THE FIELD OF REALS WITH MULTISUMMABLE SERIES AND THE EXPONENTIAL FUNCTION

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Abstract. We show that the field of real numbers with multisummable real power series is model complete, o-minimal and polynomially bounded. Further expanding by the exponential function yields again a model complete and o-minimal structure which is exponentially bounded, and in which the Gamma function on the positive real line is definable.

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Introduction

We begin by rephrasing a few definitions concerning model completeness and o-minimality in a setting suitable for this paper. This allows us to state our first main result, Theorem A, in a way that is at the same time precise and widely accessible.

Let $\mathcal{F}$ be a collection of functions $f : \mathbb{R}^m \to \mathbb{R}$ for various $m \in \mathbb{N} := \{0, 1, 2, \dots\}$. A set $A \subseteq \mathbb{R}^n$ is **0-definable** in the **structure** $\mathbb{R}_{\mathcal{F}} := (\mathbb{R}, <, 0, 1, +, -, \cdot, \mathcal{F})$ (the **expansion** of the ordered field of real numbers by $\mathcal{F}$) if A belongs to the smallest collection of subsets of the spaces $\mathbb{R}^0 := \{0\}$, $\mathbb{R}^1 = \mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\dots$, which

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(i) contains the graphs of addition, multiplication, and of all functions in $\mathcal{F}$,
(ii) contains all diagonals $\{(x_1,\dots,x_m):\ x_i=x_j\}\subseteq\mathbb{R}^m$, $1\le i<j\le m$, and
(iii) is closed under taking cartesian products, finite intersections, complements and images under projection maps $(x_1,\dots,x_{m+1})\mapsto(x_1,\dots,x_m):\mathbb{R}^{m+1}\longrightarrow\mathbb{R}^m$.

The collection of 0-definable sets is automatically closed under many other basic operations, like taking finite unions and topological closures. A set $A\subseteq\mathbb{R}^n$ is said to be **definable** in $\mathbb{R}_{\mathcal{F}}$ if $A=S_a$ for some $S\subseteq\mathbb{R}^{m+n}$ that is 0-definable in $\mathbb{R}_{\mathcal{F}}$ and some $a\in\mathbb{R}^m$, where $S_a=\{y\in\mathbb{R}^n:\ (a,y)\in S\}$. A map $f:A\longrightarrow\mathbb{R}^n$ with $A\subseteq\mathbb{R}^m$ is said to be definable (0-definable) in $\mathbb{R}_{\mathcal{F}}$ if its graph $\Gamma(f)\subseteq\mathbb{R}^{m+n}$ is definable (0-definable) in $\mathbb{R}_{\mathcal{F}}$. The definable sets form the smallest collection of subsets of the spaces $\mathbb{R}^n$ that satisfies (i), (ii) and (iii) above and that includes all one-point sets $\{r\}$ with $r\in\mathbb{R}$.

The structure $\mathbb{R}_{\mathcal{F}}$ is said to be **model complete** if in the above definition of "0-definable set" the operation of taking complements is superfluous. (See [3] for the model-theoretic significance of "model completeness".) Model completeness of $\mathbb{R}_{\mathcal{F}}$ often has the very strong consequence that $\mathbb{R}_{\mathcal{F}}$ is **o-minimal**, that is, each subset of $\mathbb{R}^n$ which is definable in $\mathbb{R}_{\mathcal{F}}$ has only finitely many connected components. (See [4] and [5] for background and results on o-minimal structures.) Four basic examples of model complete and o-minimal expansions of the field of real numbers are:

(1) $(\mathbb{R},<,0,1,+,-,\cdot)$ itself, where $\mathcal{F}=\emptyset$ (Tarski),
(2) $\mathbb{R}_{\text{an}}$, where $\mathcal{F}$ is the set of all functions $f:\mathbb{R}^m\longrightarrow\mathbb{R}$, $m\in\mathbb{N}$, whose restriction to $[-1,1]^m$ is real analytic and which are identically zero outside $[-1,1]^m$ (Gabrielov [13], see also [9, Section 2]),
(3) $(\mathbb{R},<,0,1,+,-,\cdot,\exp)$ with $\mathcal{F}=\{\exp\}$ (Wilkie [24]),
(4) $(\mathbb{R}_{\text{an}},\exp)$, obtained by taking the union of the classes $\mathcal{F}$ from examples (2) and (3), see [9] and [10].

(The choice of the compact cubes $[-1,1]^m$ in example (2) is only a convenient normalization, and the condition on vanishing outside $[-1,1]^m$ merely serves to make the functions in $\mathcal{F}$ everywhere defined.)

In [8] it was shown that the Gamma function on $(0,\infty)$, and also the Riemann zeta function on $(1,\infty)$ are not definable in $(\mathbb{R}_{\text{an}},\exp)$, though for quite different reasons. In this paper we construct model complete and o-minimal expansions of the real field in which these functions *are* definable. Towards this goal we consider in Section 2 (a variant of) Tougeron's class $\mathcal{G}$ of "Gevrey" functions from [21] which gives rise to the structure $\mathbb{R}_{\mathcal{G}}$. Here we only define the one-variable functions belonging to $\mathcal{G}$. Going to the complex plane, we let $R>0$, $0<\phi<\pi$, and $0<\kappa\le1$, and define $\mathcal{G}(R,\phi,\kappa)$ to be the set of all holomorphic functions $f:S\longrightarrow\mathbb{C}$ on the open sector $S=S(R,\phi,\kappa):=\{z\in\mathbb{C}:\ 0<|z|<R,|\arg z|<\kappa\phi\}$ satisfying:

(i) there exist constants $A,\ B>0$ (depending on f) such that $\left|f^{(n)}(z)\right|/n!\le AB^n(n!)^\kappa$ for all $n\in\mathbb{N}$ and $z\in S$ (**Gevrey condition**);
(ii) $\lim_{z\to0}f^{(n)}(z)$ exists in $\mathbb{C}$, for each $n\in\mathbb{N}$.

It is easily checked that $\mathcal{G}(R,\phi,\kappa)$ contains all $\mathbb{C}$-valued constant functions on S and is a ring (under pointwise addition and multiplication of functions) closed under taking derivatives. We extend $f\in\mathcal{G}(R,\phi,\kappa)$ to a continuous function on $S\cup\{0\}$ by setting $f(0):=\lim_{z\to0}f(z)$, and define the **Taylor series** of f at 0 to be the formal power series

$$\widehat{f}(Z) := \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} Z^n \in \mathbb{C}[\![Z]\!].$$

The map $f \mapsto \widehat{f} : \mathcal{G}(R, \phi, \kappa) \longrightarrow \mathbb{C}[\![Z]\!]$ is clearly a differential ring homomorphism. Watson and Nevanlinna [16] proved that this homomorphism is injective if $\phi > \pi/2$. (Here the assumption $\phi > \pi/2$ is essential.)

Example. The integral

$$\int_0^{\infty} \frac{e^{-t}}{1+zt} dt := \lim_{T \to +\infty} \int_0^T \frac{e^{-t}}{1+zt} dt$$

converges for every $z \in \mathbb{C} \setminus (-\infty, 0)$ and defines a function in $\mathcal{G}(R, \phi, 1)$ for any $R > 0$ and $\phi \in (0, \pi)$, whose Taylor series at 0 is the divergent power series $\sum_{n=0}^{\infty} (-1)^n n! Z^n$. Borel summation of this divergent power series returns the value of the integral for $\mathrm{Re}(z) > 0$.

Over the last twenty years this **quasianalyticity property** (the injectivity of the Taylor series homomorphism above) has been extended by Ramis, Ecalle, and others in the context of the theory of multisummability; see [2, 15]. We only need a small part of this theory and restrict our attention to the functions relevant for our purpose, namely those that are real valued and defined on a segment $[0, R] \subseteq \mathbb{R}$.

Thus, for $R > 0$ we let $\mathcal{G}(R)$ be the set of all functions $f : [0, R] \longrightarrow \mathbb{R}$ for which there exists an $\widetilde{R} > R$, an angle $\phi \in (\frac{\pi}{2}, \pi)$, numbers $\kappa_1, \dots, \kappa_n \in (0, 1]$ and functions $f_1 \in \mathcal{G}(\widetilde{R}, \phi, \kappa_1), \dots, f_n \in \mathcal{G}(\widetilde{R}, \phi, \kappa_n)$ such that $f_1, \dots, f_n$ are real valued on $[0, \widetilde{R})$ and

$$f(x) = f_1(x) + \cdots + f_n(x) \quad \text{for } 0 \leq x \leq R.$$

(The Taylor series at 0 of these functions are multisummable in the positive real direction, cf. Balser [2].) It follows easily from this definition that $\mathcal{G}(R)$ contains all $\mathbb{R}$-valued constant functions on $[0, R]$, and that if $f \in \mathcal{G}(R)$, then f is C^{∞} on $[0, R]$, real analytic on $(0, R]$, and $f' \in \mathcal{G}(R)$. Much less obvious is that $\mathcal{G}(R)$ is a ring (under pointwise addition and multiplication) and that the Taylor map

$$f \mapsto \widehat{f} := \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} X^n : \mathcal{G}(R) \longrightarrow \mathbb{R}[\![X]\!]$$

is injective; see [2, 15, 21] for various proofs of this injectivity. This quasianalyticity property is the main fact from multisummability theory that we use in this paper without giving the proof. We now define the one-variable functions of our class $\mathcal{G}$ to be those functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ that are identically 0 outside $[0, 1]$ and satisfy $f\big|[0, 1] \in \mathcal{G}(1)$.

We actually need a rather subtle generalization of these rings $\mathcal{G}(R)$ to several variables, which is due to Tougeron [21]. In Section 2 we develop a simple version of Tougeron's idea which suffices for our purpose. There we associate to every m-tuple $R = (R_1, \dots, R_m)$ of positive real numbers a set $\mathcal{G}(R)$ of functions $f : [0, R_1] \times \cdots \times [0, R_m] \longrightarrow \mathbb{R}$. It is crucial for us that the family of all $\mathcal{G}(R)$ has certain "substitution" properties, which allow the use of "blowing-up". We establish these properties in Sections 4 and 5. The m-variable functions of our class $\mathcal{G}$ are then defined to be the functions $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ that are identically 0 outside $[0,1]^m$ and satisfy $f\big|[0,1]^m \in \mathcal{G}(1,\dots,1)$.

We can now state the first main result, proved in Sections 6 and 7.

Theorem A. *The structure $\mathbb{R}_{\mathcal{G}}$ is model complete and o-minimal. If $\epsilon > 0$ and the function $f : (0,\epsilon) \longrightarrow \mathbb{R}$ is definable in $\mathbb{R}_{\mathcal{G}}$, then there are integers p, d with $d > 0$ and a function $g \in \mathcal{G}(R)$ for some $R > 0$ such that $f(x) = x^{p/d} g\left(x^{1/d}\right)$ for all sufficiently small $x > 0$.*

The second part of the theorem gives a kind of Puiseux expansion for one-variable functions definable in $\mathbb{R}_{\mathcal{G}}$. In particular, it implies (by a change of variables) that the structure $\mathbb{R}_{\mathcal{G}}$ is polynomially bounded. (The structure $\mathbb{R}_{\mathcal{F}}$ is said to be **polynomially bounded** if for each function $f : \mathbb{R} \longrightarrow \mathbb{R}$ that is definable in $\mathbb{R}_{\mathcal{F}}$ there exists a $d \in \mathbb{N}$ and an $R > 0$ such that $|f(x)| < x^d$ for all $x > R$. See [11] for numerous consequences of polynomial boundedness.)

All subsets of $\mathbb{R}^n$ that are definable in $\mathbb{R}_{\text{an}}$ are also definable in $\mathbb{R}_{\mathcal{G}}$. By Nielsen [17, Section 79] the function ϕ on $(1,\infty)$ given by $\log\Gamma(x) = \left(x - \frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \phi(x)$ is definable in $\mathbb{R}_{\mathcal{G}}$. Theorem B below implies that $(\mathbb{R}_{\mathcal{G}}, \exp)$ is model-complete and o-minimal, thus providing a model-complete and o-minimal expansion of the real field in which the Gamma function on $(0,\infty)$ is definable.

Similarly, in [12] we constructed a polynomially bounded, model complete and o-minimal expansion $\mathbb{R}_{\text{an}^*}$ of the real field in which the function $\zeta(-\log x)$ on $(0,1)$ is definable, where ζ denotes the Riemann zeta function. Thus the Riemann zeta function on $(1,\infty)$ is definable in $(\mathbb{R}_{\text{an}^*}, \exp)$, and Theorem B below implies that this structure is model complete and o-minimal. (This answers a question left open in [12].)

Up to this point we have defined from scratch all notions needed to state our results, but we now adopt the usual model-theoretic framework, which allows stronger and more precise formulations. In particular "definable" will mean "definable with parameters" unless otherwise indicated. Let $\widetilde{\mathbb{R}}$ denote any polynomially bounded o-minimal expansion of $(\mathbb{R}, <, 0, 1, +, -, \cdot)$. Also, let K be the **field of exponents** of $\widetilde{\mathbb{R}}$, that is,

$$K = \left\{ r \in \mathbb{R} : \text{ the function } x \mapsto x^r : (0,\infty) \longrightarrow \mathbb{R} \text{ is 0-definable in } \widetilde{\mathbb{R}} \right\}.$$

Then K is a subfield of $\mathbb{R}$, for example, $K = \mathbb{Q}$ for $\widetilde{\mathbb{R}} = \mathbb{R}_{\mathcal{G}}$ by the second part of Theorem A. Finally, let L denote the language of $\widetilde{\mathbb{R}}$ and $T := \mathrm{Th}\big(\widetilde{\mathbb{R}}\big)$ in the language L. For convenience in stating Theorem B, we assume that each 0-definable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the interpretation in $\widetilde{\mathbb{R}}$ of an n-ary function symbol of L. (We can always achieve this by passing to an expansion by definitions of $\widetilde{\mathbb{R}}$.) In particular, we have for each $r \in K$ a unary function symbol f_r whose interpretation in $\widetilde{\mathbb{R}}$ is the function that is identically zero on $(-\infty, 0]$ and that assigns to each $x > 0$ the number x^r.

We can now state the other main result of this paper, proved in Section 9.

Theorem B. *Suppose $\exp\big|[0,1]$ is definable in $\widetilde{\mathbb{R}}$. Then the structure $\big(\widetilde{\mathbb{R}}, \exp, \log\big)$ admits quantifier elimination and its complete theory is axiomatized by*

(i) *T;*
(ii) *the Ressayre axioms for $\exp$; and*
(iii) *for each $r \in K$, the axiom $\forall x(\exp(rx) = f_r(\exp(x)))$.*

Remarks. In this theorem and below we make the convention that log is everywhere defined by setting $\log(x) := 0$ for $x \le 0$. The "Ressayre axioms for exp" (cf. [9, 19]) are the universal closures of the following formulas:

(1) $\exp(x+y) = \exp(x)\exp(y)$;
(2) $x < y \to \exp(x) < \exp(y)$;
(3) $\big(x > 0 \to \exp(\log x) = x\big) \wedge \big(x \le 0 \to \log x = 0\big)$;
(4) $x > n^2 \to \exp(x) > x^n$, for each $n \in \mathbb{N}$;
(5) $0 \le x \le 1 \to \exp(x) = E(x)$,

where E is the unary function symbol of L whose interpretation in $\widetilde{\mathbb{R}}$ is given by $E(x) = \exp(x)$ if $0 \le x \le 1$ and $E(x) = 0$ otherwise. Note that by the assumption preceding the theorem there is indeed a function symbol E as in (5) above. Also, the axioms (iii) for $r \in \mathbb{Q}$ can be omitted in Theorem B, since they are deducible from the axioms (i) and (ii).

Among the consequences of Theorem B are (see Section 9):

(1) any function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ that is 0-definable in $\big(\widetilde{\mathbb{R}}, \exp\big)$ is given piecewise by terms in the language $L(\exp, \log)$;
(2) $\big(\widetilde{\mathbb{R}}, \exp\big)$ is o-minimal; and
(3) $\big(\widetilde{\mathbb{R}}, \exp\big)$ is **exponentially bounded**, that is, for each function $f : \mathbb{R} \longrightarrow \mathbb{R}$ that is definable in $\big(\widetilde{\mathbb{R}}, \exp\big)$ there exists an n such that $|f(x)| < \exp_n(x)$ for all sufficiently large x. (Here $\exp_n$ denotes the n-fold iterate of exp.)

Theorem B is clearly a generalization of the main result in [9] on $(\mathbb{R}_{\mathrm{an}}, \exp)$. The proof there also goes through, except for one difficulty: the simple axiomatization of $\mathrm{Th}(\mathbb{R}_{\mathrm{an}})$ given in [9, Section 3] enabled us to embed its models into power series models and then read off an otherwise non-trivial (but crucial) valuation theoretic property of its models. This route is not available for $\widetilde{\mathbb{R}}$, not even in the special case of $\mathbb{R}_{\mathcal{G}}$. Therefore, we have to establish this valuation theoretic property in a different way. After doing so we can turn things around and use this property to show that some of the generalized power series fields from [9, Section 3] have natural expansions to models of $\mathrm{Th}(\mathbb{R}_{\mathcal{G}})$. This is carried out in Section 10, where we also indicate how the results from [8] on power series models of $(\mathbb{R}_{\mathrm{an}}, \exp)$ extend to $(\mathbb{R}_{\mathcal{G}}, \exp)$. (The extension is not straightforward.)

In this introduction we have focused on a careful description of the main results of this paper. As to the underlying ideas, those of Tougeron [21] were essential for Theorem A, but for various complicated reasons we had to reformulate and strengthen (in Sections 1–4) several technical lemmas in [21]. Section 5 on compositional inversion is needed later for the second part of Theorem A. In Section 6 we have all ingredients ready so that we can almost literally copy the long inductive proof of the main theorem of [12], which is analogous to Theorem A. Rather than repeating that proof and increasing the length of the paper by a third, we have just mentioned the needed modifications in a few sentences. Thus familiarity with [12], which served as a warm-up for the present paper, is required for Sections 6 and 7. Theorem B is independent of Theorem A, and is obtained in Section 9, using results from [7], [6] and [9] in an essential way. The results in Section 10 on power series models depend in addition on [8] and [9].

General conventions. We let m and n range over $\mathbb{N} := \{0, 1, 2, \dots\}$.

If A is any set, we denote by $|A|$ the cardinality of A.

Given a subset A of a topological space S we let $\mathrm{cl}(A)$, $\mathrm{int}(A)$, $\mathrm{bd}(A) := \mathrm{cl}(A) \setminus \mathrm{int}(A)$ and $\mathrm{fr}(A) := \mathrm{cl}(A) \setminus A$ denote the closure, interior, boundary and frontier of A in S respectively, if the ambient space S is clear from context. (Usually we have $S = \mathbb{R}^n$ for some n.)

All rings are assumed to be commutative with $1 \neq 0$. If A is a ring, we denote by $A[\![Y]\!]$ the ring of power series in $Y = (Y_1, \dots, Y_n)$ with coefficients in A. Similarly, if A is a normed ring, then, for $s \in (0, \infty)$, $A\{Y\}_s$ denotes the subring of $A[\![Y]\!]$ consisting of all power series $\sum a_\beta Y^\beta$ such that $\sum |a_\beta| s^\beta < \infty$.

1. Blow-up Height

For the inductive proof of [12, Section 8] to go through in the present setting, we shall need a modified notion of blow-up height for power series, because we are only dealing here with integer exponents as opposed to real exponents in [12].

Let A be a ring and let $X = (X_1, \dots, X_m)$ be a tuple of distinct indeterminates with $m \geq 2$. Given distinct $i,\ j \in \{1, \dots, m\}$, we define the corresponding **singular blow-up substitution** to be the (injective) A-algebra homomorphism $s_{ij} : A[\![X]\!] \longrightarrow A[\![X]\!]$ given by

$$s_{ij}(X_k) := \begin{cases} X_k & \text{if } 1 \leq k \leq m \text{ and } k \neq i \\ X_i X_j & \text{if } k = i. \end{cases}$$

1.1. Definition. We assign to every pair of monomials X^α, X^β with $\alpha, \beta \in \mathbb{N}^m$ a pair $\mathrm{b}_X(X^\alpha, X^\beta) \in \mathbb{N}^2$, called the **blow-up height** of the pair (X^α, X^β), also denoted by $\mathrm{b}\left(X^\alpha, X^\beta\right)$ if $X = (X_1, \dots, X_m)$ is clear from context, as follows:

Special case: $\gcd\left(X^\alpha, X^\beta\right) = 1$**.** (Here "gcd" stands for "greatest common divisor".)

Subcase 1: $X^\alpha = 1$ or $X^\beta = 1$. Then we put $\mathrm{b}\left(X^\alpha, X^\beta\right) := (0, 0)$.

Subcase 2: $X^\alpha \neq 1$ and $X^\beta \neq 1$. Let

$$\begin{aligned} M &:= \max\{\alpha_i : \ 1 \leq i \leq m\}, \quad d := |\{i : \ \alpha_i = M\}|, \\ N &:= \max\{\beta_j : \ 1 \leq j \leq m\}, \quad e := |\{j : \ \beta_j = N\}|, \end{aligned}$$

and put $\mathrm{b}\left(X^\alpha, X^\beta\right) := (M + N, d + e)$.

General case. This is reduced to the special case by setting

$$\mathrm{b}\left(X^\alpha, X^\beta\right) := \mathrm{b}\left(X^{\alpha - \omega}, X^{\beta - \omega}\right),$$

where $X^\omega = \gcd\left(X^\alpha, X^\beta\right)$.

1.2. Lemma. (1) $\mathrm{b}\left(X^\alpha, X^\beta\right) = (0, 0)$ *if and only if* $X^\alpha | X^\beta$ *or* $X^\beta | X^\alpha$.

(2) *If* $\mathrm{b}\left(X^\alpha, X^\beta\right) = (0, 0)$ *then* $\mathrm{b}\left(s_{ij}(X^\alpha), s_{ij}(X^\beta)\right) = (0, 0)$.

(3) $\mathrm{b}\left(X^\alpha, X^\beta\right) = \mathrm{b}\left(X^\beta, X^\alpha\right)$.

(4) *If* $\mathrm{b}\left(X^\alpha, X^\beta\right) \neq (0, 0)$, *then there are distinct* $i,\ j \in \{1, \dots, m\}$ *such that*

$$\mathrm{b}\left(s_{ij}(X^\alpha), s_{ij}(X^\beta)\right) < \mathrm{b}\left(X^\alpha, X^\beta\right) \quad \text{and} \quad \mathrm{b}\left(s_{ji}(X^\alpha), s_{ji}(X^\beta)\right) < \mathrm{b}\left(X^\alpha, X^\beta\right).$$

Proof. (1),(2) and (3) are easy, so we prove (4).

Let $\mathrm{b}\left(X^\alpha, X^\beta\right) \neq (0,0)$. Assume first that we are in Subcase 2 of the previous definition, and use the same notations. Take $i,\ j \in \{1,\dots,m\}$ with $\alpha_i = M$ and $\beta_j = N$; note that $i \neq j$, because $\gcd\left(X^\alpha, X^\beta\right) = 1$. Assume that $M \le N$ (the case $N \le M$ is handled in the same way). Then $s_{ij}(X^\alpha) = X^\alpha X_j^M$ and $s_{ij}(X^\beta) = X^\beta$. Dividing $X^\alpha X_j^M$ and X^β by their gcd X_j^M, we see that $\mathrm{b}\left(s_{ij}(X^\alpha), s_{ij}(X^\beta)\right) < \mathrm{b}\left(X^\alpha, X^\beta\right)$. Also, $s_{ji}(X^\alpha) = X^\alpha$ and $s_{ji}(X^\beta) = X^\beta X_i^N$. Dividing X^α and $X^\beta X_i^N$ by their gcd X_i^M, we see that $\mathrm{b}\left(s_{ji}(X^\alpha), s_{ji}(X^\beta)\right) < \mathrm{b}\left(X^\alpha, X^\beta\right)$.

