultiplicative if

$$g(1) = 0 \quad \text{and} \quad g(xy) \geq g(x) + g(y)$$

Tignol-Wadsworth gauges

Define abelian groups

$$A_{\geq \gamma} := \{a \in A \mid g(a) \geq \gamma\}, \qquad A_{>\gamma} := \{a \in A \mid g(a) > \gamma\}$$

$$A_\gamma := A_{\geq\gamma}/A_{>\gamma}$$

Similarly (for v):

$$F_{\geq\gamma}, \qquad F_{>\gamma}, \qquad F_\gamma$$

If g is surmultiplicative v-value function on A, then

$$\mathrm{gr}(A) := \bigoplus_{\gamma\in\Gamma} A_\gamma \quad \text{is a graded algebra over} \quad \mathrm{gr}(F) := \bigoplus_{\gamma\in\Gamma} F_\gamma$$

A_0: residue ring of g; $\qquad F_0 = F_v$: residue field of v

Tignol-Wadsworth gauges

$g : A \longrightarrow \Gamma \cup \{\infty\}$ is a *v*-gauge if

(1) g is a surmultiplicative v-value function

(2) $\mathrm{gr}(A)$ is a graded semisimple $\mathrm{gr}(F)$-algebra

(3) $[\mathrm{gr}(A) : \mathrm{gr}(F)] = [A : F]$

Theorem (Tignol-Wadsworth 2010[14])

Let g be a surmultiplicative v-value function on A. Then g is a Morandi value function on A with defect $\delta(g) = 1 \iff g$ is a gauge on A with simple residue ring A_0.

In this case, the gauge ring $A_{\geq 0}$ is a Dubrovin valuation ring of A.

Positive cones and gauges on (A, σ)

Fix $\mathscr{P} \in X_{(A,\sigma)}$ with $P = \mathscr{P}_F \in X_F$. (We say "$\mathscr{P}$ is over P".)

Assume $1 \in \mathscr{P}$ and let (inspired by Holland 1980[10])

$$R_{\mathscr{P}} := \{a \in A \mid \exists m \in \mathbb{Q} \quad \sigma(a)a \preccurlyeq_{\mathscr{P}} m\}$$
$$I_{\mathscr{P}} := \{a \in A \mid \forall m \in \mathbb{Q}_{>0} \quad \sigma(a)a \preccurlyeq_{\mathscr{P}} 1/m\}$$

Properties

- $R_{\mathscr{P}}$ is a subring of A and $\sigma(R_{\mathscr{P}}) \subseteq R_{\mathscr{P}}$
- $I_{\mathscr{P}}$ is a two-sided ideal of $R_{\mathscr{P}}$ and $\sigma(I_{\mathscr{P}}) \subseteq I_{\mathscr{P}}$
- $R_{\mathscr{P}}^{\times} = \{a \in A \mid \exists r, s \in \mathbb{Q}_{>0} \quad r \preccurlyeq_{\mathscr{P}} \sigma(a)a \preccurlyeq_{\mathscr{P}} s\}$
- $R_{\mathscr{P}} \cap F = R_P$

Positive cones and gauges on (A, σ)

Theorem (A.-U. 2017[7])

Let $\mathscr{P} \in X_{(A,\sigma)}$ with $P = \mathscr{P}_F \in X_F$. Assume that $1 \in \mathscr{P}$. Then

(1) *$\mathscr{P}$ induces a v_P-gauge $w_{\mathscr{P}}$ on A with $R_{\mathscr{P}} = A_{\geq 0}$ and $I_{\mathscr{P}} = A_{>0}$*

(2) *$w_{\mathscr{P}}$ is σ-special, i.e. $w_{\mathscr{P}}(\sigma(x)x) = 2w_{\mathscr{P}}(x)\ \forall x \in A$*

(3) *$\mathscr{P}$ induces a prepositive cone $\overline{\mathscr{P}}$ on $(A_0, \overline{\sigma})$ (A_0: residue ring of $w_{\mathscr{P}}$; $\overline{\sigma}$: involution induced by σ)*

Sketch of Proof of Part (1).

(a) Extend scalars to real closure at P:[2]

$$(A \otimes F_P, \sigma \otimes \mathrm{id}) \cong (\mathbb{M}_n(F_P), t)$$

Then $\mathscr{P}$ over P extends to PSD over $\mathbb{P}$ on $(\mathbb{M}_n(F_P), t)$, where $\mathbb{P} = F_P^2$: unique ordering on F_P.

Let $R_{\mathbb{P}}$ be the valuation ring of F_P induced by $\mathbb{P}$ and $v_{\mathbb{P}}$ the corresponding valuation.

[2] To simplify the presentation I only consider the split orthogonal case.