In general, write $X^\alpha = X^\omega X^{\alpha-\omega}$ and $X^\beta = X^\omega X^{\beta-\omega}$ with $X^\omega := \gcd\left(X^\alpha, X^\beta\right)$, and take distinct $i,\ j \in \{1,\dots,m\}$ such that

$$\mathrm{b}\left(s_{ij}(X^{\alpha-\omega}), s_{ij}(X^{\beta-\omega})\right) < \mathrm{b}\left(X^\alpha, X^\beta\right) \quad \text{and} \quad \mathrm{b}\left(s_{ji}(X^{\alpha-\omega}), s_{ji}(X^{\beta-\omega})\right) < \mathrm{b}\left(X^\alpha, X^\beta\right).$$

The identity $s_{ij}(X^\alpha) = s_{ij}(X^\omega)s_{ij}(X^{\alpha-\omega})$ implies

$$\mathrm{b}\left(s_{ij}(X^\alpha), s_{ij}(X^\beta)\right) = \mathrm{b}\left(s_{ij}(X^{\alpha-\omega}), s_{ij}(X^{\beta-\omega})\right),$$

hence $\mathrm{b}\left(s_{ij}(X^\alpha), s_{ij}(X^\beta)\right) < \mathrm{b}\left(X^\alpha, X^\beta\right)$. Similarly $\mathrm{b}\left(s_{ji}(X^\alpha), s_{ji}(X^\beta)\right) < \mathrm{b}\left(X^\alpha, X^\beta\right)$. □

Next we consider a finite collection $\mathcal{M} = \left\{X^{\alpha(1)}, \dots, X^{\alpha(K)}\right\}$ of K distinct monomials in X, and define

$$s_{ij}(\mathcal{M}) := \left\{s_{ij}(X^{\alpha(1)}), \dots, s_{ij}(X^{\alpha(K)})\right\}.$$

We associate to $\mathcal{M}$ the triple $\mathrm{b}_X(\mathcal{M}) \in \mathbb{N}^3$ defined as follows: if $\mathrm{b}_X\left(X^{\alpha_k}, X^{\alpha_l}\right) \neq (0,0)$ for some $k,\ l \in \{1,\dots,K\}$, let $p \in \mathbb{N}$ be the number of pairs (k,l) such that $1 \le k < l \le K$ and $\mathrm{b}_X\left(X^{\alpha_k}, X^{\alpha_l}\right) \neq (0,0)$ (so $p > 0$), and let $q \in \mathbb{N}^2$ be the lexicographic minimum of $\left\{\mathrm{b}_X\left(X^{\alpha_k}, X^{\alpha_l}\right) :\ 1 \le k < l \le K, \mathrm{b}_X\left(X^{\alpha_k}, X^{\alpha_l}\right) \neq (0,0)\right\}$. Then put

$$\mathrm{b}_X(\mathcal{M}) := \begin{cases} (0,0,0) & \text{if } \mathrm{b}_X\left(X^{\alpha_k}, X^{\alpha_l}\right) = (0,0) \text{ for all } k,\ l \in \{1,\dots,K\} \\ (p,q) & \text{otherwise.} \end{cases}$$

Again, if $X = (X_1,\dots,X_m)$ is clear from context we just write $\mathrm{b}(\mathcal{M})$ for $\mathrm{b}_X(\mathcal{M})$. Order $\mathbb{N}^3$ lexicographically in what follows.

Note that $\mathrm{b}(\mathcal{M}) = (0,0,0)$ means that $\mathcal{M}$ is linearly ordered by divisibility.

1.3. Lemma. (1) *If $\mathcal{M}' \subseteq \mathcal{M}$ then $\mathrm{b}(\mathcal{M}') \le \mathrm{b}(\mathcal{M})$.*

(2) *If $\mathrm{b}(\mathcal{M}) \neq (0,0,0)$, then there are distinct $i,\ j \in \{1,\dots,m\}$ such that*

$$\mathrm{b}(s_{ij}(\mathcal{M})) < \mathrm{b}(\mathcal{M}) \quad \textit{and} \quad \mathrm{b}(s_{ji}(\mathcal{M})) < \mathrm{b}(\mathcal{M}).$$

Proof. Along the lines of [12, Lemma 4.10]. □

To avoid too many nested parentheses, we will write $s_{ij}f$ instead of $s_{ij}(f)$ below.

Consider a finite collection $\mathcal{F} \subseteq A[\![X]\!]$ of power series. For distinct $i,\ j \in \{1,\dots,m\}$ we put $s_{ij}(\mathcal{F}) = \{s_{ij}f :\ f \in \mathcal{F}\}$, and let $\mathrm{b}_X(\mathcal{F}) := \mathrm{b}_X(\mathcal{M})$, where $\mathcal{M}$ is the (finite) set of monomials that are minimal (with respect to the divisibility relation among monomials in X) among the members of $\mathcal{F}$. The elements of $\mathcal{M}$ are called the **minimal monomials of** $\mathcal{F}$, and $\mathrm{b}_X(\mathcal{F})$ is the **blow-up height** of $\mathcal{F}$. (As before we write $\mathrm{b}(\mathcal{F})$ if X is clear from context.)

Note that each $f \in \mathcal{F}$ can be written as $f = \sum X^\omega g_\omega$, where each $g_\omega \in A[\![X]\!]$ and the sum is over the finitely many $\omega \in \mathbb{N}^m$ with $X^\omega \in \mathcal{M}$. The following is proved along the lines of [12, Lemma 4.12].

1.4. Proposition. (1) *If* $\mathrm{b}(\mathcal{F}) \neq (0,0,0)$*, then there are distinct* $i,\ j \in \{1,\dots,m\}$ *such that* $\mathrm{b}(s_{ij}(\mathcal{F})) < \mathrm{b}(\mathcal{F})$ *and* $\mathrm{b}(s_{ji}(\mathcal{F})) < \mathrm{b}(\mathcal{F})$.

(2) *If* $\mathrm{b}(\mathcal{F}) = (0,0,0)$*, then each nonzero* $f \in \mathcal{F}$ *is of the form* $f = X^\omega g$ *with* $\omega \in \mathbb{N}^m$, $g \in A[\![X]\!]$ *and* $g(0) \neq 0$. □

2. Gevrey Functions in Several Variables

This section and the next two are based largely on parts of Tougeron [21], adding here and there extra precision and elaboration when needed for our purpose. It is perhaps surprising that the domains of the m-variable functions introduced below are in general not cartesian products of 1-variable domains. This, and other curious features, are forced on us because we need our family of function algebras to be closed under various operations and substitutions.

By convention, let $\arg(z) \in (-\pi, \pi]$ for $z \in \mathbb{C}$ with $\arg(0) := 0$, and put $z^k := |z|^k e^{ik\arg(z)}$ for $z \in \mathbb{C}$ and $k \geq 0$. Fix $m \geq 0$.

Notation. For $k = (k_1, \dots, k_m) \in [0,\infty)^m$ and $z = (z_1, \dots, z_m) \in \mathbb{C}^m$ we put

$$\Sigma k := k_1 + \dots + k_m,$$
$$k \cdot |\arg(z)| := k_1|\arg(z_1)| + \dots + k_m|\arg(z_m)|,$$
$$z^k := z_1^{k_1} \cdots z_m^{k_m} \in \mathbb{C},$$
$$|z| := \sup\{|z_i| :\ i = 1, \dots, m\}.$$

For a nonempty set $S \subseteq \mathbb{C}^m$ and $z \in \mathbb{C}^m$, we let

$$d(z,S) := \inf\{|z-u| :\ u \in S\}.$$

We also write $z' := (z_1, \dots, z_{m-1}) \in \mathbb{C}^{m-1}$ for any $z \in \mathbb{C}^m$, if $m > 0$. If $\alpha \in \mathbb{N}^m$, we put $\alpha! := \alpha_1! \cdots \alpha_m!$.

A tuple $R = (R_1, \dots, R_m) \in (0,\infty)^m$ is called a **polyradius**. If $R, \widetilde{R} \in (0,\infty)^m$ are polyradii, we write $R \leq \widetilde{R}$ if $R_i \leq \widetilde{R}_i$ for each i (and similarly with "$<$" in place of "$\leq$"). For any polyradius $R \in (0,\infty)^m$, we put

$$[0,R) := [0,R_1) \times \dots \times [0,R_m) \subseteq \mathbb{R}^m,$$
$$[0,R] := [0,R_1] \times \dots \times [0,R_m].$$

Finally, we let $\mathcal{K} := \{0\} \cup [1,\infty)$.

2.1. Definition. Let $R \in (0,\infty)^m$ be a polyradius, $\phi \in (0,\pi)$ and $k \in \mathcal{K}^m$, and put

$$D(R) := \{z \in \mathbb{C}^m :\ |z_i| < R_i \text{ for } i = 1, \dots, m\}, \qquad \textbf{(polydisk)}$$
$$S^k(R,\phi) := \{z \in D(R) :\ k \cdot |\arg(z)| < \phi\} \qquad \textbf{(generalized sector)}.$$

Note that if $m = 1$ and $k \geq 1$, then $S^k(R,\phi) = S(R,\phi,1/k) \cup \{0\}$, where $S(R,\phi,1/k)$ is the open sector defined in the introduction. The reason for allowing $k_i = 0$ is that we need our class of generalized sectors to be closed under taking cartesian products with discs $D(\rho) \subseteq \mathbb{C}$, $\rho > 0$. For example, if $m > 0$ and $k = (k', 0)$ with $k' \in \mathcal{K}^{m-1}$, then $S^k(R, \phi) = S^{k'}(R', \phi) \times D(R_m)$.

Next, for $p \in \mathbb{N}$ we put (see Fig. 2.1)

$$D_p^k(R) := \left\{ z \in D(R) : \ |z^k| < \frac{R^k}{p+1} \right\}, \quad \text{and}$$
$$S_p^k(R, \phi) := S^k(R, \phi) \cup D_p^k(R).$$

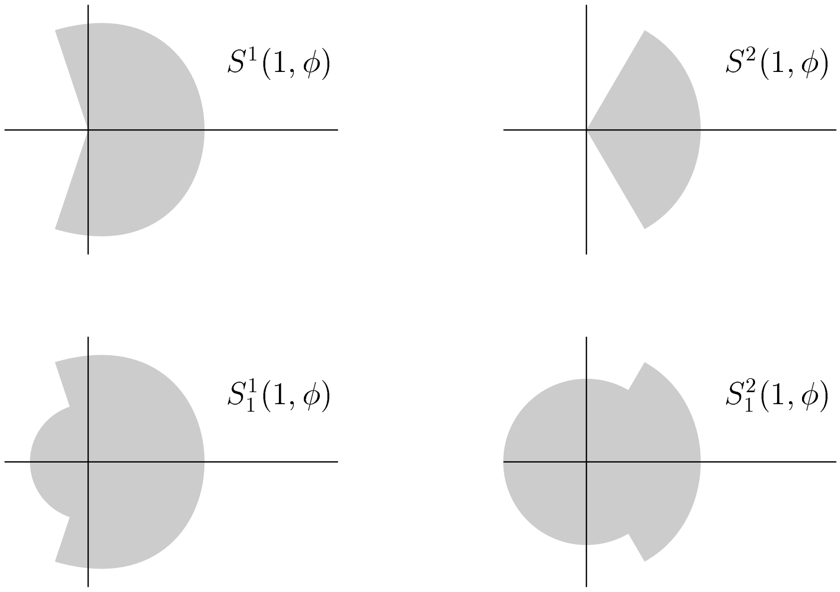

Fig. 2.1 (with $\frac{\pi}{2} < \phi < \pi$)

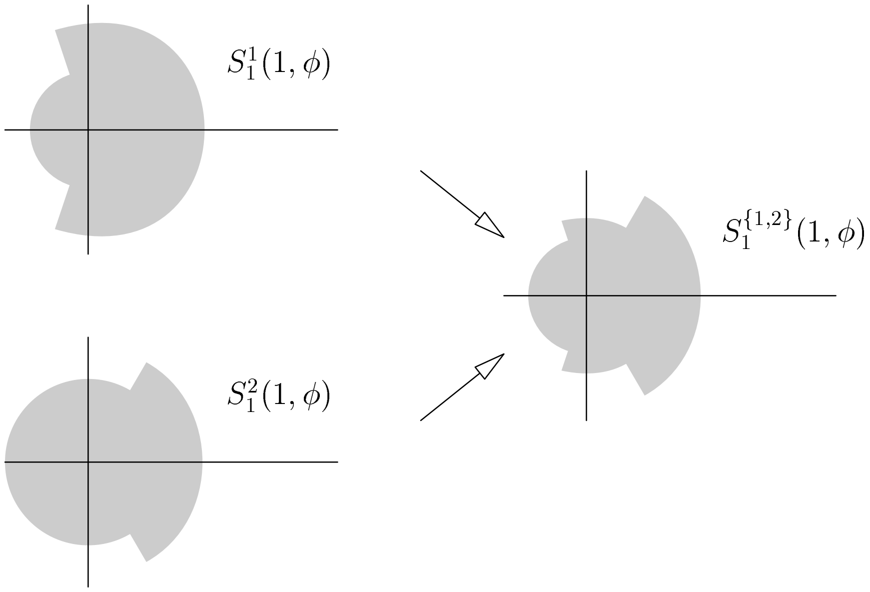

Fig. 2.2 (with $\frac{\pi}{2} < \phi < \pi$)

For any nonempty finite set $K \subseteq \mathcal{K}^m$ we let (see Fig. 2.2)

$$S^K(R,\phi) := \bigcap_{k\in K} S^k(R,\phi),$$

and for $p \in \mathbb{N}$ we let

$$S_p^K(R,\phi) := \bigcap_{k\in K} S_p^k(R,\phi).$$

For later use we observe that

$$\operatorname{int}(S^K(R,\phi)) = \bigcap_{k\in K} \operatorname{int}(S^k(R,\phi)),$$

$S^K(R,\phi)$ and $\operatorname{int}(S^K(R,\phi))$ are convex, and

$$S^K(R,\phi) \subseteq \operatorname{cl}(\operatorname{int}(S^K(R,\phi))).$$

Thus, if $z \in S_p^K(R,\phi)$ and $t \in [0,1]$, then $tz \in S_p^K(R,\phi)$; in particular, $S_p^K(R,\phi)$ is connected. Note also that $S^K(R,\phi) \cap \mathbb{R}^m = [0,R)$ whenever $K \subseteq [1,\infty)^m$. On the other hand, $S^K(R,\phi) = D(R)$ if $K = \{0\}$.

Given a set U and a function $f:\ U \longrightarrow \mathbb{C}$, we let $\|f\|_U := \sup_{z\in U} |f(z)| \in [0,\infty]$.

2.2. Definition. Let $\tau = (K,R,r,\phi)$, where $K \subseteq \mathcal{K}^m$ is nonempty and finite, $R \in (0,\infty)^m$, $r \in (1,\infty)$ and $\phi \in (0,\pi)$. To simplify notations we put

$$S(\tau) := S^K(R,\phi) \quad \text{and} \quad S_p(\tau) := S_p^K(R,\phi) \text{ for } p \in \mathbb{N},$$

and if τ is clear from context we even write S and S_p for $S(\tau)$ and $S_p(\tau)$. (Similarly, if a tuple $\widetilde{\tau}$ as above is given, we write $\widetilde{S}$ and $\widetilde{S}_p$ for $S(\widetilde{\tau})$ and $S_p(\widetilde{\tau})$.)

Note that $\left(S_p\right)_{p\in\mathbb{N}}$ is a descending sequence of open neighbourhoods of S. If for each p we have a bounded holomorphic function $f_p : S_p \longrightarrow \mathbb{C}$ such that $\sum_p \|f_p\|_{S_p} \cdot r^p < \infty$, then $\sum f_p$ clearly converges uniformly on S to a (bounded and continuous) function $f : S \longrightarrow \mathbb{C}$; we indicate this state of affairs by $f =_\tau \sum f_p$. Clearly, f is then holomorphic on $\operatorname{int}(S)$; in particular, if $K = \{0\}$, then $f : D(R) \longrightarrow \mathbb{C}$ is holomorphic.

We let $\mathcal{G}_\tau$ be the set of all functions $f : S \longrightarrow \mathbb{C}$ such that $f =_\tau \sum f_p$ for some sequence (f_p), and for $f \in \mathcal{G}_\tau$ we put

$$\|f\|_\tau := \inf\left\{\sum \|f_p\|_{S_p} \cdot r^p\right\},$$

where the infimum is over all sequences (f_p) such that $f =_\tau \sum f_p$.

With these notations we have:

2.3. Lemma. (1) *If $g : D(R) \longrightarrow \mathbb{C}$ is bounded and holomorphic, then $g\big|S$ belongs to $\mathcal{G}_\tau$; in particular, for each polynomial $P \in \mathbb{C}[X_1,\dots,X_m]$ the function $z \mapsto P(z) : S \longrightarrow \mathbb{C}$ belongs to G_τ.*

(2) *$\|f\|_S \le \|f\|_\tau$ for all $f \in \mathcal{G}_\tau$, and $\left\|g\big|S\right\|_\tau = \left\|g\big|S\right\|_S$ for all bounded and holomorphic $g : D(R) \longrightarrow \mathbb{C}$.*

(3) *$\left(\mathcal{G}_\tau, \|\cdot\|_\tau\right)$ is a complex Banach algebra under pointwise addition and multiplication of functions.*

The proof is routine and left to the reader.

Next, we establish some basic facts about derivatives of functions in $\mathcal{G}_\tau$.

2.4. Definition. Let $U \subseteq \mathbb{C}^m$ be such that $U \subseteq \mathrm{cl}(\mathrm{int}(U))$. We shall say that a continuous function $f : U \longrightarrow \mathbb{C}$ is C^∞ if $f\big|\mathrm{int}(U)$ is holomorphic and for each multi-index $\alpha \in \mathbb{N}^m$ the partial derivative $\partial^\alpha(f\big|\mathrm{int}\, U)/\partial z^\alpha$ extends to a continuous function $f^{(\alpha)} : U \longrightarrow \mathbb{C}$. Below we use the notation $f^{(\alpha)}$ in this sense; thus, if U is open, then $f^{(\alpha)} = \partial^\alpha f/\partial z^\alpha$.

Let $m \geq 1$. Along with $\tau = (K, R, r, \phi)$ we now consider a tuple $\widetilde{\tau} = (K, \widetilde{R}, \widetilde{r}, \widetilde{\phi})$, where $\widetilde{R} \in (0, \infty)^m$, $\widetilde{R} < R$, $1 < \widetilde{r} < r$ and $0 < \widetilde{\phi} < \phi$. We keep these assumptions about τ and $\widetilde{\tau}$ until 2.14; in particular $\widetilde{S} = S(\widetilde{\tau}) \subseteq S = S(\tau)$.

2.5. Lemma. *There is a $\delta \in (0, 1)$ such that $d(z, \mathrm{bd}\, S) \geq \delta|z|$ for all $z \in \widetilde{S}$.*

Proof. Since $\widetilde{S} \subseteq \mathrm{cl}(\mathrm{int}\, \widetilde{S})$, it suffices to show the existence of such a constant δ for points $z \in \mathrm{int}(\widetilde{S})$. Multiplying $z \in \mathrm{int}(\widetilde{S})$ by a suitable positive scalar, we may further restrict to points $z \in \mathrm{int}(\widetilde{S})$ with $|z| > \min\{R_i/2 : i = 1, \dots, m\}$. If no such constant δ would exist for those z, then $d(z, \mathrm{bd}\, S)$ could assume arbitrarily small positive values for such z, which is impossible. ☐

In general a function $f \in \mathcal{G}_\tau$ is not C^∞ on its entire domain S (at least, we have no reason to think so), but below we show that its restriction $f\big|\widetilde{S}$ is C^∞. We keep the notations from above and also put

$$\kappa := \begin{cases} \max\{1/(\Sigma k) : \ k \in K, \ k \neq 0\} & \text{if } K \neq \{0\}, \\ 1 & \text{if } K = \{0\}. \end{cases}$$

Thus $0 < \kappa \leq 1$ and $\kappa \cdot \Sigma k \geq 1$ for all $k \in K \setminus \{0\}$.

2.6. Lemma. *Let $f \in \mathcal{G}_\tau$. Then $f\big|\widetilde{S}$ is C^∞, and there are constants $A, B > 0$ independent of f such that for all $\alpha \in \mathbb{N}^m$ and $z \in \widetilde{S}$,*

$$\frac{\left|f^{(\alpha)}(z)\right|}{\alpha!} \leq AB^{\Sigma\alpha}(\Sigma\alpha)^{\kappa\cdot\Sigma\alpha}\, \|f\|_\tau\,,$$

where we have written $f^{(\alpha)}(z)$ instead of $(f\big|\widetilde{S})^{(\alpha)}(z)$.

Proof. We first derive a distance estimate. Put $\lambda := \min\{R_i : \ i = 1, \dots, m\}$. We claim that

$$d(0, \mathrm{bd}(S_p)) \geq \frac{\lambda}{(p+1)^\kappa}, \quad \text{for all } p \in \mathbb{N}. \tag{2.1}$$

To see this, let $p \in \mathbb{N}$, $z \in \mathbb{C}^m$ with $|z| < \lambda/(p+1)^\kappa$, and $k \in K$; we have to show that $z \in S_p^k := S_p^k(R, \phi)$. We clearly have $|z_i| < \lambda \leq R_i$ for $i = 1, \dots, m$, so $z \in D(R)$. If $k = 0$, then $D(R) = S_p^k$, and therefore $z \in S_p^k$. Suppose that $k \neq 0$. Then $|z| < \lambda/(p+1)^\kappa$ gives

$$\left|z^k\right| = |z_1|^{k_1} \cdots |z_m|^{k_m} < \frac{R_1^{k_1}}{(p+1)^{\kappa k_1}} \cdots \frac{R_m^{k_m}}{(p+1)^{\kappa k_m}} = \frac{R^k}{(p+1)^{\kappa\cdot\Sigma k}} \leq \frac{R^k}{p+1},$$

so $z \in D_p^k(R) \subseteq S_p^k$, as required.

Next, with δ as in Lemma 2.5 we claim that

$$d(z,\mathrm{bd}(S_p)) \geq \frac{\delta\lambda}{2(p+1)^\kappa} \quad \text{for } z \in \widetilde{S}. \tag{2.2}$$

To see this, suppose first that $z \in \widetilde{S}$ with $|z| \leq \lambda/2(p+1)^\kappa$. Then by (2.1) we obtain $d(z,\mathrm{bd}(S_p)) \geq \lambda/2(p+1)^\kappa \geq \delta\lambda/2(p+1)^\kappa$, as desired. Suppose next that $z \in \widetilde{S}$ with $|z| \geq \lambda/2(p+1)^\kappa$. Then by Lemma 2.5

$$d(z,\mathrm{bd}(S_p)) \geq d(z,\mathrm{bd}(S)) \geq \delta|z| \geq \frac{\delta\lambda}{2(p+1)^\kappa},$$

which establishes (2.2).

Now choose a sequence (f_p) such that $f =_\tau \sum f_p$ and $\sum \|f\|_{S_p} \cdot r^p \leq 2\,\|f\|_\tau$. Then we have for $\alpha \in \mathbb{N}^m$ and $z \in \widetilde{S}$ that

$$\begin{aligned}
\frac{\left|f_p^{(\alpha)}(z)\right|}{\alpha!} &\leq \frac{\|f_p\|_{S_p}}{d(z,\mathrm{bd}(S_p))^{-\Sigma\alpha}} && \text{(by the Cauchy estimates for derivatives)}\\
&\leq \left(\frac{2}{\delta\lambda}\right)^{\Sigma\alpha}(p+1)^{\kappa\cdot\Sigma\alpha}\,\|f_p\|_{S_p} && \text{(by (2.2))}\\
&\leq 2\left(\frac{2}{\delta\lambda}\right)^{\Sigma\alpha}(p+1)^{\kappa\cdot\Sigma\alpha}\; r^{-p}\,\|f\|_\tau\,.
\end{aligned}$$

Hence for each fixed $\alpha \in \mathbb{N}^m$ the series $\sum f_p^{(\alpha)}$ converges uniformly on $\widetilde{S}$ to a bounded continuous function, which moreover equals $(f\big|\mathrm{int}\,\widetilde{S})^{(\alpha)}$ on $\mathrm{int}(\widetilde{S})$. Thus $f\big|\widetilde{S}$ is C^∞, and

$$\frac{\left|f^{(\alpha)}(z)\right|}{\alpha!} \leq 2\left(\frac{2}{\delta\lambda}\right)^{\Sigma\alpha}\left(\sum_p (p+1)^{\kappa\cdot\Sigma\alpha}\; r^{-p}\right)\|f\|_\tau \quad \text{for } z \in \widetilde{S}. \tag{2.3}$$

Finally, let $s < 1$ be such that $rs > 1$ and $\log s > -\kappa$; we claim that there is a constant $L > 0$ such that

$$(p+1)^{\kappa\cdot\Sigma\alpha} s^p \leq (L\cdot\kappa\cdot\Sigma\alpha)^{\kappa\cdot\Sigma\alpha} \quad \text{for all } p \text{ and all } \alpha.$$

To see this, note that any constant $L > 0$ works if $\alpha = 0$. For $\alpha \neq 0$, the function $x \mapsto (x+1)\cdot s^{x/\kappa\cdot\Sigma\alpha}$ for $x > 0$ attains its maximum at $x = -(\kappa\cdot\Sigma\alpha/\log s) - 1$; evaluating the function for this value of x and raising to the power $\kappa\cdot\Sigma\alpha$ now yields the desired inequality with $L := -1/\log s$.

Therefore, the right-hand side of (2.3) is bounded above by $AB^{\Sigma\alpha}(\Sigma\alpha)^{\kappa\cdot\Sigma\alpha}\,\|f\|_\tau$ for positive reals A and B which do not depend on f, α or z. ☐

Remark. Each point $z \in S$ belongs to some subsector $\widetilde{S}$ as above, and clearly $f^{(\alpha)}(z)$ does not depend on the choice of such a subsector. This justifies our notation of $f^{(\alpha)}(z)$ for $(f\big|\widetilde{S})^{(\alpha)}(z)$. We shall also write $\partial f/\partial z_i$ instead of $f^{(\alpha)}$ if $\alpha_i = 1$ and $\alpha_j = 0$ for $j \neq i$.

2.7. Lemma. *There is a constant $\epsilon > 0$ such that*

$$d(z,\mathrm{bd}(S_p)) \geq \frac{\epsilon}{(p+1)^\kappa} \quad \textit{for all } p \in \mathbb{N} \textit{ and } z \in \widetilde{S}_p.$$

Proof. We choose for each $k \in K \setminus \{0\}$ a constant $\epsilon_k > 0$ such that $d\big(z, \mathrm{bd}(S_p^k)\big) \geq \epsilon_k/(p+1)^\kappa$ for all $z \in S^k\big(\widetilde{R}, \widetilde{\phi}\big)$ and $p \in \mathbb{N}$. (The existence of such an ϵ_k follows from the estimate (2.2) for $K = \{k\}$.) Next, we choose a constant $C > 0$ such that $|z^k - w^k| \leq C|z - w|$ for all $z, w \in D(R)$ and $k \in K$. Now let $\epsilon > 0$ be such that $\epsilon < \min\{R_i - R_i' :\ i = 1, \dots, m\}$, $\epsilon < \epsilon_k$ for all $k \in K \setminus \{0\}$, and $C\epsilon < R^k - \big(\widetilde{R}\big)^k$ for all $k \in K \setminus \{0\}$. We show that the lemma holds for this value of ϵ.

Let $p \in \mathbb{N}$, $z \in \widetilde{S}_p$ and $|z - w| < \epsilon/(p+1)^\kappa$; it suffices to derive that $w \in S_p$. First, let $k \in K \setminus \{0\}$. Then either $z \in S^k\big(\widetilde{R}, \widetilde{\phi}\big)$ or $z \in D_p^k\big(\widetilde{R}\big)$. If $z \in S^k\big(\widetilde{R}, \widetilde{\phi}\big)$, then we use $\epsilon < \epsilon_k$ to obtain $w \in S_p^k := S_p^k(R, \phi)$. If $z \in D_p^k\big(\widetilde{R}\big)$, then $z \in D\big(\widetilde{R}\big)$ and $|z - w| < \epsilon$, so $w \in D(R)$ and

$$\left|z^k - w^k\right| \leq \frac{C\epsilon}{(p+1)^\kappa} \leq \frac{C\epsilon}{p+1} < \frac{R^k - \big(\widetilde{R}\big)^k}{p+1}.$$

Hence $|w^k| < R^k/(p+1)$, and therefore $w \in D_p^k(R) \subseteq S_p^k$. Second, if $k = 0 \in K$, we use $S_p^k(R, \phi) = D(R)$ and $S_p^k\big(\widetilde{R}, \widetilde{\phi}\big) = D\big(\widetilde{R}\big)$ to conclude that $w \in S_p^k$. We have now shown that $w \in S_p^k$ for all $k \in K$, and therefore $w \in S_p$. This concludes the proof of the lemma. $\square$

2.8. Lemma. *If $f \in \mathcal{G}_\tau$ and $\alpha \in \mathbb{N}^m$, then $f^{(\alpha)}\big|\widetilde{S}$ belongs to $\mathcal{G}_{\widetilde{\tau}}$ and satisfies*

$$\left\|f^{(\alpha)}\big|\widetilde{S}\right\|_{\widetilde{\tau}} \leq C_\alpha \left\|f\right\|_\tau$$

for some positive constant C_α independent of $f \in \mathcal{G}_\tau$.

Proof. Let $f \in \mathcal{G}_\tau$, say $f =_\tau \sum f_p$. We show that $f^{(\alpha)}\big|\widetilde{S} =_{\widetilde{\tau}} \sum \left(f_p^{(\alpha)}\big|\widetilde{S}_p\right)$ for $\alpha \in \mathbb{N}^m$, so that in particular $f^{(\alpha)}\big|\widetilde{S}$ belongs to $\mathcal{G}_{\widetilde{\tau}}$. Let $p \in \mathbb{N}$ and $z \in \widetilde{S}_p$. Then

$$\begin{aligned} \frac{\left|f_p^{(\alpha)}(z)\right|}{\alpha!} &\leq \frac{\|f_p\|_{S_p}}{d(z, \mathrm{bd}\, S_p)^{-\Sigma\alpha}} && \text{(by the Cauchy estimates on derivatives)} \\ &\leq \epsilon^{-\Sigma\alpha}(p+1)^{\kappa\cdot\Sigma\alpha}\left\|f_p\right\|_{S_p} && \text{(by Lemma 2.7),} \end{aligned}$$

so that

$$\left\|f_p^{(\alpha)}\big|\widetilde{S}_p\right\|_{\widetilde{S}_p} \leq B_\alpha(p+1)^{\kappa\cdot\Sigma\alpha}\left\|f_p\right\|_{S_p}$$

for some positive constant B_α independent of f and p. Hence

$$\begin{aligned} \sum_p \left\|f_p^{(\alpha)}\big|\widetilde{S}_p\right\|_{\widetilde{S}_p} \cdot (\widetilde{r})^p &\leq B_\alpha \sum_p \left\|f_p\right\|_{S_p} \cdot r^p \cdot (p+1)^{\kappa\cdot\Sigma\alpha} \cdot \left(\frac{\widetilde{r}}{r}\right)^p \\ &\leq C_\alpha \sum_p \left\|f_p\right\|_{S_p} \cdot r^p < \infty \end{aligned}$$

for some positive constant C_α independent of f. In combination with the proof of Lemma 2.6, this gives $f^{(\alpha)}\big|\widetilde{S} =_{\widetilde{\tau}} \sum f_p^{(\alpha)}\big|\widetilde{S}_p$, as well as the desired bound on $\left\|f^{(\alpha)}\big|\widetilde{S}\right\|_{\widetilde{\tau}}$. $\square$

In the next lemma, let $k' := (k_1, \dots, k_{m-1})$ for $k \in K$, $K' := \{k' :\ k \in K\}$ and $R' := (R_1, \dots, R_{m-1})$, and put $\tau' := (K', R', r, \phi)$ and $S' := S^{K'}(R', \phi) \subseteq \mathbb{C}^{m-1}$.

2.9. Lemma. *Let $f \in \mathcal{G}_\tau$.*

(1) *There exists a* $g \in \mathcal{G}_{\widetilde{\tau}}$ *such that* $f(z) - f(z', 0) = z_m \cdot g(z)$ *for all* $z \in \widetilde{S}$, *and there is a constant* $C > 0$ *independent of* f *such that* $\|g\|_{\widetilde{\tau}} \leq C \|f\|_\tau$.

(2) *For every* $r \in [0, R_m)$ *the function* $g : S' \longrightarrow \mathbb{C}$ *defined by* $g(z') := f(z', r)$ *belongs to* $\mathcal{G}_{\tau'}$.

Proof. (1) Assume that $f =_\tau \sum f_p$. Define $g_p : \widetilde{S}_p \longrightarrow \mathbb{C}$ for each p by

$$g_p(z) := \int_0^1 \frac{\partial f_p}{\partial z_m}(z', tz_m) dt.$$

Then each g_p is holomorphic and $\|g_p\|_{\widetilde{S}_p} \leq \|\partial f_p / \partial z_m\|_{S_p}$. From the previous lemma we get $\partial f / \partial z_m =_{\widetilde{\tau}} \sum \partial f_p / \partial z_m$, and the function $g : \widetilde{S} \longrightarrow \mathbb{C}$ defined by $g(z) := \sum g_p(z)$ satisfies $g =_{\widetilde{\tau}} \sum g_p$ with $\|g\|_{\widetilde{\tau}} \leq \|\partial f / \partial z_m\|_{\widetilde{\tau}} \leq C \|f\|_\tau$ for some $C > 0$ not depending on f. Moreover, for all $z \in \widetilde{S}$ and each p we clearly have

$$f_p(z) - f_p(z', 0) = z_m \cdot g_p(z),$$

and taking the sum over all p on both sides gives the desired equality.

(2) Let $r \in [0, R_m)$. Then for $k \in K$ and $z' \in \mathbb{C}^{m-1}$ we have $k \cdot |\arg(z', r)| = k' \cdot |\arg(z')|$, so that $z' \in S'$ implies $(z', r) \in S$. Moreover, if $|(z')^{k'}| < (R')^{k'}/(p+1)$ with $p \in \mathbb{N}$, then

$$\left|(z', r)^k\right| = \left|(z')^{k'}\right| \left|r^{k_m}\right| < \frac{(R')^{k'} \cdot r^{k_m}}{p+1} < \frac{R^k}{p+1},$$

which shows that $z' \in S'_p$ implies $(z', r) \in S_p$. Therefore, $f =_\tau \sum f_p$ gives $g =_{\tau'} \sum g_p$, where $g_p : S'_p \longrightarrow \mathbb{C}$ is given by $g_p(z') := f_p(z', r)$. □

2.10. Remark. The following property is the reason for allowing the parameters k to range over $\mathcal{K}^m$ rather than just $[1, \infty)^m$. Let $\tau = (K, R, r, \phi)$ be as above and $R_{m+1} > 0$; put $\widetilde{\tau} := (\widetilde{K}, \widetilde{R}, r, \phi)$, where $\widetilde{K} := \{(k, 0) : k \in K\}$ and $\widetilde{R} := (R_1, \ldots, R_m, R_{m+1})$. Note that $z' \in S$ for all $z \in \widetilde{S}$, and similarly $z' \in S_p$ for all $z \in \widetilde{S}_p$. Thus, for each $f \in \mathcal{G}_\tau$ the function $g : \widetilde{S} \longrightarrow \mathbb{C}$ defined by $g(z) := f(z')$ belongs to $\mathcal{G}_{\widetilde{\tau}}$ and satisfies $\|g\|_{\widetilde{\tau}} \leq \|f\|_\tau$.

2.11. Definition. We let $\mathcal{T}_m$ be the collection of all tuples $\tau = (K, R, r, \phi)$ with $K \subseteq \mathcal{K}^m$ nonempty and finite, $R \in (0, \infty)^m$, $r > 1$ and $\phi \in (\frac{\pi}{2}, \pi)$. (We restrict ϕ to be greater than $\pi/2$ to obtain quasianalyticity, see 2.18.) If $\tau = (K, R, r, \phi)$ and $\widetilde{\tau} = (\widetilde{K}, \widetilde{R}, \widetilde{r}, \widetilde{\phi})$ belong to $\mathcal{T}_m$, we put $\widetilde{\tau} \leq \tau$ if and only if $K \subseteq \widetilde{K}$, $\widetilde{R} \leq R$, $\widetilde{r} \leq r$ and $\widetilde{\phi} \leq \phi$. Clearly $\leq$ defines a partial order on $\mathcal{T}_m$ such that if τ, $\widetilde{\tau} \in \mathcal{T}_m$, then there exists a $\sigma \in \mathcal{T}_m$ such that $\sigma \leq \tau$ and $\sigma \leq \widetilde{\tau}$; that is, $\mathcal{T}_m$ is a directed set.

2.12. Remark. Let $\tau, \widetilde{\tau} \in \mathcal{T}_m$ with $\widetilde{\tau} \leq \tau$. Then $\widetilde{S} \subseteq S$, and if $f \in \mathcal{G}_\tau$ with $f =_\tau \sum f_p$ for some sequence (f_p), then $f|\widetilde{S} =_{\widetilde{\tau}} \sum (f_p|\widetilde{S}_p)$; in particular $f|\widetilde{S} \in \mathcal{G}_{\widetilde{\tau}}$ and $\left\|f|\widetilde{S}\right\|_{\widetilde{\tau}} \leq \|f\|_\tau$.

2.13. Definition. Let τ, $\widetilde{\tau} \in \mathcal{T}_m$ with $\widetilde{\tau} \leq \tau$. By the previous remark, the restriction map $f \mapsto f|\widetilde{S}$ maps $\mathcal{G}_\tau$ into $\mathcal{G}_{\widetilde{\tau}}$, and it follows from the Identity Theorem for holomorphic functions that this map is injective. The family $(\mathcal{G}_\tau : \tau \in \mathcal{T}_m)$ equipped with the partial order $\leq$ on $\mathcal{T}_m$ and the restriction maps is a directed system. Since the restriction maps are injective, there is no harm in identifying each $\mathcal{G}_\tau$ with its image in $\mathcal{G}_{\widetilde{\tau}}$ whenever $\widetilde{\tau} \le \tau$. Under these identifications the direct limit of the system $(\mathcal{G}_\tau : \tau \in \mathcal{T}_m)$ is simply given by

$$\mathcal{G}_m := \bigcup_{\tau \in \mathcal{T}_m} \mathcal{G}_\tau.$$

Then $\mathcal{G}_m$ is a $\mathbb{C}$-algebra with derivations $\partial/\partial z_i$ (by Lemma 2.8), and we extend each norm $\|\cdot\|_\tau$ to all of $\mathcal{G}_m$ by setting $\|f\|_\tau := \infty$ if $f \notin \mathcal{G}_\tau$.

2.14. Lemma. *Let $f \in \mathcal{G}_m$.*

1. *If $f(0) = 0$, then $\lim_\tau \|f\|_\tau = 0$, where the limit is over the downward directed set $\mathcal{T}_m$.*
2. *If $f(0) \neq 0$, then f is a unit in $\mathcal{G}_m$.*
3. *If $f(0) = 0$, then there are $f_i \in \mathcal{G}_m$ for $i = 1, \dots, m$ such that $f = z_1 \cdot f_1 + \cdots + z_m \cdot f_m$, where $z_i \in \mathcal{G}_\tau \subseteq \mathcal{G}_m$ denotes the coordinate function $(z_1, \dots, z_m) \mapsto z_i : S \longrightarrow \mathbb{C}$ for $\tau \in \mathcal{T}_m$.*

Proof. (1) Let $\tau = (K, R, r, \phi) \in \mathcal{T}_m$ with $f \in \mathcal{G}_\tau$, and let $f =_\tau \sum f_p$; subtracting from each f_p the constant $f_p(0)$, we reduce to the case that $f_p(0) = 0$ for all p. Let $\epsilon > 0$, and take an $N \in \mathbb{N}$ such that $\sum_{p>N} \|f_p\|_{S_p} \cdot r^p < \epsilon/2$. By decreasing R and replacing each f_p by the corresponding restriction, we get in addition $\sum_{p \le N} \|f_p\|_{S_p} \cdot r^p < \epsilon/2$. Then $\|f\|_\tau < \epsilon$, which proves (1).

(2) We may assume that $f(0) = 1$, so that $f = 1 - g$ with $g \in \mathcal{G}_m$ and $g(0) = 0$. By (1) there is a $\tau \in \mathcal{T}_m$ such that $f, g \in \mathcal{G}_\tau$ with $\|g\|_\tau < 1$. Since $\mathcal{G}_\tau$ is a Banach algebra, f has inverse $1 + g + g^2 + g^3 + \cdots$ in $\mathcal{G}_\tau$, and hence in $\mathcal{G}_m$.

(3) By induction on m, using Lemma 2.9. □

2.15. Definition. The **Taylor series at** 0 of a function $f \in \mathcal{G}_\tau$ $(\tau \in \mathcal{T}_m)$ is

$$\widehat{f}(X) := \sum_{\kappa \in \mathbb{N}^m} \frac{f^{(\kappa)}(0)}{\kappa!} X^\kappa \in \mathbb{C}[\![X]\!].$$

The map $f \mapsto \widehat{f} : \mathcal{G}_\tau \longrightarrow \mathbb{C}[\![X]\!]$ is clearly a $\mathbb{C}$-algebra homomorphism. These so-called **Taylor maps** satisfy $\widehat{f|S} = \widehat{f}$ for $f \in \mathcal{G}_{\widetilde{\tau}}$ and $\tau \le \widetilde{\tau} \in \mathcal{T}_m$, and thus extend for each m to a $\mathbb{C}$-algebra homomorphism $f \mapsto \widehat{f} : \mathcal{G}_m \longrightarrow \mathbb{C}[\![X]\!]$, which respects the standard derivations: $\widehat{\partial f/\partial z_i} = \partial \widehat{f}/\partial X_i$ for $i = 1, \dots, m$.

The Taylor map $\mathcal{G}_\tau \longrightarrow \mathbb{C}[\![X]\!]$ is injective for $\tau \in \mathcal{T}_m$ (here the condition $\phi > \pi/2$ is essential). This result is important for us, and it is easily obtained from the special case $m = 1$, as we shall see. The case $m = 1$, however, requires an equivalent description of the functions in $\mathcal{G}_1$, so that we can invoke the fundamental quasianalyticity theorem from multisummability theory. The next two lemmas provide this equivalence, which is also needed to relate the one-variable functions in $\mathcal{G}$ as defined in the Introduction to the functions in $\mathcal{G}_1$ of this section.

2.16. Lemma. *Let $K = \{k_1, \dots, k_n\} \subseteq [1, \infty)$, $n > 0$, $0 < \widetilde{R} < R$, $1 < \widetilde{r} < r$, $0 < \widetilde{\phi} < \phi < \pi$, and put $\tau := (K, R, r, \phi)$, $\widetilde{\tau} := (K, \widetilde{R}, \widetilde{r}, \widetilde{\phi})$. Then for any $f \in \mathcal{G}_\tau$ there exist*

$f_j \in \mathcal{G}(\widetilde{R}, \widetilde{\phi}, 1/k_j)$ *for* $j = 1, \dots, n$ *such that*

$$f(z) = f_1(z) + \cdots + f_n(z) \text{ for all } z \in \widetilde{S}.$$

Proof. Let $f \in \mathcal{G}_\tau$. If $n = 1$, then by Lemmas 2.6 and 2.8 we have $f\big|S(\widetilde{R}, \widetilde{\phi}, 1/k_1) \in \mathcal{G}(\widetilde{R}, \widetilde{\phi}, 1/k_1)$, and we are done. In general we may assume that $k_1 < \cdots < k_n$, and we put $\tau_i := \left(\{k_i\}, \widetilde{R}, \widetilde{r}, \widetilde{\phi}\right)$. We are therefore done after showing the following

Claim. *There are* $f_i \in \mathcal{G}_{\tau_i}$ *for each* i *such that* $f(z) = f_1(z) + \cdots + f_n(z)$ *for all* $z \in \widetilde{S}$.

To see this, we proceed by induction on n; the case $n = 1$ is trivial, so we assume that $n > 1$. Write $f =_\tau \sum s_p$; decreasing R if necessary, we may assume that each s_p extends to a continuous function on $\mathrm{cl}(S_p)$. Put $K' := K \setminus \{k_n\}$, $\tau' := (K', \widetilde{R}, \widetilde{r}, \widetilde{\phi})$, $S' := S^{K'}(\widetilde{R}, \widetilde{\phi})$ and $S^* := S^{k_n}(\widetilde{R}, \widetilde{\phi})$. For each p we parametrize $\mathrm{bd}(S_p)$ counterclockwise, and we consider $\gamma_p := \mathrm{bd}(S_p) \cap D_p^{k_n}(R)$ and $\delta_p := \mathrm{bd}(S_p) \setminus \gamma_p$ with the induced parametrization. (Note that the lengths of γ_p and δ_p are uniformly bounded for $p \in \mathbb{N}$.) For each p we define

$$g_p(z) := \frac{1}{2\pi i} \int_{\delta_p} \frac{s_p(w)}{w - z} dw \qquad \text{for } z \in S'_p$$

and

$$h_p(z) := \frac{1}{2\pi i} \int_{\gamma_p} \frac{s_p(w)}{w - z} dw \qquad \text{for } z \in S^*_p.$$

By Cauchy's theorem $s_p(z) = g_p(z) + h_p(z)$ for all $z \in \widetilde{S}_p$. Using Lemma 2.7, we get by an argument similar to the proof of Lemma 2.8 that

$$\sum \|g_p\|_{S'_p} \cdot \widetilde{r}^{\,p} < \infty \quad \text{and} \quad \sum \|h_p\|_{S^*_p} \widetilde{r}^{\,p} < \infty.$$

Therefore, the function $g : S' \longrightarrow \mathbb{C}$ given by $g(z) := \sum g_p(z)$ belongs to $\mathcal{G}_{\tau'}$, and the function $h : S^* \longrightarrow \mathbb{C}$ given by $h(z) := \sum h_p(z)$ belongs to $\mathcal{G}(\tau_n)$. The lemma now follows easily from the inductive hypothesis. $\square$

The proofs of the next two results involve references to the literature on multisummability (where notation may differ somewhat from ours).

2.17. Lemma. *Let* $k \geq 1$, $0 < \widetilde{R} < R$, $1 < \widetilde{r} < r$ *and* $\pi/2 < \widetilde{\phi} < \phi < \pi$, *and put* $\sigma := (\{k\}, \widetilde{R}, \widetilde{r}, \widetilde{\phi})$. *Then for any* $f \in \mathcal{G}(R, \phi, 1/k)$ *there exists a* $g \in \mathcal{G}_\sigma$ *such that* $f(z) = g(z)$ *for all* $z \in S(\widetilde{R}, \widetilde{\phi}, 1/k)$.

Proof. See Tougeron [21, Lemma 2.8 and Proposition 2.9]. $\square$

2.18. Proposition. *For* $\tau \in \mathcal{T}_m$ *the Taylor map* $f \mapsto \widehat{f} : \mathcal{G}_\tau \longrightarrow \mathbb{C}[\![X]\!]$ *is injective.*

Proof. In view of Lemma 2.16, the case $m = 1$ follows from the fundamental quasianalyticity result of multisummability theory, see [2, 15, 21]. We now proceed by induction on m. Let $m > 1$ and assume the proposition holds for lower values of m. Let $\tau \in \mathcal{T}_m$ and $f \in \mathcal{G}_\tau$ with $\widehat{f} = 0$. We have to derive that $f = 0$. Let $\tau' := (K', R', r, \phi) \in \mathcal{T}_{m-1}$ with $K' := \{(k_1, \dots, k_{m-2}, k_{m-1} + k_m) : k \in K\}$ and $R' := (R_1, \dots, R_{m-1})$, and let $\widetilde{\tau} := \left(\widetilde{K}, \widetilde{R}, \widetilde{r}, \widetilde{\phi}\right) \in \mathcal{T}_1$ with $\widetilde{K} := \{k_1 + \cdots + k_m : k \in K\}$, $\widetilde{R} < R_m$, $\widetilde{r} \in (1, r)$ and $\widetilde{\phi} < \phi$.

By Lemma 2.8 and (repeated application of) Lemma 2.9(2), for each $\alpha \in \mathbb{N}^{m-1}$ the function $g_\alpha : \widetilde{S} \longrightarrow \mathbb{C}$ defined by $g_\alpha(z_m) := f^{(\alpha,0)}(0,\dots,0,z_m)$ belongs to $\mathcal{G}_{\widetilde{\tau}}$. Moreover, $\widehat{f} = 0$ implies that $\widehat{f^{(\alpha,0)}} = 0$, so that $\widehat{g_\alpha} = 0$. The inductive hypothesis therefore gives $g_\alpha = 0$.

On the other hand by Lemma 2.9(2), for each $t \in [0,\widetilde{R})$ the function $h_t : S(\tau') \longrightarrow \mathbb{C}$ defined by $h_t(z') := f(z',t)$ belongs to $\mathcal{G}_{\tau'}$. The above now implies that $h_t^{(\alpha)}(0) = g_\alpha(t) = 0$ for each $\alpha \in \mathbb{N}^{m-1}$, so that $\widehat{h_t} = 0$. Thus by the inductive hypothesis $h_t = 0$. Therefore $f(z) = 0$ for each $z \in S(\tau') \times [0,\widetilde{R})$, and by analytic continuation $f = 0$. □

Here is an easy consequence.