(b) Consider the "Holland ring" R_{PSD} and ideal I_{PSD}.

Then R_{PSD} is a Dubrovin valuation ring of $\mathbb{M}_n(F_P)$ and $J(R_{\text{PSD}}) = I_{\text{PSD}}$.

In fact: $R_{\text{PSD}} = \mathbb{M}_n(R_{\mathbb{P}})$.

(c) R_{PSD} is integral over $Z(R_{\text{PSD}}) = R_{\text{PSD}} \cap F_P = R_{\mathbb{P}}$.

(d) Morandi $\implies$ $\exists$ Morandi value function

$$w_{\text{PSD}} : \mathbb{M}_n(F_P) \longrightarrow \Gamma \cup \{\infty\}$$

with $R_{\text{PSD}} = B_{w_{\text{PSD}}}$. In fact:

$$w_{\text{PSD}}\big((a_{ij})\big) = \min_{i,j}\{v_{\mathbb{P}}(a_{ij})\}$$

(e) w_{PSD} is a surmultiplicative $v_{\mathbb{P}}$-value function.

(f) $\text{char}(F_{v_{\mathbb{P}}}) = 0 \implies$ defect $\delta(w_{\text{PSD}}) = 1$.

(g) Tignol-Wadsworth $\implies$ w_{PSD} is a $v_{\mathbb{P}}$-gauge on $\mathbb{M}_n(F_P)$.

(h) $w_{\mathscr{P}} := w_{\mathrm{PSD}}|_A$ is a v_P-gauge on A. (Requires some work !)

(i) $R_{\mathscr{P}} = A_{\geq 0}$ and $I_{\mathscr{P}} = A_{>0}$: easy computation. ■

Problems we are currently working on

- In part (3), is $\overline{\mathscr{P}}$ maximal, i.e., a positive cone ?
- A notion of compatibility between gauges and positive cones ?
- Baer-Krull: Can we lift positive cones from the residue algebra with involution to (A, σ) ?
- Residue hermitian forms ?
- Bröcker-Prestel local-global principle ?

Thank you !

References

[1] V. Astier and T. Unger, Signatures of hermitian forms and the Knebusch trace formula, Math. Ann. **358** (2014), no. 3-4, 925–947.

[2] V. Astier and T. Unger, Signatures of hermitian forms and "prime ideals" of Witt groups, Adv. Math. **285** (2015), 497–514.

[3] V. Astier and T. Unger, Signatures of hermitian forms, positivity, and an answer to a question of Procesi and Schacher (2016), http://arxiv.org/abs/1511.06330.

[4] V. Astier and T. Unger, Positive cones on algebras with involution (2017), http://arxiv.org/abs/1609.06601.

[5] V. Astier and T. Unger, Signatures, sums of hermitian squares and positive cones on algebras with involution (2017), http://arxiv.org/abs/1706.01264.

[6] V. Astier and T. Unger, Stability index of algebras with involution, Contemporary Mathematics, vol. 697, American Mathematical Society, 2017 (to appear).

[7] V. Astier and T. Unger, Positive cones and gauges on algebras with involution, in preparation.

[8] N. I. Dubrovin, Noncommutative valuation rings in simple finite-dimensional algebras over a field, Mat. Sb. (N.S.) **123** (165) (1984), 496–509 (Russian); English transl.: Math. USSR Sb. **51** (1985), 493–505.

[9] Yu. L. Ershov, Co-Henselian extensions and Henselization of division algebras, Algebra i Logika **27** (1988), 649–658 (Russian); English transl.: Algebra and Logic **27** (1988), 401–407.

[10] S. S. Holland, Jr., $*$-Valuations and ordered $*$-fields, Trans. Amer. Math. Soc. **262** (1980), 219–243.

[11] P. J. Morandi, The Henselization of a valued division algebra, J. Algebra **122** (1989), 232–243.

[12] P. J. Morandi, Value functions on central simple algebras, Trans. Amer. Math. Soc. **315** (1989), 605–622.

[13] C. Procesi and M. Schacher, A non-commutative real Nullstellensatz and Hilbert's 17th problem. Ann. of Math. (2), **104**(3) (1976), 395–406.

[14] J.-P. Tignol and A. R. Wadsworth. Value functions and associated graded rings for semisimple algebras. Trans. Amer. Math. Soc., **362**(2) (2010), 687–726.

[15] J.-P. Tignol and A. R. Wadsworth. Valuations on algebras with involution. Math. Ann., **351**(1) (2011), 109–148.

[16] J.-P. Tignol and A. R. Wadsworth. Value functions on simple algebras, and associated graded rings. Springer Monographs in Mathematics. Springer, Cham, 2015.

[17] A. R. Wadsworth, Dubrovin valuation rings and Henselization, Math. Ann. **283** (1989), 301–328.