2.19. Corollary. *Let $\tau \in \mathcal{T}_m$ and $f \in \mathcal{G}_\tau$. Then*

$$f\big([0,R)\big) \subseteq \mathbb{R} \quad \text{if and only if} \quad \widehat{f} \in \mathbb{R}[\![X]\!].$$

Proof. The forward direction is clear. For the converse, note that $S = S(\tau)$ is closed under the conjugation map $z = (z_1,\dots,z_m) \mapsto \overline{z} := (\overline{z}_1,\dots,\overline{z}_m)$ from $\mathbb{C}^m$ onto itself, that the function $f^* : S \longrightarrow \mathbb{C}$ defined by $f^*(z) := \overline{f(\overline{z})}$ belongs to $\mathcal{G}_\tau$, and that if $\widehat{f} = \sum c_\alpha X^\alpha$, then $\widehat{f^*} = \sum \overline{c_\alpha} X^\alpha$. Hence, if $\widehat{f} \in \mathbb{R}[\![X]\!]$, then $\widehat{f} = \widehat{f^*}$, which by Proposition 2.18 implies $f = f^*$, and thus $f\big([0,R)\big) \subseteq \mathbb{R}$. □

Remark. Suppose that in Lemma 2.16 we have $\widetilde{\phi} > \pi/2$ and that $f \in \mathcal{G}_\tau$ takes real values on $[0,R)$. Then the f_j in Lemma 2.16 can also be taken to be real valued on $[0,\widetilde{R})$. This follows from Proposition 2.18 by replacing each f_j by $(f_j + f_j^*)/2$, which is real valued on $[0,\widetilde{R})$.

2.20. Definition. Given a polyradius $R = (R_1,\dots,R_m)$ we let $\mathcal{G}(R)$ be the set of all functions $f : [0,R] \longrightarrow \mathbb{R}$ for which there exist a tuple $\widetilde{\tau} = (\widetilde{K},\widetilde{R},\widetilde{r},\widetilde{\phi}) \in \mathcal{T}_m$ with polyradius $\widetilde{R} > R$ and a function $\widetilde{f} \in \mathcal{G}_{\widetilde{\tau}}$ such that $f(x) = \widetilde{f}(x)$ for all $x \in [0,R]$.

Thus, by the results above $\mathcal{G}(R)$ is a ring (under pointwise addition and multiplication of functions) which contains all real constant functions on $[0,R]$. Also, if $f \in \mathcal{G}(R)$, then f is of class C^∞ and its derivatives $\partial f/\partial x_i$ belong to $\mathcal{G}(R)$.

These $\mathcal{G}(R)$ were mentioned in the introduction. We even gave there a precise definition of $\mathcal{G}(R)$ for $m = 1$, which by 2.16, 2.17 and the remark following 2.19 is equivalent to the definition given here for $m = 1$.

Having available the rings $\mathcal{G}(R)$, we now define the class $\mathcal{G}$ of real valued functions on $\mathbb{R}^m$ ($m = 0,1,\dots$) as in the introduction. This leads to the structure

$$\mathbb{R}_{\mathcal{G}} := (\mathbb{R}, <, 0, 1, +, -, \cdot, \mathcal{G}).$$

The primitives of the structure $\mathbb{R}_{\text{an}}$ are definable in $\mathbb{R}_{\mathcal{G}}$ because of the following: if $R = (R_1,\dots,R_m)$ and $\widetilde{R} = (\widetilde{R}_1,\dots,\widetilde{R}_m)$ are polyradii with $R < \widetilde{R}$ and f is a holomorphic function on $D(\widetilde{R})$ that is real valued on $[0,\widetilde{R})$, then the restriction of f to $[0,R]$ belongs to $\mathcal{G}(R)$.

The remaining part of this section is not needed until Section 5.

2.21. Remark and Definition. Let $\tau = (K, R, r, \phi) \in \mathcal{T}_1$ and $\widetilde{\tau} = \left(K, \widetilde{R}, \widetilde{r}, \widetilde{\phi}\right) \in \mathcal{T}_1$ be such that $R > \widetilde{R}$, $r > \widetilde{r}$ and $\phi > \widetilde{\phi}$. Let $g \in \mathcal{G}_\tau$, and assume that $g =_\tau \sum g_p$ and that each g_p extends to a continuous function on $\mathrm{cl}(S_p)$. (This last assumption is harmless in applications, since we can always achieve it by decreasing R and ϕ.)

Fix a $q \in \mathbb{N}$. By Taylor's Theorem with remainder (see e.g. [22, Section 5.4]), we have for each $p \in \mathbb{N}$, every $a \in \widetilde{S}_p$ and every $z \in \mathbb{C}$ with $a + z \in \widetilde{S}_p$ that

$$g_p(a+z) = g_p(a) + g_p'(a)z + \cdots + \frac{g_p^{(q-1)}(a)}{(q-1)!} z^{q-1} + z^q \cdot R_q g_p(a, a+z), \tag{2.4}$$

where we define for $a, b \in \widetilde{S}_p$:

$$R_q g_p(a, b) := \frac{1}{2\pi i} \int_{\mathrm{bd}(S_p)} \frac{g_p(w)}{(w-a)^q (w-b)} dw,$$

with $\mathrm{bd}(S_p)$ oriented counterclockwise. In the following lemma we show how to take the sum over all p on both sides of (2.4) above.

Let $\omega := \left(K_\omega, R_\omega, \widetilde{r}, \widetilde{\phi}\right) \in \mathcal{T}_2$ with $K_\omega := \{(k,0) : \ k \in K\} \cup \{(0,k) : \ k \in K\}$ and $R_\omega := (\widetilde{R}, \widetilde{R})$. Then $S(\omega) = \widetilde{S} \times \widetilde{S}$, and we get

2.22. Lemma. *There is a function $R_q g \in \mathcal{G}_\omega$ such that for all nonzero $a \in \widetilde{S}$ and all $z \in \mathbb{C}$ with $a + z \in \widetilde{S} \setminus \{0\}$ we have*

$$g(a+z) = g(a) + g'(a)z + \cdots + \frac{1}{(q-1)!} g^{(q-1)}(a) z^{q-1} + z^q \cdot R_q g(a, a+z),$$

and which satisfies $\|R_q g\|_\omega \le C \cdot \|g\|_\tau$, where $C > 0$ depends only on q, τ and $\widetilde{\tau}$.

Proof. Let κ be the quantity associated to K after Lemma 2.5. By Lemma 2.7 there is an $\epsilon > 0$ (depending only on τ and $\widetilde{\tau}$) such that $|w - z| \ge \epsilon/(p+1)^\kappa$ for all $z \in \widetilde{S}_p$ and all $w \in \mathrm{bd}(S_p)$. Hence for all $a, b \in \widetilde{S}_p$ we have

$$|R_q g_p(a,b)| = \frac{1}{2\pi} \left| \int_{\mathrm{bd}(S_p)} \frac{g_p(w)}{(w-a)^q (w-b)} dw \right| \le A \, \|g_p\|_{S_p} \left(\frac{(p+1)^\kappa}{\epsilon} \right)^{q+1},$$

where $A > 0$ is a constant depending only on τ and $\widetilde{\tau}$. Now choose a $B > 0$ such that $((p+1)^\kappa/\epsilon)^{q+1} \le B(r/\widetilde{r})^p$ for all p. Then

$$\sum_p \|R_q g_p\|_{S_p(\omega)} \cdot \widetilde{r}^{\,p} \le AB \sum_p \|g_p\|_{S_p} \cdot r^p < \infty.$$

Thus the lemma follows with $R_q g : \widetilde{S} \times \widetilde{S} \longrightarrow \mathbb{C}$ defined by $R_q g(a,b) := \sum_{p=0}^\infty R_q g_p(a,b)$. □

3. Mixed Series

Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be distinct indeterminates, and put $X := (X_1, \ldots, X_m)$ and $Y := (Y_1, \ldots, Y_n)$. We also let $X' = (X_1, \ldots, X_{m-1})$ if $m \ge 1$ and $Y' = (Y_1, \ldots, Y_{n-1})$ if $n \ge 1$. Similarly, $\tau = (K, R, r, \phi)$ will denote an element of $\mathcal{T}_m$ and ρ an element of $(0, \infty)^n$; we put $S := S(\tau)$ and $D := D(\rho)$. These notations will be in force throughout this paper, unless we explicitly indicate otherwise.

We let $\mathbb{C}\{X\}_\tau$ be the image of the Taylor map $\mathcal{G}_\tau \longrightarrow \mathbb{C}[\![X]\!]$. We transfer the norm on $\mathcal{G}_\tau$ to its isomorphic copy $\mathbb{C}\{X\}_\tau$ by setting $\|\widehat{f}\,\|_\tau := \|f\|_\tau$, and we transfer evaluation to $\mathbb{C}\{X\}_\tau$ by setting $\widehat{f}(x) := f(x)$ for $x \in S(\tau)$. It is also convenient to define $\mathbb{C}\{X\}_\mathcal{G}$ to be the image of the Taylor map $\mathcal{G}_m \longrightarrow \mathbb{C}[\![X]\!]$, so $\mathbb{C}\{X\}_\mathcal{G}$ is the union of its subrings $\mathbb{C}\{X\}_\tau$ for $\tau \in \mathcal{T}_m$.

The goal of this and the next sections is to show that after making certain substitutions for the indeterminate X_m of a series $f(X) \in \mathbb{C}\{X\}_\mathcal{G}$ with $m \geq 1$, we obtain a "mixed series" $g(X', Y_1) = \sum_{\nu\in\mathbb{N}} g_\nu(X')Y_1^n$ that is "convergent in Y_1" and whose coefficients $g_\nu(X')$ belong to $\mathbb{C}\{X'\}_\mathcal{G}$ (all expressions in quotes will be made precise below). We have to choose the right collection of such mixed series, so that (among other things) all the series obtained by making these substitutions are among our collection, and the collection "has Weierstrass Preparation" (see Proposition 3.7).

3.1. Definition. Given $f \in \mathbb{C}\{X\}_\tau[\![Y]\!]$, we write $f(X, Y) = \sum_{\beta\in\mathbb{N}^n} c_\beta(X)Y^\beta$ with all c_β belonging to $\mathbb{C}\{X\}_\tau$, and we put

$$\|f\|_{\tau,\rho} := \sum \|c_\beta\|_\tau \, \rho^\beta \in [0, \infty].$$

Thus the set of $f \in \mathbb{C}\{X\}_\tau[\![Y]\!]$ with $\|f\|_{\tau,\rho} < \infty$ is just the subalgebra $\mathbb{C}\{X\}_\tau\{Y\}_\rho$ of the $\mathbb{C}$-algebra $\mathbb{C}\{X\}_\tau[\![Y]\!]$ (see "conventions" in Section 1). It is in fact a complex Banach algebra with respect to the norm $\|\cdot\|_{\tau,\rho}$, and we shall denote it by $\mathbb{C}\{X; Y\}_{\tau,\rho}$. We will also refer to each X_i as a **Gevrey variables** and to each Y_j as a **convergent variables**, and we use " ; " to indicate the separation between the two types of variables in $\mathbb{C}\{X; Y\}_{\tau,\rho}$ (whereas " , " merely indicates a separation between variables of the same type).

It follows from Remark 2.12 that $\mathbb{C}\{X; Y\}_{\widetilde{\tau},\widetilde{\rho}} \subseteq \mathbb{C}\{X; Y\}_{\tau,\rho}$ whenever $\widetilde{\tau} \in \mathcal{T}_m$ with $\widetilde{\tau} \geq \tau$ and $\widetilde{\rho} \in (0, \infty)^n$ with $\widetilde{\rho} \geq \rho$. Each $f(X, Y) = \sum c_\beta(X)Y^\beta \in \mathbb{C}\{X; Y\}_{\tau,\rho}$ defines the continuous function $f_{\tau,\rho} : S \times D \longrightarrow \mathbb{C}$ given by

$$f_{\tau,\rho}(x, y) := \sum c_\beta(x)y^\beta = \lim_{\nu\to\infty} \sum_{\Sigma\beta\leq\nu} c_\beta(x)y^\beta. \tag{3.1}$$

Note that $f_{\tau,\rho}$ is holomorphic on $\mathrm{int}(S) \times D$. We shall also write $f(x, y)$ instead of $f_{\tau,\rho}(x, y)$ when $f \in \mathbb{C}\{X; Y\}_{\tau,\rho}$ and $(x, y) \in S \times D$.

Assume now that $m > 0$ and $n > 0$, and let $\widetilde{\tau} = (K, \widetilde{R}, \widetilde{r}, \widetilde{\phi}) \in \mathcal{T}_m$ with $\widetilde{R} > R$, $\widetilde{r} > r$ and $\widetilde{\phi} > \phi$, and $\widetilde{\rho} \in (0, \infty)^n$ with $\widetilde{\rho} > \rho$. We shall also write $\widetilde{S}$ in place of $S(\widetilde{\tau})$ and $\widetilde{D}$ in place of $D(\widetilde{\rho})$.

3.2. Lemma. *Let $f \in \mathbb{C}\{X; Y\}_{\widetilde{\tau},\widetilde{\rho}}$. Then*

(1) *for each $i \in \{1, \dots, m\}$ the partial derivative $\partial f/\partial X_i$ belongs to $\mathbb{C}\{X; Y\}_{\tau,\rho}$ and satisfies $(\partial f/\partial X_i)(x, y) = \partial f_{\tau,\rho}/\partial x_i(x, y)$ for all $(x, y) \in \mathrm{int}(S) \times D$;*
(2) *for each $j \in \{1, \dots, n\}$ the partial derivative $\partial f/\partial Y_j$ belongs to $\mathbb{C}\{X;\ Y\}_{\tau,\rho}$ and satisfies $(\partial f/\partial Y_j)(x, y) = \partial f_{\tau,\rho}/\partial y_j(x, y)$ for all $(x, y) \in \mathrm{int}(S) \times D$.*

(Note that $\partial f_{\tau,\rho}/\partial x_i(x, y)$ and $\partial f_{\tau,\rho}/\partial y_j(x, y)$ in (1) and (2) are defined since $f_{\tau,\rho}$ is holomorphic on $\mathrm{int}(S) \times D$.)

Proof. Let $f(X,Y) = \sum_\beta c_\beta(X)Y^\beta$. By Lemma 2.8 we have $\partial c_\beta/\partial X_i \in \mathbb{C}\{X\}_\tau$ for each $\beta \in \mathbb{N}^n$, and there is a constant $C > 0$ such that $\|\partial c_\beta/\partial X_i\|_\tau \le C\,\|c_\beta\|_{\widetilde{\tau}}$ for each β. Thus $\sum \|\partial c_\beta/\partial X_i\|_\tau \cdot \rho^\beta \le C\,\|f\|_{\widetilde{\tau},\widetilde{\rho}} < \infty$. The equation $(\partial f/\partial X_i)(x,y) = \partial f_{\tau,\rho}/\partial x_i(x,y)$ for $(x,y) \in \mathrm{int}(S) \times D$ obviously holds if f has finite support (as a power series in Y with coefficients in $\mathbb{C}[\![X]\!]$). Therefore, it follows for general f from (3.1) and the Weierstrass Convergence Theorem.

(2) follows similarly, using standard arguments for convergent power series in place of Lemma 2.8. □

3.3. Remark and Definition. Let $f \in \mathbb{C}\{X;Y\}_{\widetilde{\tau},\widetilde{\rho}}$. Then it follows easily from Lemma 3.2 that $f_{\tau,\rho}$ is C^∞ (in the sense of Definition 2.4). We define

$$\mathbb{C}\{X;Y\}_{\tau^+,\rho^+} := \bigcup \{\mathbb{C}\{X;Y\}_{\widetilde{\tau},\widetilde{\rho}} :\ \widetilde{\tau} > \tau \text{ and } \widetilde{\rho} > \rho\}\,;$$

then $\mathbb{C}\{X;Y\}_{\tau^+,\rho^+}$ is a $\mathbb{C}$-subalgebra of $\mathbb{C}\{X;\ Y\}_{\tau,\rho}$, and $\mathbb{C}\{X;Y\}_{\tau^+,\rho^+}$ is closed under the operators $\partial/\partial X_i$ and $\partial/\partial Y_j$, in contrast to $\mathbb{C}\{X;Y\}_{\tau,\rho}$.

Next, we put

$$\mathbb{C}\{X;Y\}_{\mathcal{G}} := \bigcup \{\mathbb{C}\{X;Y\}_{\tau,\rho} :\ \tau \in \mathcal{T}_m \text{ and } \rho \in (0,\infty)^n\}\,,$$

and we extend each norm $\|\cdot\|_{\tau,\rho}$ to all of $\mathbb{C}\{X;Y\}_{\mathcal{G}}$ by putting $\|f\|_{\tau,\rho} := \infty$ whenever $f \notin \mathbb{C}\{X;Y\}_{\tau,\rho}$. This definition also makes sense when $m = 0$ or $n = 0$, and for $n = 0$ it just gives $\mathbb{C}\{X\}_{\mathcal{G}}$ with the maps $\|\cdot\|_\tau$ as defined in the beginning of this section. So below we do allow again m and n to be 0, and we partially order the cartesian product $\mathcal{T}_m \times (0,\infty)^n$ by the product ordering.

Generalizing Lemma 2.14, we then obtain

3.4. Lemma. *Let $f \in \mathbb{C}\{X;Y\}_{\mathcal{G}}$.*

(1) *If $f(0,0) = 0$, then $\lim_{(\tau,\rho)} \|f\|_{\tau,\rho} = 0$, where the limit is over the downward directed set $\mathcal{T}_m \times (0,\infty)^n$.*
(2) *If $f(0,0) \neq 0$, then f is a unit in $\mathbb{C}\{X;Y\}_{\mathcal{G}}$.* □

The next lemma says in particular that no new series have been introduced by the mixed series, that is, $\mathbb{C}\{X;Y\}_{\mathcal{G}} \subseteq \mathbb{C}\{X,Y\}_{\mathcal{G}}$. If $m \ge 1$ we let $\tau' := (K', R', r, \phi)$ with $K' := \{(k_1,\dots,k_{m-1}) :\ k \in K\}$ and $R' := (R_1,\dots,R_{m-1})$, and we let $\rho^* := (R_m, \rho)$.

3.5. Lemma. *Assume that $m \ge 1$. Then $\mathbb{C}\{X';X_m,Y\}_{\tau',\rho^*} \subseteq \mathbb{C}\{X;\ Y\}_{\tau,\rho}$, and $\|f\|_{\tau,\rho} \le \|f\|_{\tau',\rho^*}$ for $f \in \mathbb{C}\{X';X_m,Y\}_{\tau',\rho^*}$.*

Proof. Let $f(X,Y) = \sum c_{n,\beta}(X')X_m^nY^\beta \in \mathbb{C}\{X';X_m,Y\}_{\tau',\rho^*}$. By Remark 2.10, each $c_{n,\beta}$ belongs to $\mathbb{C}\{X\}_\tau$ and satisfies $\|c_{n,\beta}\|_\tau \le \|c_{n,\beta}\|_{\tau'}$. By Lemma 2.3, each X_m^n belongs to $\mathbb{C}\{X\}_\tau$ and satisfies $\|X_m^n\|_\tau \le R_m^n$. Hence, for each $\beta \in \mathbb{N}^n$ we have

$$\sum_n \|c_{n,\beta} \cdot X_m^n\|_\tau \le \sum_n \|c_{n,\beta}\|_{\tau'} \cdot R_m^n \le \frac{\|f\|_{\tau',\rho^*}}{\rho^\beta} < \infty,$$

so the completeness of $\mathbb{C}\{X\}_\tau$ implies that each $d_\beta(X) := \sum_n c_{n,\beta}(X') \cdot X_m^n$ belongs to $\mathbb{C}\{X\}_\tau$ and satisfies $\|d_\beta\|_\tau \le \sum_n \|c_{n,\beta}\|_{\tau'} \cdot R_m^n$. Therefore $\sum_\beta \|d_\beta\|_\tau \cdot \rho^\beta \le \|f\|_{\tau',\rho^*} < \infty$, which proves the lemma. □

3.6. Definition. Let $n > 0$ and $d \in \mathbb{N}$. A power series $f \in \mathbb{C}\{X;Y\}_{\mathcal{G}}$ is called **regular in $\boldsymbol{Y_n}$ of order** d, if $f(0,0,Y_n) = uY_n^d +$ terms of higher degree in Y_n, with $u \in \mathbb{C}$ and $u \neq 0$.

In other words, we consider $f \in \mathbb{C}\{X;Y\}_{\mathcal{G}}$ to be regular in Y_n if f, considered as an element of $\mathbb{C}[\![X,Y]\!]$, is regular in Y_n in the usual sense. (Instead, one might want to define $f \in \mathbb{C}\{X;Y\}_{\mathcal{G}}$ to be regular in Y_n if f, considered as an element of $A[\![Y]\!]$ with $A = \mathbb{C}[\![X]\!]$, is regular in Y_n; but this definition would be too restrictive for our purposes.)

We can now state the Weierstrass Preparation theorem. It is proved along the lines of well known similar theorems, using Lemma 3.4; we refer the reader to [21].

3.7. Proposition. *Let $f \in \mathbb{C}\{X;Y\}_{\mathcal{G}}$, and assume that $n > 0$ and f is regular in Y_n of order d. Then the series f factors uniquely as $f = gh$, where $g \in \mathbb{C}\{X;Y\}_{\mathcal{G}}$ is a unit and $h \in \mathbb{C}\{X;Y'\}_{\mathcal{G}}[Y_n]$ is monic in Y_n of degree d.* ☐

4. Substitutions

In this section we show that the algebras $\mathbb{C}\{X;Y\}_{\mathcal{G}}$ are closed under certain substitutions needed in carrying out the arguments of Section 6. Of particular importance is the fact that the "regular blow-up substitutions" and the "shift substitutions" as defined below change a Gevrey variable into a convergent variable.

Throughout this section we let i range over $\{1,\ldots,m\}$ and j over $\{1,\ldots,n\}$.

4.1. Definition. A **substitution** $\sigma : (X,Y) \longrightarrow \mathbb{C}[\![Z]\!]$, where $Z = (Z_1,\ldots,Z_p)$ is a tuple of distinct indeterminates, is by definition a map $\sigma : \{X_1,\ldots,X_m,Y_1,\ldots,Y_n\} \longrightarrow \mathbb{C}[\![Z]\!]$. If $\sigma(X_i)(0) = 0$ and $\sigma(Y_j)(0) = 0$ for each i and j, then σ extends to a unique $\mathbb{C}$-algebra homomorphism $\mathbb{C}[\![X,Y]\!] \longrightarrow \mathbb{C}[\![Z]\!]$ which we also denote by σ. We then usually write σf in place of $\sigma(f)$ for $f \in \mathbb{C}[\![X,Y]\!]$; in other words, if $f = f(X,Y) \in \mathbb{C}[\![X,Y]\!]$, then $\sigma f = f(\sigma(X_1),\ldots,\sigma(X_m),\sigma(Y_1),\ldots,\sigma(Y_n))$. If all $\sigma(X_i)$ and $\sigma(Y_j)$ lie in a subring A of $\mathbb{C}[\![Z]\!]$, then we refer to σ also as a substitution $\sigma : (X,Y) \longrightarrow A$.

Let $V = (V_1,\ldots,V_\mu)$, $W = (W_1,\ldots,W_\nu)$, $\widetilde{\tau} \in \mathcal{T}_\mu$ and $\widetilde{\rho} \in (0,\infty)^\nu$, and write $\widetilde{S} := S(\widetilde{\tau})$ and $\widetilde{D} := D(\widetilde{\rho})$. If $\sigma : (X,Y) \longrightarrow \mathbb{C}\{V;W\}_{\widetilde{\tau},\widetilde{\rho}}$ is a substitution, then σ gives rise to a map $\sigma_{\widetilde{\tau},\widetilde{\rho}} : \widetilde{S} \times \widetilde{D} \longrightarrow \mathbb{C}^{m+n}$ defined by

$$\sigma_{\widetilde{\tau},\widetilde{\rho}}(v,w) := \big(\sigma(X_1)(v,w),\ldots,\sigma(Y_n)(v,w)\big).$$

Once more we shall simply write $\sigma(v,w)$ in place of $\sigma_{\widetilde{\tau},\widetilde{\rho}}(v,w)$ for $(v,w) \in \widetilde{S} \times \widetilde{D}$.

4.2. Lemma. *Let $\sigma : (X,Y) \longrightarrow \mathbb{C}\{V;W\}_{\widetilde{\tau}^+,\widetilde{\rho}^+}$ be a substitution, and assume that $\sigma\big(\widetilde{S} \times \widetilde{D}\big) \subseteq S \times D$, $\sigma(X_i)(0) = 0$ for each i and $\sigma(Y_j)(0) = 0$ for each j. Then for every $f \in \mathbb{C}\{X;Y\}_{\tau^+,\rho^+}$ the function $f_{\tau,\rho} \circ \sigma_{\widetilde{\tau},\widetilde{\rho}}$ is C^∞ on $\widetilde{S} \times \widetilde{D}$, and the Taylor series at 0 of $f_{\tau,\rho} \circ \sigma_{\widetilde{\tau},\widetilde{\rho}}$ is given by σf.*

Proof. The fact that $f_{\tau,\rho} \circ \sigma_{\widetilde{\tau},\widetilde{\rho}}$ is C^∞ on $\widetilde{S} \times \widetilde{D}$ is clear, so we prove the second assertion. Let $f \in \mathbb{C}\{X;Y\}_{\tau^+,\rho^+}$. For each $d \in \mathbb{N}$ write

$$f(X,Y) = P_d(X,Y) + \sum_{\Sigma\alpha+\Sigma\beta=d} X^\alpha Y^\beta \cdot h_{\alpha,\beta},$$

where $P_d(X,Y) \in \mathbb{C}[X,Y]$ is of degree $\le d$ and each $h_{\alpha,\beta}$ belongs to $\mathbb{C}\{X;Y\}_{\tau^+,\rho^+}$. Since the lemma clearly holds with P_d in place of f, it follows that the Taylor series of $f_{\tau,\rho} \circ \sigma_{\widetilde{\tau},\widetilde{\rho}}$ equals $\sigma P_d + \sum_{\Sigma\alpha+\Sigma\beta=d} \sigma(X_1)^{\alpha_1} \cdots \sigma(Y_n)^{\beta_n} \cdot h^*_{\alpha,\beta}$ for certain $h^*_{\alpha,\beta} \in \mathbb{C}[\![X,Y]\!]$. By letting d tend to infinity we obtain the desired result. □

To illustrate the process of showing that our class of algebras is closed under a given substitution, we first consider a fairly easy case. Assume that $m > 1$ and let the substitution $\sigma : (X,Y) \longrightarrow \mathbb{C}[\![X',Y]\!]$ be given by

$$\begin{aligned} \sigma(X_i) &:= X_i \quad \text{for } i = 1,\dots,m-1, \\ \sigma(X_m) &:= X_{m-1}, \\ \sigma(Y_j) &:= Y_j \quad \text{for each } j. \end{aligned}$$

Thus for $f = f(X,Y) \in \mathbb{C}[\![X,Y]\!]$ we have $\sigma f = f(X_1,\dots,X_{m-1},X_{m-1},Y)$. We now let $\widetilde{\tau} := (\widetilde{K},\widetilde{R},r,\phi) \in \mathcal{T}_{m-1}$ with $\widetilde{K} := \{(k_1,\dots,k_{m-2},k_{m-1}+k_m) : k \in K\}$ and $\widetilde{R} := (R_1,\dots,R_{m-2},\min\{R_{m-1},R_m\})$. Clearly $\sigma : (X,Y) \longrightarrow \mathbb{C}\{X';Y\}_{\widetilde{\tau}^+,\rho^+}$.

4.3. Lemma. *We have $\sigma(\widetilde{S} \times D) \subseteq S \times D$, and if $f \in \mathbb{C}\{X;Y\}_{\tau^+,\rho^+}$ then the series σf belongs to $\mathbb{C}\{X';Y\}_{\widetilde{\tau},\rho}$ and satisfies $(\sigma f)_{\widetilde{\tau},\rho} = f_{\tau,\rho} \circ \sigma_{\widetilde{\tau},\rho}$ and $\|\sigma f\|_{\widetilde{\tau},\rho} \le \|f\|_{\tau,\rho}$.*

Proof. We distinguish two cases.

Case 1: $n = 0$. We write $\widetilde{k} := (k_1,\dots,k_{m-2},k_{m-1}+k_m)$ for $k \in K$. Note that for each $k \in K$ and all $x' = (x_1,\dots,x_{m-1}) \in \mathbb{C}^{m-1}$ we have

$$\Sigma\widetilde{k} = \Sigma k, \ \left|(x',x_{m-1})^k\right| = \left|(x')^{\widetilde{k}}\right| \ \text{ and } k \cdot |\arg(x',x_{m-1})| = \widetilde{k} \cdot |\arg x'|.$$

Therefore, $x' \in \widetilde{S}$ implies $(x',x_{m-1}) \in S$, and $x' \in \widetilde{S}_p$ implies $(x',x_{m-1}) \in S_p$ for each p. Thus, $f_\tau =_\tau \sum f_p$ gives $f_\tau \circ \sigma_{\widetilde{\tau}} =_{\widetilde{\tau}} \sum (f_p \circ \sigma_{\widetilde{\tau}})$. By Lemma 4.2 the Taylor series at 0 of $f_\tau \circ \sigma_{\widetilde{\tau}}$ is given by σf, so Case 1 is proved.

Case 2: $n > 0$. We let $f = \sum c_\beta Y^\beta$ with each $c_\beta \in \mathbb{C}\{X\}_{\tau^+}$. By Case 1 each σc_β belongs to $\mathbb{C}\{X'\}_{\widetilde{\tau}}$ with $\|\sigma c_\beta\|_{\widetilde{\tau}} \le \|c_\beta\|_\tau$. Hence σf belongs to $\mathbb{C}\{X';Y\}_{\widetilde{\tau},\rho}$ and satisfies $\|\sigma f\|_{\widetilde{\tau},\rho} \le \|f\|_{\tau,\rho}$, and it remains to show that $(\sigma f)_{\widetilde{\tau},\rho} = f_{\tau,\rho} \circ \sigma_{\widetilde{\tau},\rho}$. For each $d \in \mathbb{N}$ we put

$$f_d(X,Y) := \sum_{\Sigma\beta \le d} c_\beta(X) Y^\beta \in \mathbb{C}\{X;Y\}_{\tau^+,\rho^+}.$$

By the same argument as before each σf_d belongs to $\mathbb{C}\{X',Y\}_{\widetilde{\tau},\rho}$, and since each f_d has finite support (as a series in Y), we get $(\sigma f_d)_{\widetilde{\tau},\rho} = (f_d)_{\tau,\rho} \circ \sigma_{\widetilde{\tau},\rho}$. Clearly $\lim_{d\to\infty} f_d(x,y) = f(x,y)$ for all $(x,y) \in S \times D$. Moreover, since $\sigma : \mathbb{C}[\![X;Y]\!] \longrightarrow \mathbb{C}[\![X',Y]\!]$ is a homomorphism, we have

$$\|\sigma f - \sigma f_d\|_{\widetilde{\tau},\rho} = \|\sigma(f - f_d)\|_{\widetilde{\tau},\rho} = \sum_{\Sigma\beta > d} \|\sigma c_\beta\|_{\widetilde{\tau}}\, \rho^\beta \le \sum_{\Sigma\beta > d} \|c_\beta\|_\tau\, \rho^\beta,$$

so that $\lim_{d\to\infty} \sigma f_d(x',y) = \sigma f(x',y)$ for all $(x',y) \in \widetilde{S} \times D$. Hence

$$\sigma f(x',y) = \lim_{d\to\infty} \sigma f_d(x',y) = \lim_{d\to\infty} \big((f_d)_{\tau,\rho} \circ \sigma_{\widetilde{\tau},\rho}\big)(x',y) = \big(f_{\tau,\rho} \circ \sigma_{\widetilde{\tau},\rho}\big)(x',y)$$

for all $(x',y) \in \widetilde{S} \times D$, which finishes the proof. □

4.4. Lemma. *Let the substitution $\sigma : (X,Y) \longrightarrow \mathbb{C}[\![\widetilde{X},\widetilde{Y}]\!]$ satisfy one of the three conditions below:*

(1) $\widetilde{X} = X$, $\widetilde{Y} = Y$, *and a permutation π of $\{1,\dots,m\}$ is given such that $\sigma(X_i) = X_{\pi(i)}$ for each i and $\sigma(Y_j) = Y_j$ for each j. (Note that for $f = f(X,Y) \in \mathbb{C}[\![X,Y]\!]$ we have $\sigma f = f(X_{\pi(1)},\dots,X_{\pi(m)},Y)$.) In this case we put $\widetilde{\tau} := (\pi(K),\pi(R),r,\phi)$ with $\pi(K) := \big\{(k_{\pi(1)},\dots,k_{\pi(m)}) : \ k \in K\big\}$, $\pi(R) := (R_{\pi(1)},\dots,R_{\pi(m)})$ and $\widetilde{\rho} := \rho$;*

(2) $\widetilde{X} = X$, $\widetilde{Y} = Y$, *$\sigma(X_i) = a_i X_i$ with $a_i \in (0,\infty)$ for each i, and $\sigma(Y_j) = Y_j$ for each j. (Note that for $f = f(X,Y) \in \mathbb{C}[\![X,Y]\!]$ we have $\sigma f = f(a_1X_1,\dots,a_mX_m,Y)$.) In this case we put $\widetilde{\tau} := (K,R/a,r,\phi)$ with $R/a := (R_1/a_1,\dots,R_m/a_m)$, and $\widetilde{\rho} := \rho$;*

(3) $\widetilde{X} = X$, $\widetilde{Y} = Z = (Z_1,\dots,Z_p)$, *$\sigma(X_i) = X_i$ for each i, and a polyradius $s \in (0,\infty)^p$ is given such that $\sigma(Y_j) \in \mathbb{C}\{Z\}_s$ with $\sigma(Y_j)(0) = 0$ and $\|\sigma(Y_j)\|_s < \rho_j$, for each j. (Note that for $f = f(X,Y) \in \mathbb{C}[\![X,Y]\!]$ we have $\sigma f = f(X_1,\dots,X_m,\sigma(Y_1),\dots,\sigma(Y_n))$.) In this case we put $\widetilde{\tau} := \tau$ and $\widetilde{\rho} := s$.*

Then $\sigma\big(\widetilde{S} \times \widetilde{D}\big) \subseteq S \times D$, and if $f \in \mathbb{C}\{X;Y\}_{\tau^+,\rho^+}$ then the series σf belongs to $\mathbb{C}\{\widetilde{X};\ \widetilde{Y}\}_{\widetilde{\tau},\widetilde{\rho}}$ and satisfies $(\sigma f)_{\widetilde{\tau},\widetilde{\rho}} = f_{\tau,\rho} \circ \sigma_{\widetilde{\tau},\widetilde{\rho}}$ and $\|\sigma f\|_{\widetilde{\tau},\widetilde{\rho}} \le \|f\|_{\tau,\rho}$.

Proof. We obtain (1) and (2) as in the previous proof, while (3) is standard; we leave the details to the reader. □

The following application of Lemma 4.4 is of interest in connection with Proposition 3.7. Let $f = \sum c_\beta Y^\beta \in \mathbb{C}\{X;Y\}_{\mathcal{G}}$ with $f \neq 0$, $n > 0$. Assume that there is an $\alpha \in \mathbb{N}^m$ such that $f = X^\alpha \cdot g$ with $g = \sum g_\beta Y^\beta \in \mathbb{C}\{X;Y\}_{\mathcal{G}}$ and $g_\beta(0) \neq 0$ for at least one $\beta \in \mathbb{N}^n$. (Note that this always holds if $m = 1$.) Take a minimal $d \in \mathbb{N}$ such that there is a $\beta \in \mathbb{N}^n$ with $d = \Sigma\beta$ and $g_\beta(0) \neq 0$. Consider a **linear substitution** $\theta : (X,Y) \longrightarrow \mathbb{C}[\![X,Y]\!]$ of the form

$$\begin{aligned} \theta(X_i) &:= X_i \quad \text{for each } i, \\ \theta(Y_j) &:= Y_j + c_jY_n \quad \text{with } c_j \in \mathbb{C} \quad \text{for } j = 1,\dots,n-1, \\ \theta(Y_n) &:= Y_n. \end{aligned}$$

Then $(\theta g)(0,0,Y_n) = g(0,c_1Y_n,\dots,c_{n-1}Y_n,Y_n) = P(c_1,\dots,c_{n-1})Y_n^d + \text{terms of higher degree}$ in Y_n, where P is a nonzero polynomial in $c_1,\dots,c_{n-1}$ depending only on f (but not on $c_1,\dots,c_{n-1}$). By Lemma 4.4(iii) we have $\theta g \in \mathbb{C}\{X;Y\}_{\mathcal{G}}$. We therefore get

4.5. Lemma. *Let $f_1,\dots,f_l$ be nonzero series belonging to $\mathbb{C}\{X;Y\}_{\mathcal{G}}$, and assume that $f_k = X^{\alpha_k} \cdot g_k$ for some suitable $\alpha_k \in \mathbb{N}^m$ and $g_k \in \mathbb{C}\{X;Y\}_{\mathcal{G}}$ satisfying $g_k(0,Y) \neq 0$, $k = 1,\dots,l$. Then there are infinitely many linear substitutions θ as above such that $\theta f_k = X^{\alpha_k} \cdot h_k$ with each $h_k \in \mathbb{C}\{X;Y\}_{\mathcal{G}}$ regular in Y_n.* □

We now return to dealing with substitutions. Assume that $m > 1$ and consider the **singular blow-up substitution** $s : (X,Y) \longrightarrow \mathbb{C}[\![X,Y]\!]$ given by

$$\begin{aligned} s(X_i) &:= X_i \quad \text{for } i = 1,\dots,m-1, \\ s(X_m) &:= X_{m-1}X_m, \\ s(Y_j) &:= Y_j \quad \text{for each } j. \end{aligned}$$

Thus for $f = f(X,Y) \in \mathbb{C}[\![X,Y]\!]$ we have $sf = f(X_1,\dots,X_{m-1},X_{m-1}X_m,Y)$. Let $\widetilde{\tau} := (\widetilde{K},\widetilde{R},r,\phi) \in \mathcal{T}_m$ with $\widetilde{R}_i := \min\{R_i,\sqrt{R_i}\}$ for each i and

$$\widetilde{K} := \left\{\widetilde{k} :\ k \in K\right\} \quad \text{with} \quad \widetilde{k} := (k_1,\dots,k_{m-2},k_{m-1}+k_m,k_m).$$

4.6. Lemma. *We have $s\big(\widetilde{S}\times D\big) \subseteq S \times D$, and if $f \in \mathbb{C}\{X;Y\}_{\tau^+,\rho^+}$ then the series sf belongs to $\mathbb{C}\{X;Y\}_{\widetilde{\tau},\rho}$ and satisfies $(sf)_{\widetilde{\tau},\rho} = f_{\tau,\rho}\circ s_{\widetilde{\tau},\rho}$ and $\|sf\|_{\widetilde{\tau},\rho} \le \|f\|_{\tau,\rho}$.*

Proof. As in the proof of Lemma 4.3 we reduce to the case that $n=0$. Let $x \in \widetilde{S}$; we need to show that $s(x) \in S$. For each $k \in K$ we have $\widetilde{k}\cdot|\arg(x)| < \phi$, so that

$$\begin{aligned} k\cdot|\arg(s(x))| &= k'\cdot|\arg(x')| + k_m\cdot|\arg(x_{m-1}x_m)| \\ &\le k'\cdot|\arg(x')| + k_m\cdot|\arg(x_{m-1})| + k_m\cdot|\arg(x_m)| \\ &= \widetilde{k}\cdot|\arg(x)| < \phi. \end{aligned}$$

Therefore $s(x) \in S$. Next, let $x \in \widetilde{S}_p$ for some p. Then

$$\left|s(x)^k\right| = \left|(x')^{k'}\right|\cdot\left|x_{m-1}^{k_m}x_m^{k_m}\right| = \left|x^{\widetilde{k}}\right| < \frac{\widetilde{R}^{\widetilde{k}}}{p+1} \le \frac{R^k}{p+1},$$

so $s(x) \in S_p$. The case $n=0$ now follows along the lines of Case 1 in the proof of Lemma 4.3. $\square$

Remark. Let $m > 0$. Given $f \in \mathbb{C}\{X;Y\}_{\mathcal{G}}$ and $0 < d \in \mathbb{N}$, the series $f(X',X_m^d,Y)$ also belongs to $\mathbb{C}\{X;Y\}_{\mathcal{G}}$: by Remark 2.10 we have $f \in \mathbb{C}\{X,X_{m+1};\ Y\}_{\mathcal{G}}$, so by repeated application of the above lemma $f(X',X_m^{d-1}X_{m+1},Y) \in \mathbb{C}\{X,X_{m+1};Y\}_{\mathcal{G}}$, and then $f(X',X_m^d,Y) \in \mathbb{C}\{X;Y\}_{\mathcal{G}}$ by Lemma 4.3.

Next, assume that $m > 1$, let $\lambda > 0$ and consider the **regular blow-up substitution** $r_\lambda : (X,Y) \longrightarrow \mathbb{C}[\![X,Y]\!]$ given by

$$\begin{aligned} r_\lambda(X_i) &:= X_i \quad \text{for } i=1,\dots,m-1, \\ r_\lambda(X_m) &:= X_{m-1}(\lambda+X_m), \\ r_\lambda(Y_j) &:= Y_j \quad \text{for each } j. \end{aligned}$$

Thus for $f = f(X,Y) \in \mathbb{C}[\![X,Y]\!]$ we have $\sigma f = f(X_1,\dots,X_{m-1},X_{m-1}(\lambda+X_m),Y)$. Let $\widetilde{\phi} \in (\frac{\pi}{2},\phi)$ and $\rho_0 > 0$ be such that $k_m|\arg(\lambda+v)| < \phi-\widetilde{\phi}$ for all $v \in D(2\rho_0)$ and all $k \in K$. (Note that such $\widetilde{\phi}$ and ρ_0 exist.) Let $l := \max\{k_m : k \in K\}$ and $\widetilde{\tau} := (\widetilde{K},\widetilde{R},r,\widetilde{\phi}) \in \mathcal{T}_{m-1}$, where

$$\begin{aligned} &\widetilde{K} := \left\{\widetilde{k} :\ k \in K\right\} \quad \text{with} \quad \widetilde{k} := (k_1,\dots,k_{m-2},k_{m-1}+k_m), \\ &\widetilde{R} := (R_1,\dots,R_{m-2},\widetilde{R}_{m-1}) \quad \text{with} \quad \widetilde{R}_{m-1} := \min\left\{R_{m-1},R_m,\frac{R_m}{(\lambda+2\rho_0)^l}\right\}, \end{aligned}$$

and let $\widetilde{\rho} := (\rho_0,\rho)$. Thus $\widetilde{S} \subseteq \mathbb{C}^{m-1}$ and $\widetilde{D} \subseteq \mathbb{C}^{1+n}$. Note that $\Sigma k = \Sigma\widetilde{k}$ for each $k \in K$.

4.7. Lemma. *We have $r_\lambda\big(\widetilde{S}\times\widetilde{D}\big) \subseteq S\times D$, and if $f \in \mathbb{C}\{X;Y\}_{\tau^+,\rho^+}$ then the series $r_\lambda f$ belongs to $\mathbb{C}\{X';X_m,Y\}_{\widetilde{\tau},\widetilde{\rho}}$ and satisfies $(r_\lambda f)_{\widetilde{\tau},\widetilde{\rho}} = f_{\tau,\rho}\circ(r_\lambda)_{\widetilde{\tau},\widetilde{\rho}}$ and $\|r_\lambda f\|_{\widetilde{\tau},\widetilde{\rho}} \le 2\,\|f\|_{\tau,\rho}$.*

Proof. We first consider the special case $n = 0$. In this case we have

Claim 1: $r_\lambda\big(\widetilde{S} \times D(2\rho_0)\big) \subseteq S$.

Proof. It is enough to consider the case $K = \{k\}$. Let $u \in \widetilde{S}$ and $v \in D(2\rho_0)$. By definition $\widetilde{k} \cdot |\arg(u)| < \widetilde{\phi}$, so with $k' := (k_1, \dots, k_{m-1})$ we get

$$\begin{aligned} k \cdot |\arg(r_\lambda(u,v))| &= k' \cdot |\arg(u)| + k_m |\arg(u_{m-1}(\lambda + v))| \\ &\le k' \cdot |\arg(u)| + k_m |\arg(u_{m-1})| + k_m |\arg(\lambda + v)| \\ &\le \widetilde{k} \cdot |\arg(u)| + \phi - \widetilde{\phi} < \phi, \end{aligned}$$

which finishes the proof of Claim 1.

Claim 2: $r_\lambda\big(\widetilde{S}_p \times D(2\rho_0)\big) \subseteq S_p$ *for each* p.

Proof. Again we consider the case $K = \{k\}$. Let $u \in D_p^{\widetilde{k}}(\widetilde{R})$ and $v \in D(2\rho_0)$; it suffices to show that $r_\lambda(u,v) \in D_p^k(R)$. By definition $\left|u^{\widetilde{k}}\right| < \widetilde{R}^{\,\widetilde{k}}/(p+1)$, so that

$$\left|r_\lambda(u,v)^k\right| = \left|u^{\widetilde{k}}\right| \cdot |\lambda + v|^{k_m} < \frac{\widetilde{R}^{\,\widetilde{k}}(\lambda + 2\rho_0)^{k_m}}{p+1} \le \frac{(R')^{k'}(\widetilde{R}_m(\lambda + 2\rho_0))^{k_m}}{p+1} \le \frac{R^k}{p+1},$$

which finishes the proof of Claim 2.

Note that Claim 2 implies in particular that for every $u \in \widetilde{S}_p$ and every $z \in \mathbb{C}$ with $|z| \le 2\rho_0 |u_{m-1}|$ we have $(u, \lambda u_{m-1} + z) \in S_p$. Thus by the Cauchy inequalities for derivatives, we get for any holomorphic function $h : S_p \longrightarrow \mathbb{C}$, any $u \in \widetilde{S}_p$ and any $\nu \in \mathbb{N}$,

$$\left|\frac{\partial^\nu h}{\partial x_m^\nu}(u, \lambda u_{m-1})\right| \cdot |u_{m-1}|^\nu \le \frac{\nu!}{(2\rho_0)^\nu} \cdot \|h\|_{S_p}. \tag{4.1}$$

Next, a simple calculation shows that for all $(u,v) \in \widetilde{S}_p \times D(2\rho_0)$ we have

$$\frac{\partial^\nu (h \circ r_\lambda)}{\partial v^\nu}(u,v) = \frac{\partial^\nu h}{\partial x_m^\nu}(u, u_{m-1}(\lambda + v)) \cdot u_{m-1}^\nu. \tag{4.2}$$

We now return to the proof of the lemma. Let $f_\tau =_\tau \sum f_p$ for some sequence (f_p). For convenience, we write below f_∞, S_∞ and $\widetilde{S}_\infty$ for f_τ, S and $\widetilde{S}$.

For every $p \in \mathbb{N} \cup \{\infty\}$ and $u \in \widetilde{S}_p$ the function $h_{u,p} : D(2\rho_0) \longrightarrow \mathbb{C}$ defined by $h_{u,p}(v) := (f_p \circ r_\lambda)(u,v)$ is holomorphic, and its Taylor series at 0 is given by

$$\widehat{h_{u,p}}(T) = \sum_{\nu \in \mathbb{N}} \frac{1}{\nu!} \frac{\partial^\nu (f_p \circ r_\lambda)}{\partial v^\nu}(u,0) \cdot T^\nu = \sum_\nu f_{p,\nu}(u) T^\nu, \tag{4.3}$$

where $f_{p,\nu} : \widetilde{S}_p \longrightarrow \mathbb{C}$ is defined by

$$f_{p,\nu}(u) := \frac{1}{\nu!} \frac{\partial^\nu (f_p \circ r_\lambda)}{\partial v^\nu}(u,0).$$

Note that by (4.2) we have $f_{p,\nu}(u) = \frac{1}{\nu!}(\partial^\nu f_p / \partial x_m^\nu)(u, \lambda u_{m-1}) \cdot u_{m-1}^\nu$. We therefore get from (4.1) that

$$\|f_{p,\nu}\|_{\widetilde{S}_p} \le \frac{\|f_p\|_{S_p}}{(2\rho_0)^\nu}.$$

Thus, $f_\tau =_\tau \sum f_p$ implies $f_{\infty,\nu} =_{\widetilde{\tau}} \sum f_{p,\nu}$ with $\|f_{\infty,\nu}\|_{\widetilde{\tau}} \leq \|f\|_\tau /(2\rho_0)^\nu$ for each ν. Hence the series $\psi := \sum_{\nu=0}^\infty \widehat{f_{\infty,\nu}} X_m^\nu$ belongs to $\mathbb{C}\{X'; X_m\}_{\widetilde{\tau},\rho_0}$ and satisfies $\|\psi\|_{\widetilde{\tau},\rho_0} \leq 2\,\|f\|_\tau$. Moreover, by (4.3) with $p = \infty$ and by the definition of $h_{u,p}$, we have $(\psi)_{\widetilde{\tau},\rho_0} = f_\tau \circ (r_\lambda)_{\widetilde{\tau},\rho_0}$. Thus, from Lemma 4.2 we get $r_\lambda f = \psi$. This finishes the proof for the case $n = 0$. The case $n > 0$ now follows as in the proof of Lemma 4.3. $\square$

Last, assume that $m > 1$, let $\lambda \in (0, R_m)$ and consider the **shift substitution** $t_\lambda : (X, Y) \longrightarrow \mathbb{C}[\![X, Y]\!]$ given by

$$
\begin{aligned}
t_\lambda(X_i) &:= X_i \quad \text{for } i = 1, \ldots, m-1,\\
t_\lambda(X_m) &:= \lambda + X_m,\\
t_\lambda(Y_j) &:= Y_j \quad \text{for each } j.
\end{aligned}
$$

Note that $t_\lambda(X_m)(0) \neq 0$; therefore, a substitution lemma involving t_λ will have to be formulated differently from the earlier substitution lemmas. Nevertheless, an easy adaptation of the previous proof gives the following.

Let $\widetilde{\phi} \in (\frac{\pi}{2}, \phi)$, and let $\rho_0 < \min\{\lambda, R_m - \lambda\}$ be such that $k_m|\arg(\lambda + v)| < \phi - \widetilde{\phi}$ for all $v \in D(2\rho_0)$ and all $k \in K$. (Note that such $\widetilde{\phi}$ and ρ_0 exist.) Let $\widetilde{\tau} := (\widetilde{K}, \widetilde{R}, r, \widetilde{\phi})$ with $\widetilde{K} := \{(k_1, \ldots, k_{m-1}) :\ k \in K\}$ and $\widetilde{R} := (R_1, \ldots, R_{m-1})$, and let $\widetilde{\rho} := (\rho_0, \rho)$.

4.8. Lemma. *We have $t_\lambda\big(\widetilde{S} \times \widetilde{D}\big) \subseteq S \times D$. Let $f \in \mathbb{C}\{X; Y\}_{\tau^+,\rho^+}$, and define*

$$
t_\lambda f(X, Y) := \sum \frac{f^{(\alpha,\nu,\beta)}(0, \lambda, 0)}{\alpha!\,\nu!\,\beta!} \cdot (X')^\alpha X_m^\nu Y^\beta,
$$

where the sum is over all $(\alpha, \nu, \beta) \in \mathbb{N}^{m-1} \times \mathbb{N} \times \mathbb{N}^n$. Then $t_\lambda f$ belongs to $\mathbb{C}\{X'; X_m, Y\}_{\widetilde{\tau},\widetilde{\rho}}$ and satisfies $(t_\lambda f)_{\widetilde{\tau},\widetilde{\rho}} = f_{\tau,\rho} \circ (t_\lambda)_{\widetilde{\tau},\widetilde{\rho}}$. $\square$

Let now $a \in \mathbb{R}^{m+n}$ be such that $0 \leq a_i < R_i$ for each i and $|a_{m+j}| < \rho_j$ for each j. Put $m' := |\{i :\ a_i = 0\}|$, and choose a permutation π of $\{1, \ldots, m\}$ such that $\pi^{-1}\big(\{i :\ a_i = 0\}\big) = \{1, \ldots, m'\}$. Let $T_a : (X, Y) \longrightarrow \mathbb{C}[\![X, Y]\!]$ be the substitution given by

$$
\begin{aligned}
T_a(X_i) &:= a_i + X_{\pi(i)} \quad \text{for each } i,\\
T_a(Y_j) &:= a_{m+j} + Y_j \quad \text{for each } j.
\end{aligned}
$$

Below we write $X' := (X_1, \ldots, X_{m'})$ and $X'' := (X_{m'+1}, \ldots, X_m)$. Applying Lemmas 3.5, 4.4 and 4.8 repeatedly, we obtain

4.9. Corollary. *There is a $\widetilde{\tau} \in \mathcal{T}_{m'}$ and a $\widetilde{\rho} \in (0, \infty)^{m-m'+n}$ such that $T_a\big(\widetilde{S} \times \widetilde{D}\big) \subseteq S \times D$, and if $f \in \mathbb{C}\{X; Y\}_{\tau^+,\rho^+}$ then the series*

$$
T_a f := \sum \frac{f^{(\alpha,\gamma,\beta)}(a)}{\alpha!\,\gamma!\,\beta!} \cdot (X')^\alpha (X'')^\gamma Y^\beta
$$

(where the sum is over all $(\alpha, \gamma, \beta) \in \mathbb{N}^{m'} \times \mathbb{N}^{m-m'} \times \mathbb{N}^n$) belongs to $\mathbb{C}\{X'; X'', Y\}_{\widetilde{\tau},\widetilde{\rho}}$ and satisfies $(T_a f)_{\widetilde{\tau},\widetilde{\rho}} = f_{\tau,\rho} \circ (T_a)_{\widetilde{\tau},\widetilde{\rho}}$. $\square$

5. Composition and Inversion

The goal of this section is to study compositions and compositional inverses, as in Propositions 5.3 and 5.6 below. (These results will not be needed until Section 7.) Let T be an indeterminate not among $X_1, \dots, X_m, Y_1, \dots, Y_n$. We continue letting i range over $\{1, \dots, m\}$ and j over $\{1, \dots, n\}$.

We first consider substituting a series of $\mathbb{C}\{T\}_{\mathcal{G}}$ for a convergent variable of a series in $\mathbb{C}\{X;Y\}_{\mathcal{G}}$. Let $n > 0$, and let the substitution $\sigma : (X,Y) \longrightarrow \mathbb{C}[\![X, T, Y_2, \dots, Y_n]\!]$ be given by

$$\begin{aligned} \sigma(X_i) &:= X_i \quad \text{for each } i, \\ \sigma(Y_1) &:= g(T), \\ \sigma(Y_j) &:= Y_j \quad \text{for } j = 2, \dots, n. \end{aligned}$$

where $g \in \mathbb{C}\{T\}_{\mathcal{G}}$ is such that $g(0) = 0$. (Thus for $f = f(X,Y) \in \mathbb{C}[\![X,Y]\!]$ we have $\sigma f = f(X_1, \dots, X_m, g(T), Y_2, \dots, Y_n)$.) Let $\widetilde{Y} := (Y_2, \dots, Y_n)$ and put $\widetilde{\rho} := (\rho_2, \dots, \rho_n)$.

5.1. Lemma. *There is a $\widetilde{\tau} \in \mathcal{T}_{m+1}$ such that $\sigma : (X,Y) \longrightarrow \mathbb{C}\{X, T; \widetilde{Y}\}_{\widetilde{\tau},\widetilde{\rho}}$ and $\sigma(\widetilde{S} \times \widetilde{D}) \subseteq S \times D$, and such that if $f \in \mathbb{C}\{X;Y\}_{\tau,\rho}$, then σf belongs to $\mathbb{C}\{X, T;\ \widetilde{Y}\}_{\widetilde{\tau},\widetilde{\rho}}$ and satisfies $(\sigma f)_{\widetilde{\tau},\widetilde{\rho}} = f_{\tau,\rho} \circ \sigma_{\widetilde{\tau},\widetilde{\rho}}$ and $\|\sigma f\|_{\widetilde{\tau},\rho} \leq \|f\|_{\tau,\rho}$.*

Proof. We assume here for simplicity that $n = 1$; the general case is handled similarly. In particular, $\rho \in \mathbb{R}$ and Y is a single indeterminate. Let $f = \sum_{\nu=0}^{\infty} c_\nu Y^\nu \in \mathbb{C}\{X;Y\}_{\tau,\rho}$. By Remark 2.10 and Lemmas 2.14, 3.5 and 4.4(1), there is a $\tau_1 \in \mathcal{T}_{m+1}$ such that g belongs to $\mathbb{C}\{X,T\}_{\tau_1}$ and satisfies $\|g\|_{\tau_1} \leq \rho_1$. Similarly, there is a $\tau_2 \in \mathcal{T}_{m+1}$ such that each series c_ν belongs to $\mathbb{C}\{X,T\}_{\tau_2}$ and satisfies $\|c_\nu\|_{\tau_2} \leq \|c_\nu\|_\tau$. Take a $\widetilde{\tau} \in \mathcal{T}_{m+1}$ with $\widetilde{\tau} \leq \tau_1$ and $\widetilde{\tau} \leq \tau_2$. Then $g \in \mathbb{C}\{X,T\}_{\widetilde{\tau}}$ with $\|g\|_{\widetilde{\tau}} \leq \rho_1$ and each $c_\nu \in \mathbb{C}\{X,T\}_{\widetilde{\tau}}$ with $\|c_\nu\|_{\widetilde{\tau}} \leq \|c_\nu\|_\tau$. Since $\mathbb{C}\{X,T\}_{\widetilde{\tau}}$ is a Banach algebra, it follows that the (well defined) series

$$\sigma f(X,T) = \sum_\nu c_\nu(X) \cdot g(T)^\nu$$

belongs to $\mathbb{C}\{X,T\}_{\widetilde{\tau}}$ and satisfies $\|\sigma f\|_{\widetilde{\tau}} \leq \|f\|_{\tau,\rho}$. Similar to Case 2 in the proof of Lemma 4.3 we get $(\sigma f)_{\widetilde{\tau}} = f_{\tau,\rho} \circ \sigma_{\widetilde{\tau},\widetilde{\rho}}$, so Part (1) is proved. (Note that we do not use Lemma 4.2 here.) $\square$

Next, let $m > 0$, $d \in \mathbb{N}$, $d > 0$ and $\lambda > 0$, and consider the substitution $\sigma : (X,Y) \longrightarrow \mathbb{C}[\![X,T,Y]\!]$ given by

$$\begin{aligned} \sigma(X_i) &:= X_i \quad \text{for } i = 1, \dots, m-1, \\ \sigma(X_m) &:= X_m^d(\lambda + T), \\ \sigma(Y_j) &:= Y_j \quad \text{for each } j. \end{aligned}$$

(Note that for $f = f(X,Y) \in \mathbb{C}[\![X,Y]\!]$ we have $\sigma f = f(X_1, \dots, X_{m-1}, X_m^d(\lambda+T), Y)$.) Let $\widetilde{\phi} \in (\frac{\pi}{2}, \phi)$ and $\rho_0 > 0$ be such that $\frac{\pi}{2} < \widetilde{\phi} < \phi$ and $k_m|\arg(a+v)| \leq \phi - \widetilde{\phi}$ for all $v \in D(2\rho_0)$. (Note that such $\widetilde{\phi}$ and ρ_0 exist.) Let $\widetilde{\tau} := (\widetilde{K}, \widetilde{R}, r, \widetilde{\phi}) \in \mathcal{T}_m$ with $\widetilde{K} :=$

$\{(k_1,\dots,k_{m-1},dk_m):\ k\in K\}$, $\widetilde{R}:=(R_1,\dots,R_{m-1},\widetilde{R}_m)$ with

$$\widetilde{R}_m := \min\left\{R_m,\ R_m^{1/d},\ \frac{R_m}{\lambda+2\rho_0},\ \frac{R_m^{1/d}}{\lambda+2\rho_0}\right\},$$

and let $\widetilde{\rho}:=(\rho_0,\rho)$.

5.2. Lemma. *We have $\sigma\big(\widetilde{S}\times\widetilde{D}\big)\subseteq S\times D$, and if $f\in\mathbb{C}\{X;Y\}_{\tau^+,\rho^+}$ then the series σf belongs to $\mathbb{C}\{X;T,Y\}_{\widetilde{\tau},\widetilde{\rho}}$ and satisfies $(\sigma f)_{\widetilde{\tau},\widetilde{\rho}}=f_{\tau,\rho}\circ\sigma_{\widetilde{\tau},\widetilde{\rho}}$ and $\|\sigma f\|_{\widetilde{\tau},\widetilde{\rho}}\le 2\,\|f\|_{\tau,\rho}$.*

Proof. We proceed along the lines of the proof of Lemma 4.7, giving only an outline here. As in the proof of Lemma 4.3 we first consider the case $n=0$. In this case we define $h:\widetilde{S}\times D(2\rho_0)\longrightarrow\mathbb{C}$ by $h(x,v):=f(x',x_m^d(\lambda+v))$. Then for every fixed $x\in\widetilde{S}$ the function $h_x:D(2\rho_0)\longrightarrow\mathbb{C}$ defined by $h_x(v):=h(x,v)$ is holomorphic, and its Taylor series at the origin is given by

$$\widehat{h_x}(T)=\sum_{\nu\in\mathbb{N}}\frac{h_x^{(\nu)}(0)}{\nu!}T^\nu.$$

Each function $x\mapsto h_\nu(x):=\frac{1}{\nu!}h_x^{(\nu)}(0)$ with $\nu\in\mathbb{N}$ belongs to $\mathcal{G}_{\widetilde{\tau}}$ and satisfies $\|h_\nu\|_{\widetilde{\tau}}\le\|f\|_\tau/(2\rho_0)^\nu$, that is,

$$\sum_{\nu=0}^{\infty}\|h_\nu\|_{\widetilde{\tau}}\cdot\rho_0^\nu\le 2\,\|f\|_\tau\,.$$

Moreover $\sigma f=\sum\widehat{h_\nu}T^\nu$, so the lemma is proved for $n=0$. The case $n>0$ is now an easy consequence. ☐

We next consider a $g\in\mathbb{C}\{T\}_{\mathcal{G}}$ such that $g(T)=T^d(\lambda+h(T))$ for some $d\in\mathbb{N}$, $\lambda>0$ and $h\in\mathbb{C}\{T\}_{\mathcal{G}}$ with $h(0)=0$, and we let $\sigma:(X,Y)\longrightarrow\mathbb{C}[\![X,Y]\!]$ be given by

$$\begin{aligned}\sigma(X_i)&:=X_i\quad\text{for } i=1,\dots,m-1,\\ \sigma(X_m)&:=g(X_m),\\ \sigma(Y_j)&:=Y_j\quad\text{for each } j.\end{aligned}$$

If $d=0$ we assume in addition that $\lambda<R_m$.

5.3. Proposition. *There is a $\widetilde{\tau}\in\mathcal{T}_m$ and a $\widetilde{\rho}\in(0,\infty)^n$ such that $\sigma:(X,Y)\longrightarrow\mathbb{C}\{X;Y\}_{\widetilde{\tau}^+,\widetilde{\rho}^+}$ and $\sigma\big(\widetilde{S}\times\widetilde{D}\big)\subseteq S\times D$, and such that if $f\in\mathbb{C}\{X;Y\}_{\tau^+,\rho^+}$ then there is a series $\sigma f\in\mathbb{C}\{X;Y\}_{\widetilde{\tau},\widetilde{\rho}}$ satisfying $(\sigma f)_{\widetilde{\tau},\widetilde{\rho}}=f_{\tau,\rho}\circ\sigma_{\widetilde{\tau},\widetilde{\rho}}$.*

Proof. If $g(0)=0$, then we apply Lemmas 5.2 and 5.1. If $g(0)\neq0$, then by Lemma 4.8 there is a series $t_\lambda f\in\mathbb{C}\{X';X_m,Y\}_{\mathcal{G}}$ such that $t_\lambda f$ is the Taylor series at 0 of $f_{\tau,\rho}\circ t_\lambda$. Now apply Lemma 5.1 with h in place of g to $t_\lambda f$. ☐

Finally, we prove that a compositional inverse exists in $\mathbb{C}\{T\}_{\mathcal{G}}$ for certain elements of $\mathbb{C}\{T\}_{\mathcal{G}}$. First, let the substitution $\sigma : (T, Y) \longrightarrow \mathbb{C}[\![T, Y_2, \dots, Y_n]\!]$ be given by

$$\begin{aligned}\sigma(T) &:= T,\\ \sigma(Y_1) &:= g(T),\\ \sigma(Y_j) &:= Y_j \quad \text{for } j = 2, \dots, n.\end{aligned}$$

where $g \in \mathbb{C}\{T\}_\tau$ is such that $g(0) = 0$ and $\|g\|_\tau \le \rho_1$. (Note that for $f = f(X, Y) \in \mathbb{C}[\![X, Y]\!]$ we have $\sigma f = f(T, g(T), Y_2, \dots, Y_n)$.) Let $\widetilde{Y} := (Y_2, \dots, Y_n)$ and put $\widetilde{\rho} := (\rho_2, \dots, \rho_n)$.

5.4. Lemma. *We have $\sigma : (T, Y) \longrightarrow \mathbb{C}\{T; \widetilde{Y}\}_{\tau,\widetilde{\rho}}$ and $\sigma(\widetilde{S} \times \widetilde{D}) \subseteq S \times D$, and whenever $f \in \mathbb{C}\{T; Y\}_{\tau,\rho}$, then σf belongs to $\mathbb{C}\{T; \widetilde{Y}\}_{\tau,\widetilde{\rho}}$ and satisfies $(\sigma f)_{\tau,\widetilde{\rho}} = f_{\tau,\rho} \circ \sigma_{\tau,\widetilde{\rho}}$ and $\|\sigma f\|_{\tau,\widetilde{\rho}} \le \|f\|_{\tau,\rho}$.*

Proof. For simplicity we just consider the case $n = 1$ (so $\rho \in \mathbb{R}$ and Y is a single indeterminate). Let $f = \sum_{\nu=0}^{\infty} c_\nu Y^\nu \in \mathbb{C}\{T;\ Y\}_{\tau,\rho}$. Since $\mathbb{C}\{T\}_\tau$ is a Banach algebra and $\|g\|_\tau \le \rho$, we get that $\sigma f(T) = \sum c_\nu(T) g(T)^\nu$ belongs to $\mathbb{C}\{T\}_\tau$ with $\|\sigma f\|_\tau \le \|f\|_{\tau,\rho}$. Similar to Case 2 in the proof of Lemma 4.3 we get $(\sigma f)_\tau = f_{\tau,\rho} \circ \sigma_\tau$. (Again, we do not use Lemma 4.2 here.) ☐

5.5. Definition. Let $g \in \mathbb{C}[\![T]\!]$ be such that $g(0) = 0$, and let $\sigma : (T) \longrightarrow \mathbb{C}[\![T]\!]$ be the substitution defined by $\sigma(T) := g(T)$. Given $f \in \mathbb{C}[\![T]\!]$, we shall simply write $f \circ g$ instead of σf, and we call $f \circ g$ the **composition** of f with g. These compositions behave as expected; for example, if $f \in \mathbb{C}[\![T]\!]$ with $f(0) \neq 0$, then $1/(f \circ g) = (1/f) \circ g$. Below we shall freely use facts of this nature.

5.6. Proposition. *Let $f(T) = T^d(\lambda + h(T)) \in \mathbb{C}\{T\}_{\mathcal{G}}$ be such that $d \in \mathbb{N}$ is nonzero, $\lambda > 0$ and $h \in \mathbb{C}\{T\}_{\mathcal{G}}$ with $h(0) = 0$. Then there is a $g \in \mathbb{C}\{T\}_{\mathcal{G}}$ such that $g(0) = 0$ and $f \circ g = T^d$.*

Proof. If $(f/\lambda) \circ g_1 = T^d$, where $g_1 \in \mathbb{C}\{T\}_{\mathcal{G}}$ with $g_1(0) = 0$, then $f \circ g_1 = \lambda T^d$, and hence $f \circ g = T^d$, where $g := g_1(\lambda^{-1/d} T)$ belongs to $\mathbb{C}\{T\}_{\mathcal{G}}$ by Lemma 4.4(2). Thus upon replacing f by f/λ, we may as well assume that $\lambda = 1$. Choose $\iota \in \mathcal{T}_1$ such that $f, h \in \mathbb{C}\{T\}_\iota$ and $\|T\|_\iota \le 1$.

Claim. *There are $\tau \le \iota$, $s \in (0, 1)$ and $\epsilon_n, \delta_n \in \mathbb{C}\{T\}_\tau$ for each n, such that*

$(\diamond)_n$ $\epsilon_n(0) = 0$, $\|\epsilon_n\|_\tau \le s^n$ *and* $\|\delta_n\|_\tau \le s^{n+1}$, *and with* $g_n := \sum_{i=0}^{n} T^{d-1}\epsilon_i$ *we have*

$$f \circ T(1 + g_n) = T^d\left(1 + T^d \delta_n\right).$$

Assume for the moment that the claim holds. By the completeness of $\mathbb{C}\{T\}_\tau$ the series $g := \sum_{n=0}^{\infty} T^{d-1}\epsilon_n$ belongs to $\mathbb{C}\{T\}_\tau$, so that $f \circ T(1 + g) \in \mathbb{C}\{T\}_{\mathcal{G}}$ by Proposition 5.3. By the claim, we have $f \circ T(1 + g_n) \in \mathbb{C}\{T\}_\tau$ for each n, and hence that in $\mathbb{C}\{T\}_\tau$,

$$\lim_{n\to\infty} \left(f \circ T(1 + g_n)\right) = \lim_{n\to\infty} \left(T^d \left(1 + T^d \delta_n\right)\right) = T^d.$$

On the other hand, we get from the claim that $\lim_{n\to\infty} f(t(1 + g_n(t))) = f(t(1 + g(t)))$ for all sufficiently small $t > 0$, which together with the above and Proposition 2.18 implies that $f \circ T(1 + g) = T^d$.

Proof of the Claim. Note that $f' = T^{d-1}\widetilde{f}$ with $\widetilde{f} \in \mathbb{C}\{T\}_{\mathcal{G}}$ and $\widetilde{f}(0) = d \ge 1$, so $1/\widetilde{f}$ belongs to $\mathbb{C}\{T\}_{\mathcal{G}}$ as well. Decreasing ι if necessary, we may assume that f, f' and $1/\widetilde{f}$ belong to $\mathbb{C}\{T\}_\iota$ and satisfy

$$\|f\|_\iota \leq 1 \quad \text{and} \quad \left\|1/\widetilde{f}\right\|_\iota \leq 1/4. \tag{5.1}$$

Next, by Lemma 2.22, there is an $\omega \in \mathcal{T}_2$ and a series $R_2 f \in \mathbb{C}\{X_1, X_2\}_\omega$ such that

$$f(X_1 + X_2) = f(X_1) + f'(X_1)X_2 + X_2^2 \cdot (R_2 f)(X_1, X_1 + X_2).$$

Applying Lemma 5.2 to f, $1/\widetilde{f}$ and $R_2 f$ in place of f, we obtain a $\tau \leq \iota$ and a $\rho \in (0,1)$ such that

$$\begin{aligned} &f \circ T(1+Y_1) \in \mathbb{C}\{T; Y_1\}_{\tau,\rho} \quad \text{with } \|f \circ T(1+Y_1)\|_{\tau,\rho} \leq 2\|f\|_\iota, \\ &\frac{1}{\widetilde{f} \circ T(1+Y_1)} \in \mathbb{C}\{T; Y_1\}_{\tau,\rho} \quad \text{with } \left\|\frac{1}{\widetilde{f} \circ T(1+Y_1)}\right\|_{\tau,\rho} \leq 2\left\|1/\widetilde{f}\right\|_\iota \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} &(R_2 f)(T(1+Y_1), T(1+Y_2)) \in \mathbb{C}\{T; Y_1, Y_2\}_{\tau,(\rho,\rho)} \\ &\qquad\qquad \text{with } \|(R_2 f)(T(1+Y_1), T(1+Y_2))\|_{\tau,(\rho,\rho)} \leq 4\|R_2 f\|_\omega. \end{aligned} \tag{5.3}$$

Next, we decrease ρ if necessary and choose $s \in (0, \rho)$ such that

$$\left(\frac{1}{1-\rho}\right)^{d-1} \leq 2, \quad \frac{s}{1-s} \leq \rho \quad \text{and} \quad s \leq \frac{1}{4\|R_2 f\|_\omega}. \tag{5.4}$$

Decreasing τ further if necessary, we may also assume that $\|h\|_\tau \leq s^2 < s$. We now proceed by induction on n.

Initial Step. We put $\epsilon_0 := 0$ and $\delta_0 := h$; then $\|\epsilon_0\|_\tau = 0$ and $\|\delta_0\|_\tau \leq s^2$. This proves $(\diamond)_0$.

Inductive Step. Let $n > 0$ and assume that we are given $\delta_i,\ \epsilon_i \in \mathbb{C}\{T\}_\tau$ for $i = 0, \dots, n-1$ such that $(\diamond)_i$ holds for each $i < n$. Put

$$\epsilon_n := \frac{-T \cdot \delta_{n-1}}{\widetilde{f} \circ T(1+g_{n-1})} \cdot \frac{1}{(1+g_{n-1})^{d-1}}.$$

By the inductive hypothesis, (5.1) and (5.4), we have $\|g_{n-1}\|_\tau \leq s + s^2 + \cdots + s^{n-1} \leq \rho$, so by (5.2) and Lemma 5.4 the series $1/\left(\widetilde{f} \circ T(1+g_{n-1})\right)$ belongs to $\mathbb{C}\{T\}_\tau$ and satisfies $\left\|1/\left(\widetilde{f} \circ T(1+g_{n-1})\right)\right\|_\tau \leq 2\left\|1/\widetilde{f}\right\|_\iota$. Moreover, by (5.4),

$$\left\|\frac{1}{(1+g_{n-1})}\right\|_\tau^{d-1} \leq \left(1 + \|g_{n-1}\|_\tau + \|g_{n-1}\|_\tau^2 + \cdots\right)^{d-1} \leq \left(\frac{1}{1-\rho}\right)^{d-1} \leq 2,$$

which shows that $\epsilon_n \in \mathbb{C}\{T\}_\tau$. Now we get from the inductive hypothesis (respectively, from the Initial Step if $n = 1$) that

$$\|\epsilon_n\|_\tau \leq \|\delta_{n-1}\|_\tau \cdot \frac{1}{2} \cdot 2 = \|\delta_{n-1}\|_\tau \leq \begin{cases} s^2 & \text{if } n = 1 \\ s^n & \text{if } n > 1. \end{cases} \tag{5.5}$$

In particular, $\|g_n\|_\tau \leq \rho$. Note that $T^d/f' = T/\widetilde{f}$, so that

$$\epsilon_n = \frac{-T^d \delta_{n-1}}{f' \circ T(1+g_{n-1})}.$$

Hence, by Lemma 5.4 and the inductive hypothesis, we have

$$\begin{aligned} f \circ T(1+g_n) &= f \circ T\left(1+g_{n-1}+T^{d-1}\epsilon_n\right) \\ &= f \circ T(1+g_{n-1}) + (f' \circ T(1+g_{n-1})) \cdot T^d \epsilon_n + \\ &\qquad\qquad + T^{2d}\epsilon_n^2 \cdot (R_2 f)\big(T(1+g_{n-1}), T(1+g_n)\big) \\ &= T^d \Big(1 + T^d \delta_{n-1} + \big(f' \circ T(1+g_{n-1})\big) \cdot \epsilon_n + \\ &\qquad\qquad + T^d \epsilon_n^2 \cdot (R_2 f)\big(T(1+g_{n-1}), T(1+g_n)\big)\Big) \\ &= T^d \Big(1 + T^d \epsilon_n^2 \cdot (R_2 f)\big(T(1+g_{n-1}), T(1+g_n)\big)\Big). \end{aligned}$$

Putting $\delta_n := \epsilon_n^2 \cdot (R_2 f)\big(T(1+g_{n-1}), T(1+g_n)\big)$, we therefore get $f \circ T(1+g_n) = T^d\left(1+T^d\delta_n\right)$ with $\delta_n \in \mathbb{C}\{T\}_\tau$. Morevover, by (5.3) and (5.5), we have

$$\|\delta_n\|_\tau \leq \|\epsilon_n\|_\tau^2 \cdot \left\|(R_2 f)\big(T(1+g_{n-1}), T(1+g_n)\big)\right\|_\tau \leq s^{n+1},$$

which finishes the proof of the claim. □

6. Gevrey Semianalytic Sets and Model Completeness

We fix m and n, and let $\rho \in (0,\infty)^{m+n}$ be a polyradius. We put

$$I_{m,n;\rho} := [0,\rho_1] \times \cdots \times [0,\rho_m] \times [-\rho_{m+1},\rho_{m+1}] \times \cdots \times [-\rho_{m+n},\rho_{m+n}] \subseteq \mathbb{R}^{m+n},$$

and we write $I_{m,n;\epsilon}$ instead of $I_{m,n;(\epsilon,\dots,\epsilon)}$ for positive real ϵ, and $I_{m,n;\infty} := [0,\infty)^m \times \mathbb{R}^n$.

6.1. Definition. We let

$$\mathbb{R}\{X;Y\}_{\mathcal{G},\rho} := \bigcup_{\tau,\sigma} \mathbb{C}\{X;Y\}_{\tau,\sigma} \cap \mathbb{R}[\![X,Y]\!],$$

where the union is taken over all $\tau = (K,R,r,\phi) \in \mathcal{T}_m$ with $R_i > \rho_i$ for each $i = 1,\dots,m$ and all $\sigma \in (0,\infty)^n$ with $\sigma_j > \rho_{m+j}$ for each $j = 1,\dots,n$. It follows from Section 3 that $\mathbb{R}\{X;Y\}_{\mathcal{G},\rho}$ is a subalgebra of the $\mathbb{R}$-algebra $\mathbb{R}[\![X,Y]\!]$ closed under the operators $\partial/\partial X_i$ and $\partial/\partial Y_j$.

Correspondingly, we let $\mathcal{G}_{m,n;\rho}$ be the set of all functions

$$(x,y) \mapsto f(x,y) : I_{m,n;\rho} \longrightarrow \mathbb{R}$$

with $f \in \mathbb{R}\{X;Y\}_{\mathcal{G},\rho}$. Note that $\mathcal{G}_{m,n;\rho}$ contains all $\mathbb{R}$-valued constant functions on $I_{m,n,\rho}$ and is a ring of $\mathbb{R}$-valued functions on $I_{m,n,\rho}$ under pointwise addition and multiplication of functions. (In contrast, $\mathcal{G}_{\tau,\rho}$ consists of $\mathbb{C}$-valued functions on a subset of $\mathbb{C}^{m+n}$.) If $\epsilon > 0$, we write $\mathbb{R}\{X;Y\}_{\mathcal{G},\epsilon}$ and $\mathcal{G}_{m,n;\epsilon}$ instead of $\mathbb{R}\{X;Y\}_{\mathcal{G},(\epsilon,\dots,\epsilon)}$ and $\mathcal{G}_{m,n;(\epsilon,\dots,\epsilon)}$. Next, we put

$$\mathbb{R}\{X;Y\}_{\mathcal{G}} := \bigcup_{\rho \in (0,\infty)^{m+n}} \mathbb{R}\{X;Y\}_{\mathcal{G},\rho}.$$

Note that each $f \in \mathcal{G}_{m,n;\rho}$ is C^∞ on $I_{m,n;\rho}$ and real analytic on $\mathrm{int}(I_{m,n;\rho})$, and that its partials $\partial f/\partial x_i$ and $\partial f/\partial y_j$ also belong to $\mathcal{G}_{m,n;\rho}$. The properties in Sections 3, 4 and 5 of the algebras $\mathbb{C}\{X;Y\}_{\tau^+,\rho^+}$ are easily seen to imply corresponding properties for the algebras $\mathbb{R}\{X;Y\}_{\mathcal{G},\rho}$ and $\mathcal{G}_{m,n;\rho}$. For $n=0$ we just write $\mathbb{R}\{X\}_{\mathcal{G},\rho}$ instead of $\mathbb{R}\{X;Y\}_{\mathcal{G},\rho}$. Note that $\mathcal{G}_{m,0;\rho} = \mathcal{G}(\rho)$, where the latter is defined as in 2.20.

6.2. Definition. A set $A \subseteq I_{m,n;\rho}$ is called a **basic $\mathcal{G}_{m,n;\rho}$-set** if there are $f, g_1, \dots, g_k \in \mathcal{G}_{m,n;\rho}$ such that

$$A = \{z \in I_{m,n;\rho} : \ f(z) = 0, \ g_1(z) > 0, \dots, \ g_k(z) > 0\}.$$

A **$\mathcal{G}_{m,n;\rho}$-set** is a finite union of basic $\mathcal{G}_{m,n;\rho}$-sets. Note that the $\mathcal{G}_{m,n;\rho}$-sets form a boolean algebra of subsets of $I_{m,n;\rho}$.

Given a point $a = (a_1, \dots, a_{m+n}) \in \mathbb{R}^{m+n}$ and a choice of signs $\sigma \in \{-1,1\}^m$, we let $h_{a,\sigma} : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^{m+n}$ be the bijection given by

$$h_{a,\sigma}(z) := (a_1 + \sigma_1 z_1, \dots, a_m + \sigma_m z_m, \ a_{m+1} + z_{m+1}, \dots, a_{m+n} + z_{m+n}).$$

Note that the maps $h_{a,\sigma}$ (with $a \in \mathbb{R}^{m+n}$ and $\sigma \in \{-1,1\}^m$) form a group of permutations of $\mathbb{R}^{m+n}$.

6.3. Definition. If $a \in \mathbb{R}^{m+n}$, then a set $X \subseteq \mathbb{R}^{m+n}$ is **$\mathcal{G}_{m,n}$-semianalytic at** a if there is an $\epsilon > 0$ such that for each $\sigma \in \{-1,1\}^m$ there is a $\mathcal{G}_{m,n;\epsilon}$-set $A_\sigma \subseteq I_{m,n;\epsilon}$ with

$$X \cap h_{a,\sigma}(I_{m,n;\epsilon}) = h_{a,\sigma}(A_\sigma).$$

A set $X \subseteq \mathbb{R}^{m+n}$ is **$\mathcal{G}_{m,n}$-semianalytic** if it is $\mathcal{G}_{m,n}$-semianalytic at every point $a \in \mathbb{R}^{m+n}$. For convenience, if $X \subseteq \mathbb{R}^m$ is $\mathcal{G}_{m,0}$-semianalytic we also simply say that X is $\mathcal{G}_m$-semianalytic.

6.4. Remark. (1) If $X, Y \subseteq \mathbb{R}^{m+n}$ are $\mathcal{G}_{m,n}$-semianalytic at a, then so are $X \cup Y$, $X \cap Y$ and $X \setminus Y$.
(2) Let $X \subseteq \mathbb{R}^{m+n}$ be $\mathcal{G}_{m,n}$-semianalytic, $a \in \mathbb{R}^{m+n}$ and $\sigma \in \{-1,1\}^m$. Then the set $h_{a,\sigma}(X)$ is $\mathcal{G}_{m,n}$-semianalytic. Moreover, for each $\lambda \in (0,\infty)^{m+n}$ the set $E_\lambda(X)$ is $\mathcal{G}_{m,n}$-semianalytic, where $E_\lambda : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^{m+n}$ is given by $E_\lambda(z) = (\lambda_1 z_1, \dots, \lambda_{m+n} z_{m+n})$ (here we use Lemma 4.4(ii)).
(3) If $X \subseteq \mathbb{R}^n$ is semianalytic, then X is $\mathcal{G}_{0,n}$-semianalytic.

Below we write 0 for the point $(0, \dots, 0) \in \mathbb{R}^{m+n}$. The following lemma is now proved just as in [12, Section 7] with obvious changes: "$\mathcal{R}_{\dots}$-set" is replaced by "$\mathcal{G}_{\dots}$-set", $\mathbb{R}\{X^*, Y\}$ by $\mathbb{R}\{X;Y\}_{\mathcal{G}}$ and the algebras $\mathcal{R}_{m,n,\dots}$ by $\mathcal{G}_{m,n;\dots}$. Also, the results from Sections 4, 5 and 6 there need to be replaced by the corresponding results of Sections 2, 3 and 4 here.

6.5. Lemma. (1) *Let $A \subseteq \mathbb{R}^{m+n}$ be $\mathcal{G}_{m,n}$-semianalytic at 0 and let σ be a permutation of $\{1, \dots, m\}$. Then $\sigma(A) := \big\{(x_{\sigma(1)}, \dots, x_{\sigma(m)}, x_{m+1}, \dots, x_{m+n}) : \ x \in A\big\}$ is $\mathcal{G}_{m,n}$-semianalytic at 0.*
(2) *If $n \geq 1$, then each $\mathcal{G}_{m,n}$-semianalytic subset of $\mathbb{R}^{m+n}$ is $\mathcal{G}_{m+1,n-1}$-semianalytic.*
(3) *Every $\mathcal{G}_{m,n;\rho}$-set $A \subseteq I_{m,n;\rho}$ is $\mathcal{G}_{m,n}$-semianalytic.* □

Note that Remark 6.4(3) and Lemma 6.5(2) imply in particular that every semianalytic subset of $\mathbb{R}^{m+n}$ is $\mathcal{G}_{m,n}$-semianalytic.

We now consider the system $\Lambda = (\Lambda_p)_{p\in\mathbb{N}}$, where each Λ_p is the collection of subsets of I^p with $I := [-1,1]$ defined by

$$\Lambda_p := \{A \subseteq I^p : \ A \text{ is } \mathcal{G}_p\text{-semianalytic}\}.$$

Note that if $A \subseteq I^p$ is $\mathcal{G}_{m,n}$-semianalytic with $m+n=p$, then A is also $\mathcal{G}_p$-semianalytic by Lemma 6.5(1), so $A \in \Lambda_p$. We refer the reader to [12, Section 2] for the notion of Λ-set and the Λ-Gabrielov Property.

The system Λ is easily seen to satisfy Axioms (I)-(III) listed in [12, Section 2]; the next proposition establishes Axiom (IV).

6.6. Proposition. *Every Λ-set has the Λ-Gabrielov Property.*

Proof. The proof proceeds almost literally as in [12, Section 8], with the obvious changes indicated earlier, as well as the following. Replace $\mathbb{R}\{X^*,Y\}_{\dots}$ there by $\mathbb{R}\{X;Y\}_{\mathcal{G},\dots}$ here. The quadruples $\mathrm{b}(f) \in \mathbb{N}^4$ there become quintuples $\mathrm{b}(f) \in \mathbb{N}^5$ here, defined from the blow-up height of Section 1 here rather than Section 4 there. The $\gamma > 0$ in the notations $s^\gamma_{m,m-1}$ and r^γ_λ there (for singular and regular blow-up substitutions) is simply equal to 1 here, and $s^\gamma_{m,m-1}$, r^γ_λ there correspond to s, r_λ here as defined in Section 4. The facts of the present section here are used in place of the corresponding facts from Section 7 there.

It remains to note that Remark 8.11 there becomes slightly simpler here: while the generalized power series rings are only closed under the differential operators $X_i \cdot (\partial/\partial X_i)$, the rings we are dealing with here are actually closed under $\partial/\partial X_i$. $\square$

Recall that by the remarks after Definition 6.1,

$$\mathbb{R}_{\mathcal{G}} = \left(\mathbb{R}, <, 0, 1, +, -, \cdot, \left(\widetilde{f} : \ f \in \mathcal{G}_{m,0;1}\right)\right).$$

It is clear from a remark in Definition 6.3 that every *bounded* $\mathcal{G}_p$-semianalytic set, $p \in \mathbb{N}$, is quantifier-free definable in $\mathbb{R}_{\mathcal{G}}$. We are now ready to prove the first part of Theorem A.

6.7. Theorem. *The expansion $\mathbb{R}_{\mathcal{G}}$ is model complete and o-minimal.*

Proof. The theorem follows from the previous remark in view of Propositions 6.6 above and [12, Proposition 2.9]. (Clearly $\{r\} \in \Lambda_1$ for each $r \in \mathbb{R}$, and the sets $\{(x,y,z) \in I^3 : \ xy = z\}$ and $\{(x,y,z) \in I^3 : \ x+y = z\}$ belong to Λ_3.) $\square$

As a consequence of [12, Proposition 2.9] and the way we proved the previous theorem, we also get:

6.8. Proposition. *Let $A \subseteq \mathbb{R}^m$ be bounded and definable in $\mathbb{R}_{\mathcal{G}}$. Then there is an $n \geq m$ and a bounded $\mathcal{G}_n$-semianalytic set $B \subseteq \mathbb{R}^n$ such that $A = \Pi_m(B)$, where $\Pi_m : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is the projection on the first m coordinates. More precisely, there are an $n \geq m$ and bounded $\mathcal{G}_n$-semianalytic sets $B_1, \dots, B_k \subseteq \mathbb{R}^n$ such that*

$$A = \Pi_m(B_1) \cup \cdots \cup \Pi_m(B_k)$$

and each B_i is a connected analytic (embedded) submanifold of $\mathbb{R}^n$ for which $\Pi_m|B_i$ is an immersion into $\mathbb{R}^m$. □

7. Characterizing Definable 1-variable Functions

In this section we prove the second part of Theorem A, which characterizes definable 1-variable functions. We proceed along the lines of [12, Section 9]. Below we let d and e range over $\mathbb{N} \setminus \{0\}$.

7.1. Definition. For a single indeterminate T we let

$$\mathbb{R}\left\{T^{1/d}\right\}_{\mathcal{G}} := \left\{f\left(T^{1/d}\right) : \ f \in \mathbb{R}\{T\}_{\mathcal{G}}\right\} \subseteq \mathbb{R}\left(\left(T^{\mathbb{Q}}\right)\right),$$

and we put

$$\mathrm{P}(\mathbb{R}\{T\}_{\mathcal{G}}) := \bigcup_{d \geq 1} \mathbb{R}\left\{T^{1/d}\right\}_{\mathcal{G}}.$$

Note that $\mathrm{P}(\mathbb{R}\{T\}_{\mathcal{G}})$ is a valuation ring whose maximal ideal consists of those f in $\mathrm{P}(\mathbb{R}\{T\}_{\mathcal{G}})$ satisfying $f(0) = 0$. We shall consider the fraction field of $\mathrm{P}(\mathbb{R}\{T\}_{\mathcal{G}})$ as a subfield of $\mathbb{R}\left(\left(T^{\mathbb{Q}}\right)\right)$.

Every $g \neq 0$ in the fraction field of $\mathrm{P}(\mathbb{R}\{T\}_{\mathcal{G}})$ is of the form $g = T^r h$ with $r \in \mathbb{Q}$ and $h \in \mathrm{P}(\mathbb{R}\{T\}_{\mathcal{G}})$ satisfying $h(0) \neq 0$, and we make this fraction field into an ordered field by putting $g > 0$ if $h(0) > 0$.

7.2. Lemma. *The local ring $\mathrm{P}(\mathbb{R}\{T\}_{\mathcal{G}})$ is henselian, that is, given any*

$$f(T, W) = W^n + a_1(T)W^{n-1} + \cdots + a_n(T) \in \mathrm{P}(\mathbb{R}\{T\}_{\mathcal{G}})[W]$$

with $f(0,0) = 0$ and $(\partial f/\partial W)(0,0) \neq 0$, there is an $\alpha \in \mathrm{P}(\mathbb{R}\{T\}_{\mathcal{G}})$ such that $\alpha(0) = 0$ and $f(T, \alpha(T)) = 0$.

Proof. First we observe that $\mathbb{R}\{T\}_{\mathcal{G}}$ is henselian. Let $f(T,W) \in \mathbb{R}\{T\}_{\mathcal{G}}[W]$ be as in the statement of the lemma. Considering $f(T,W)$ as an element of $\mathbb{R}\{T;W\}_{\mathcal{G}}$, this means that f is regular in W of order 1. Hence, by Weierstrass Preparation 3.7 we have $f(T,W) = u(T,W)(W - \alpha(T))$ for some unit $u \in \mathbb{R}\{T;W\}_{\mathcal{G}}$ and some $\alpha \in \mathbb{R}\{T\}_{\mathcal{G}}$, so $f(T,\alpha(T)) = 0$.

Next, each ring $\mathbb{R}\left\{T^{1/d}\right\}_{\mathcal{G}}$ for nonzero $d \in \mathbb{N}$ is isomorphic to $\mathbb{R}\{T\}_{\mathcal{G}}$, and hence is henselian. Therefore $\mathrm{P}(\mathbb{R}\{T\}_{\mathcal{G}})$ is henselian. □

7.3. Corollary. *The fraction field of $\mathrm{P}(\mathbb{R}\{T\}_{\mathcal{G}})$ is real closed.*

Proof. By the previous lemma and the remarks preceding it, using [18]. □

7.4. Lemma. *Let $\rho \in (0,\infty)^{m+n}$, $s \in (0,\infty)$, $f \in \mathbb{R}\{X;Y\}_{\mathcal{G},\rho}$ and $g_1, \ldots, g_{m+n} \in \mathbb{R}\{T\}_{\mathcal{G},s}$ be such that $g_i > 0$ for each $i = 1, \ldots, m$ and $|g_j(t)| \leq \rho_j$ for all $t \in [0,s]$ and each $j = 1, \ldots, m+n$. Then there is an $h \in \mathbb{R}\{T\}_{\mathcal{G}}$ such that $h(t) = f(g_1(t), \ldots, g_{m+n}(t))$ for every sufficiently small $t > 0$.*

Proof. For $j = 1, \ldots, n$ let $\epsilon_j := -1$ if $g_{m+j}(0) < 0$ and $\epsilon_j := 1$ otherwise. Then the series $f(X, \epsilon_1 Y_1, \ldots, \epsilon_n Y_n)$ belongs to $\mathbb{R}\{X;Y\}_{\mathcal{G},\rho}$ as well, so replacing each g_{m+j} by $\epsilon_j \cdot g_{m+j}$ we may assume that $g_{m+j}(0) \geq 0$ for each j. Now use Lemmas 5.1, 5.3 and 4.4(1). □

Given $g \in \mathrm{P}(\mathbb{R}\{T\}_{\mathcal{G}})$ we choose d, $\rho > 0$ and $\widetilde{g} \in \mathbb{R}\{T\}_{\mathcal{G},\rho}$ such that $g = \widetilde{g}(T^{1/d})$, and we put $g(x) := \widetilde{g}(x^{1/d})$ for $0 \le x \le \rho^d$. (This is unambiguous in the sense that if we make another such choice, then the same value of $g(x)$ results for those x where $g(x)$ is defined for both choices.)

7.5. Proposition (Curve Selection). *Let $B \subseteq \mathbb{R}^n$ be definable in $\mathbb{R}_{\mathcal{G}}$, and let $0 \in \mathrm{bd}(B)$. Then there exists a $g = (g_1, \dots, g_n) \in \mathrm{P}(\mathbb{R}\{T\}_{\mathcal{G}})^n$ such that $g(0) = 0$ and $g(t) \in B$ for all sufficiently small $t > 0$.*

Proof. The proof goes like the proof of [12, 9.6], with the obvious changes indicated in the proof of Proposition 6.6, and using 7.3 and 7.4 (together with the previous remark) here in place of 9.2 and 9.4 there. □

We can now deduce the second part of Theorem A.

7.6. Theorem. *Let $\epsilon > 0$ and let $f : (0, \epsilon) \longrightarrow \mathbb{R}$ be definable in $\mathbb{R}_{\mathcal{G}}$. Then there exist a $g \in P(\mathbb{R}\{T\}_{\mathcal{G}})$ and an $r \in \mathbb{Q}$ such that $f(t) = t^r g(t)$ for all sufficiently small $t > 0$.*

Proof. Assume first that $\lim_{t \to 0} f(t) = 0$. Then $(0,0) \in \mathrm{fr}(\Gamma(f))$, so by Proposition 7.5 there are $g_1, g_2 \in P(\mathbb{R}\{T\}_{\mathcal{G}})$ such that $(g_1(t), g_2(t)) \in \Gamma(f)$ for all small enough $t > 0$ and $g_1(0) = g_2(0) = 0$. Take d and $\widetilde{g}_1, \widetilde{g}_2 \in \mathbb{R}\{T\}_{\mathcal{G}}$ such that $g_1 = \widetilde{g}_1(T^{1/d})$ and $g_2 = \widetilde{g}_2(T^{1/d})$. By Lemma 5.6 there is an $h \in \mathbb{R}\{T\}_{\mathcal{G}}$ such that $h > 0$, $h(0) = 0$ and $\widetilde{g}_1 \circ h = T^d$. Then it is clear that the desired result holds for $g := (\widetilde{g}_2 \circ h)(T^{1/d})$.

If $\lim_{t \to 0} f(t) = c \in \mathbb{R}$, then the theorem follows easily from the case above by considering $f - c$ in place of f. If $\lim_{t \to 0} |f(t)| = \infty$, then the theorem follows similarly from the first case by considering $1/f$ in place of f. □

7.7. Corollary. *The expansion $\mathbb{R}_{\mathcal{G}}$ of the real field is polynomially bounded.* □

Using cell decomposition and the notion of dimension of definable sets in o-minimal structures (see for example [4]), theorem 7.6 has the following easy consequence.

7.8. Corollary. (1) *If $A \subseteq \mathbb{R}^n$ is definable in $\mathbb{R}_{\mathcal{G}}$ and $\dim(A) \le 1$, then A is in fact $\mathcal{G}_n$-semianalytic.*

(2) *If $A \subseteq \mathbb{R}^2$ is definable in $\mathbb{R}_{\mathcal{G}}$, then A is $\mathcal{G}_{1,1}$-semianalytic.* □

8. Examples of Definable and Non-Definable Functions

We indicate here two functions (mentioned already briefly in the introduction) that are definable in $\mathbb{R}_{\mathcal{G}}$, and we also give a large class of functions that are not definable in $\mathbb{R}_{\mathcal{G}}$.

8.1. Example. From Binet's second formula (see for instance [22]), we have

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log(2\pi) + \phi(z) \qquad \text{for } z \in \mathbb{C} \setminus (-\infty, 0],$$

and the function ϕ has an asymptotic expansion at $+\infty$ that is referred to in the literature as Stirling's series. The estimates on the remainder term of Stirling's series found in [17, Section 79] show that the function $\psi(z) := \phi(1/z)$ belongs in fact to $\mathcal{G}(R, \phi, 1)$ for every

$R > 0$ and $\phi \in (0, \pi)$. Also, since Γ is meromorphic on $\mathbb{C}$, the restriction $\Gamma\big|(0,1)$ is definable in $\mathbb{R}_{\text{an}}$.

Corollary. *The function*

$$x \mapsto \log \Gamma(x) - \left(x - \frac{1}{2}\right) \log x : (1, \infty) \longrightarrow \mathbb{R}$$

is definable in $\mathbb{R}_{\mathcal{G}}$. *Therefore,* $\Gamma\big|(0,\infty)$ *is definable in* $(\mathbb{R}_{\mathcal{G}}, \exp)$. □

8.2. Example. Consider the function $f : [0, \infty) \longrightarrow \mathbb{R}$ given by

$$f(x) := \int_0^\infty \frac{e^{-t}}{1 + xt} dt.$$

(In [8] it is shown that f is not definable in $(\mathbb{R}_{\text{an}}, \exp)$.) As mentioned in the introduction, $f\big|[0,1]$ belongs to $\mathcal{G}(1)$ and thus is definable in $\mathbb{R}_{\mathcal{G}}$. The function f has as its Taylor series at 0 the divergent power series $\sum(-1)^n n! X^n$ and is actually the Borel sum on $[0, \infty)$ of this power series. (For more on this, see Hardy [14].) The function f is not polynomially bounded at $+\infty$, and thus f itself is not definable in $\mathbb{R}_{\mathcal{G}}$. But as noted in [8], its restriction $f\big|[1,\infty)$ is definable in $(\mathbb{R}_{\text{an}}, \exp)$. Combining these facts, we obtain

Corollary. *The function f is definable in* $(\mathbb{R}_{\mathcal{G}}, \exp)$. □

For the remainder of this section we use the definitions and notations of [2]. Following a common abuse of notation (as in [2]), we shall write $z = re^{i\phi}$ for the point on the Riemann surface $\mathbb{C}_{\log}$ of the logarithm that has modulus $r > 0$ and argument $\phi \in \mathbb{R}$; thus $re^{i(2\pi+\phi)}$ is a different point on $\mathbb{C}_{\log}$. We denote by $\widehat{\mathcal{L}}_k$ and $\mathcal{L}_k$ the formal, respectively analytic, Laplace transform of index $k > 0$, and similarly $\widehat{\mathcal{B}}_k$ and $\mathcal{B}_k$ denote the formal, respectively analytic, Borel transform of index $k > 0$. (Such an analytic transform is defined on a subset of $\mathbb{C}_{\log}$).

8.3. Example. Let T be a single indeterminate. By Proposition 2.17 the algebra $\mathbb{C}\{T\}_{\mathcal{G}}$ consists exactly of those power series in $\mathbb{C}[\![T]\!]$ that are κ-summable in the positive real direction for some finite sequence $\kappa = (\kappa_1, \dots, \kappa_n)$ in $(0, 1]$, as defined in [2]. Conversely, every power series $f(T) \in \mathbb{C}[\![T]\!]$ that is κ-summable in the positive real direction for some arbitrary finite sequence κ in $(0, \infty)$ is of the form $f(T) = g\left(T^{1/d}\right)$ for some $g \in \mathbb{C}\{T\}_{\mathcal{G}}$ and some non-zero $d \in \mathbb{N}$. (Such a series f is also referred to simply as **multisummable in the positive real direction**.) On the other hand, Proposition 2.17 and [2] imply that for every $f \in \mathbb{C}\{T\}_\tau$ with $\tau = (K, R, r, \phi) \in \mathcal{T}_1$ the Borel sum $f_{\mathcal{B}}$ of f, as defined in [2], contains some interval $(0, \epsilon)$ with $0 < \epsilon \leq R$ in its domain and agrees there with f_τ (where we identify a positive real number s with the point $se^{i\cdot 0}$ on $\mathbb{C}_{\log}$). We therefore get

Corollary. *Let $f \in \mathbb{R}[\![T]\!]$ be multisummable in the positive real direction. Then there is an $\epsilon > 0$ such that $f_{\mathcal{B}}$ is defined on the interval $(0, \epsilon)$ and $f_{\mathcal{B}}\big|(0, \epsilon)$ is definable in* $\mathbb{R}_{\mathcal{G}}$. □

Finally, using multisummability theory and Theorem 7.6, we show that no divergent power series $\sum a_n X^n$ with all $a_n \geq 0$ can be the Taylor series at 0 of a definable C^∞ function $f : [0, \epsilon) \longrightarrow \mathbb{R}$. If $h \in \mathbb{C}[\![T]\!]$ is a convergent power series (i.e., converges in a neighbourhood of $0 \in \mathbb{C}$) with radius of convergence $a > 0$, we denote here (as in [2]) by $\mathcal{S}h : D(a) \longrightarrow \mathbb{C}$ the analytic function defined by h on the disk $D(a) \subseteq \mathbb{C}$.

8.4. Lemma. *Let $f = \sum a_n T^n$ be such that $a_n \geq 0$ for all n. Assume that f has infinite radius of convergence and that $A, B, k > 0$ are such that $f(x) \leq A\exp(Bx^k)$ for all $x > 0$. Then the power series $\widehat{\mathcal{L}}_k f := \sum \Gamma(1+n/k) a_n T^n$ is convergent and satisfies $\mathcal{S}(\widehat{\mathcal{L}}_k f)(x) = \mathcal{L}_k(\mathcal{S}f)(x)$ for all sufficiently small $x > 0$.*

Proof. The entire function $\mathcal{S}f$ lifts (when restricted to $\mathbb{C} \setminus \{0\}$) to a holomorphic function on the covering space $\mathbb{C}_{\log}$ of $\mathbb{C} \setminus \{0\}$, and we denote this lifting by $\mathcal{S}f$ as well. The growth condition on f implies that $\mathcal{S}f$ is of exponential size at most k on any sector $S(0,\alpha) \subseteq \mathbb{C}_{\log}$ with $\alpha > 0$. Hence by Theorem 1 of [2, Chapter 2] with $k_1 := +\infty$ and f in place of $\widehat{f}$ there, for every $\beta > 0$ there exists an $r = r(\beta) > 0$ such that $\mathcal{L}_k(\mathcal{S}f) \cong_k \widehat{\mathcal{L}}_k f$ on the sector $S := S(0,\beta,r) \subseteq \mathbb{C}_{\log}$.

Now choose a $\beta > 2\pi + \pi/k$ and let $r := r(\beta)$. For $z = \rho e^{i\phi} \in S' := S(\pi, \beta - \pi, r) \subseteq \mathbb{C}_{\log}$ we put $z' := \rho e^{i(\phi - 2\pi)} \in \mathbb{C}_{\log}$, and we define an analytic function $g : S' \longrightarrow \mathbb{C}$ by

$$g(z) := (\mathcal{L}_k(\mathcal{S}f))(z) - (\mathcal{L}_k(\mathcal{S}f))(z').$$

Then $g \cong_k 0$, and thus by Theorem 2 of [2, Chapter 2] with g in place of f, $\mathcal{B}_k g \cong_\infty 0$, that is, $\mathcal{B}_k g = 0$. Since $\beta - \pi > \pi/k$, we conclude from Theorem 3 of [2, Chapter 2] with g in place of f that $g = (\mathcal{L}_k \circ \mathcal{B}_k) g = \mathcal{L}_k 0 = 0$. Therefore, the function $h : D(r) \setminus (-r, 0] \longrightarrow \mathbb{C}$ defined by $h(z) := (\mathcal{L}_k(\mathcal{S}f))(z)$ extends to an analytic function $\widetilde{h} : D(r) \longrightarrow \mathbb{C}$ (here $D(r)$ is the disk in $\mathbb{C}$ defined in Section 2). Since the Taylor series of $\widetilde{h}$ is equal to $\widehat{\mathcal{L}}_k f$, this shows that the latter is convergent and satisfies $\mathcal{S}(\widehat{\mathcal{L}}_k f)(x) = \mathcal{L}_k(\mathcal{S}f)(x)$ for all sufficiently small $x > 0$. ◻

8.5. Lemma. *If $f = \sum a_n T^n$ is divergent and satisfies $a_n \geq 0$ for all n, then $f \notin \mathbb{R}\{T\}_{\mathcal{G}}$.*

Proof. Assume for a contradiction that f belongs to $\mathbb{C}\{T\}_{\mathcal{G}}$. Take q minimal such that $f \in \mathbb{C}\{T\}_\tau$ for some $\tau = (K, R, r, \phi) \in \mathcal{T}_1$ with $K = \{t_1, \dots, t_q\}$ and $t_1 < \cdots < t_q$. (We use t_i here in place of the usual k_i in order to make the notation below agree with [2].) Note that $t_q > 0$, since f is divergent. Furthermore, if $t_1 = 0$, then by the claim in Lemma 2.16 we have $f = f_1 + f_2$ with $f_1 \in \mathbb{C}\{T\}_{\tau'}$, $\tau' := (K', R, r, \phi)$, $K' := \{t_2, \dots, t_q\}$, and f_2 absolutely convergent on $D(R)$. By Lemma 2.3, f_2 belongs to $\mathbb{C}\{T\}_{\tau'}$ as well, so $f = f_1 + f_2 \in \mathbb{C}\{T\}_{\tau'}$. The minimality of q therefore implies that $t_1 > 0$.

Write $k_i := 1/t_i$ for $i = 1, \dots, q$, and put $k := (k_1, \dots, k_q)$. By Proposition 2.16 the series f is k-summable in the positive real direction (corresponding to the multidirection $d = (d_1, \dots, d_q) := (0, \dots, 0)$ in [2]). We define $\kappa = (\kappa_1, \dots, \kappa_q)$ by $\kappa_1 := k_1$ and $1/\kappa_i := 1/k_i - 1/k_{i-1}$ for $i = 2, \dots, q$, and we let

$$\widehat{g}_q := \left(\widehat{\mathcal{B}}_{\kappa_q} \circ \cdots \circ \widehat{\mathcal{B}}_{\kappa_2} \circ \widehat{\mathcal{B}}_{\kappa_1}\right) f.$$

By Theorem 1 of [2, Chapter 7] with f in place of $\widehat{f}$ (notice the misprint there: "$\widehat{g}_q := \widehat{\mathcal{B}}_{\kappa_1} \circ \cdots \circ \widehat{\mathcal{B}}_{\kappa_q} \widehat{f}$" should be "$\widehat{g}_q := \widehat{\mathcal{B}}_{\kappa_q} \circ \cdots \circ \widehat{\mathcal{B}}_{\kappa_1} \widehat{f}$"), the series $\widehat{g}_q$ is convergent and the function $g_q := \mathcal{S}(\widehat{g}_q)$ extends to a real analytic function on $(0,\infty)$ of exponential size at most κ_q.

We claim that the series $\widehat{\mathcal{L}}_{\kappa_q} \widehat{g}_q$ is convergent and satisfies $\mathcal{S}(\widehat{\mathcal{L}}_{\kappa_q} \widehat{g}_q)(x) = \mathcal{L}_{\kappa_q} g_q(x)$ for all sufficiently small $x > 0$. In fact, by the hypothesis on f and the definition of the formal Borel transform, $\widehat{g}_q = \sum b_n T^n$ with $b_n \geq 0$ for all n. Thus, if $a > 0$ is the radius of convergence of $\widehat{g}_q$, then $\lim_{x\to a^-} g_q(x) = +\infty$. Since g_q extends to a real analytic function on $(0,\infty)$, we get $a = +\infty$. But g_q is of exponential size at most κ_q, so the previous lemma (with $\widehat{g}_q$ in place of f) now implies the claim.

On the other hand $\widehat{\mathcal{L}}_{\kappa_q}\widehat{g}_q = \left(\widehat{\mathcal{B}}_{\kappa_{q-1}} \circ \cdots \circ \widehat{\mathcal{B}}_{\kappa_2} \circ \widehat{\mathcal{B}}_{\kappa_1}\right) f$, so we get from the claim and Theorem 1 of [2, Chapter 7] that f is $(k_1,\dots,k_{q-1})$-summable in the positive real direction. Hence by Proposition 2.17, f belongs to $\mathbb{C}\{T\}_{\widetilde{\tau}}$ for some $\widetilde{\tau} = (\widetilde{K},\widetilde{R},\widetilde{r},\widetilde{\phi})$ with $\widetilde{K} = \{t_1,\dots,t_{q-1}\}$, which contradicts the minimality of q. □

8.6. Corollary. *There is no C^∞ function $f : [0,\epsilon) \longrightarrow \mathbb{R}$, $\epsilon > 0$, that is definable in $\mathbb{R}_{\mathcal{G}}$ and whose Taylor series $\sum a_n T^n$ at 0 is divergent with all $a_n \geq 0$.*

Proof. Assume for a contradiction that there is such a function f. Then $\widehat{f} = \sum a_n T^n = g(T^{1/d})$ for some $g \in \mathbb{R}\{T\}_{\mathcal{G}}$ and some nonzero $d \in \mathbb{N}$. But then $g = \sum a_n T^{dn}$, which contradicts Lemma 8.5. □

Thus in contrast to $\sum(-1)^n n! T^n$, the series $\sum n! T^n$ is not the Taylor series at 0 of any C^∞ function $[0,\epsilon) \longrightarrow \mathbb{R}$, $\epsilon > 0$, that is definable in $\mathbb{R}_{\mathcal{G}}$.

9. Expanding by the Exponential Function

The goal of this section is to prove Theorem B on $(\widetilde{\mathbb{R}}, \exp)$. The only obstacle in just copying the proof for $(\mathbb{R}_{\text{an}}, \exp)$ in [9] is that we first need to know a purely valuation theoretic property of elementary extensions of $\widetilde{\mathbb{R}}$. For $\widetilde{\mathbb{R}} = \mathbb{R}_{\text{an}}$ this valuation theoretic fact was established in [9, Section 3] by showing that its elementary extensions could be "nicely" embedded into certain power series models. For general $\widetilde{\mathbb{R}}$, and even for $\mathbb{R}_{\mathcal{G}}$, this method is unavailable. Below in Proposition 9.2 we obtain this valuation theoretic property in another way.

From now on in this section we assume that $\widetilde{\mathbb{R}}$ is a polynomially bounded o-minimal expansion of the real field. (The assumption that $\exp\big|[0,1]$ is definable in $\widetilde{\mathbb{R}}$ is only needed later.) Let $\mathcal{R}$ be a model of $T := \mathrm{Th}(\widetilde{\mathbb{R}})$ and V a convex subring of $\mathcal{R}$. We let $\overline{V} = V/\mathbf{m}$ be the (ordered) residue field, and $v : \mathcal{R} \to \Gamma \cup \{\infty\}$ the induced valuation with valuation ring V.

In the next lemma we also assume that $V \neq \mathcal{R}$. Then the structure $(\mathcal{R}, V)$ is a model of the theory T_{convex} in the notations of [6] and [7]; also, given a function $f : V \to \mathcal{R}$ that is definable in $(\mathcal{R}, V)$, there exists by [6, Proposition 4.2] an element $\gamma \in \Gamma \cup \{\infty\}$ such that $v(f(x)) = \gamma$ for all sufficiently large $x \in V$, that is, for all $x \in V$ greater than some element $a \in V$ (which may of course depend on f). Denote this (unique) element γ by $v(f)$. It turns out that we need something better:

9.1. Lemma. *Let $f : V \to \mathcal{R}$ be definable in $(\mathcal{R}, V)$. Then the subset*

$$\Gamma_f := \{v(f - c) : \ c \in \mathcal{R}\}$$

of $\Gamma \cup \{\infty\}$ has a largest element.

Proof. If f is ultimately constant on V, we have $\infty \in \Gamma_f$, so we may assume f is not ultimately constant on V. By [6, Corollary 2.8], and replacing f by $-f$ if necessary, we may assume that $f(x) > 0$ for all sufficiently large $x \in V$. Similarly, replacing f by $1/f$ if necessary, we may also assume that f is ultimately strictly increasing on V. Take $p \in V$ such that f is positive and strictly increasing on $V_p := \{x \in V : \ x > p\}$.

Suppose (for a contradiction) that Γ_f has no largest element. Consider the map

$$x \mapsto v(f - f(x)) \ : \ V_p \to \Gamma_f.$$

This map is clearly increasing, and we claim that it has cofinal image in Γ_f. Otherwise there would be an element $c \in \mathcal{R}$ with $v(f-c) > v(f - f(x))$ for all $x \in V_p$. Then take $a \in V_p$ such that $v(f(y) - c) = v(f - c)$ for all $y \geq a$; in particular $v(f(a) - c) = v(f - c)$. Hence $v(f(y) - f(a)) \geq v(f - c)$ for all $y \geq a$, so $v(f - f(a)) \geq v(f - c) > v(f - f(a))$, a contradiction.

Since the above map is increasing and has cofinal image, it follows by the "Fact" in the proof of [6, Proposition 2.10] that

$$\text{cofinality}\left(\overline{V}\right) = \text{cofinality}(V_p) = \text{cofinality}(\Gamma_f).$$

Because T is polynomially bounded, the results from [6] imply that there is an elementary extension of $(\mathcal{R}, V)$ that has the same value group Γ, but whose residue field is any given elementary extension of $\overline{V}$ (viewed as model of T). By passing to such an elementary extension we do not change Γ_f and can arrange that the residue field has cofinality greater than cofinality(Γ_f), which contradicts the equality above. (A similar argument is carried out in greater detail in the proof of Proposition 4.2 of [6].) $\square$

Recall that K denotes the field of exponents of $\widetilde{\mathbb{R}}$.

Let now $\mathcal{R} \prec \mathcal{R}\langle a \rangle$ be models of T, and let V and W be convex subrings of $\mathcal{R}$ and $\mathcal{R}\langle a \rangle$ respectively, with $W \cap \mathcal{R} = V$. (We allow $V = \mathcal{R}, \ W = \mathcal{R}\langle a \rangle$.) We denote the valuations on $\mathcal{R}$ and $\mathcal{R}\langle a \rangle$ with valuation rings V and W by v and w, and their value groups by Γ_v and Γ_w. With the usual identifications we have $\Gamma_v \subseteq \Gamma_w$ (as ordered K-linear spaces), w extends v and $\overline{V} \subseteq \overline{W}$ (as ordered fields), where $\overline{V}$ and $\overline{W}$ are the residue fields of V and W.

9.2. Proposition (Valuation Property). *Suppose $\Gamma_v \neq \Gamma_w$. Then there exists an $r \in \mathcal{R}$ such that $w(a - r) \notin \Gamma_v$.*

In other words, the value group extension is witnessed by an element of the form $a - r$ with $r \in \mathcal{R}$.

Proof. We first note that $\overline{V} = \overline{W}$ by [6, Corollary 5.6], and hence that W is the convex hull of V in $\mathcal{R}\langle a \rangle$. Suppose first that we are in the (trivial) case $V = \mathcal{R}$. Then $\Gamma_v = \{0\}$. If $w(a) \neq 0$, then we can take $r = 0$. If $w(a) = 0$, then $\overline{a} := a + \mathbf{m}(W) \in \overline{W} = \overline{V}$, so there exists an $r \in V$ with $\overline{a} = \overline{r}$, so $w(a - r) > 0$, and we are done. Most other cases will be reduced in various ways to the following.

Case 1 (Main Case): $V \neq \mathcal{R}$ and there exists a $b \in \mathcal{R}\langle a \rangle$ with $V < b < |\mathcal{R} \setminus V|$ and $w(b) \notin \Gamma_v$. Then we take a function $f : \mathcal{R} \to \mathcal{R}$ that is definable in $\mathcal{R}$ such that $f(b) = a$. Let us denote the restriction of f to V also by f. In the notation of the previous lemma there is then an element $s \in \mathcal{R}$ for which $v(f-s)$ is maximal. After replacing f by $f-s$ and a by $a-s$ we reduce to the following situation:

$$v(f-c) \le v(f) < \infty \quad \text{for all } c \in \mathcal{R}. \tag{9.1}$$

Taking $c \in \mathcal{R}$ with $v(c) = v(f)$ and replacing f by f/c and a by a/c, we may assume in addition that

$$v(f) = 0. \tag{9.2}$$

We now take elements $p \in V$ and $q \in \mathcal{R}$ with $q > V$ such that f is continuous, monotone and of fixed sign on the interval (p,q): positive if $a = f(b) > 0$, negative if $a = f(b) < 0$. Take an elementary submodel $\mathcal{R}'$ of $\mathcal{R}$ such that $V = \mathcal{R}' + \mathbf{m}(V)$. By increasing p, we may assume that $p \in \mathcal{R}'$, $p > 0$. Define the function $\overline{f} : (p, +\infty)_{\mathcal{R}'} \to \mathcal{R}'$ by $\overline{f}(x) = st_{\mathcal{R}'}(f(x))$. Since $v(f) = 0$, the function $\overline{f}$ only takes nonzero values, after increasing p if necessary. Because $\mathcal{R}'$ is tame in $\mathcal{R}$, the function $\overline{f}$ is definable in $\mathcal{R}'$ by [6, Corollary 1.5]. Taking p large enough we may further assume that $\overline{f}$ is continuous, monotone, and that for all $x \in V$ with $x > p$ we have

$$f(x) - \overline{f}(st_{\mathcal{R}'}(x)) \in \mathbf{m}(V). \tag{9.3}$$

By polynomial boundedness there exists a $\lambda \in K$ and a non-zero $c \in \mathcal{R}'$ such that $\overline{f}(x) = cx^\lambda + o(x^\lambda)$ as $x \to +\infty$ in $\mathcal{R}'$, where $o(x^\lambda)/x^\lambda \to 0$ as $x \to +\infty$ in $\mathcal{R}'$.

Subcase (i): $\lambda \neq 0$. From $\left|\overline{f}(x) - cx^\lambda\right| \le \left|cx^\lambda\right|/2$ for large $x > 0$ in $\mathcal{R}'$ and (9.3) it follows that $\left|f(x) - cx^\lambda\right| \le \left|cx^\lambda\right|/2$ for large $x > 0$ in V, say for all $x > p$ in V, after increasing p if necessary. By "overspill" (using that f is definable in $\mathcal{R}$ and that V has no supremum in $\mathcal{R}$), this last inequality continues to hold "beyond" V, that is, after decreasing our element $q > V$ suitably we have $|f(x) - cx^\lambda| \le cx^\lambda/2$ for all $x \in \mathcal{R}$ with $p < x < q$. Since $\mathcal{R} \prec \mathcal{R}\langle a\rangle$, this remains true for $x = b$, and thus $w(a) = w(f(b)) = w(cb^\lambda) = \lambda w(b) + v(c) \notin \Gamma_v$, and we are done.

Subcase (ii): $\lambda = 0$. Then $\overline{f}(x) \to c$ as $x \to +\infty$ in $\mathcal{R}'$. If $\overline{f}(x) = c$ for all sufficiently large $x \in \mathcal{R}'$, then by (9.3) we have $v(f-c) > 0$, which contradicts (9.1) and (9.2) above. Hence $\overline{f}(x) \neq c$ for all sufficiently large $x \in \mathcal{R}'$. But then there is some $\mu < 0$ in K and a non-zero $d \in \mathcal{R}'$ such that $\overline{f}(x) = c + dx^\mu + o(x^\mu)$ as $x \to +\infty$ in $\mathcal{R}'$. Then we are back in Subcase (i) with $f-c$, $a-c$, μ instead of f, a, λ, and the result from that subcase gives $w(a-c) \notin \Gamma_v$. This completes the proof in the Case 1.

We now consider the general case that $V \neq \mathcal{R}$. We start by taking any $b \in \mathcal{R}\langle a\rangle$ with $w(b) \notin \Gamma_v$. After replacing b by $-b$, $1/b$ or $-1/b$ if necessary, we may assume that $b > 0$ and $w(b) < 0$. Let C be the cut in Γ_v realized by $w(b)$, that is, $C = \{\gamma \in \Gamma_v : \gamma < w(b)\}$. We distinguish several further cases.

Case 2: $C = \emptyset$. Then $w(b) < \Gamma_v$, so $b > \mathcal{R}$. Let $f : \mathcal{R} \to \mathcal{R}$ be definable in $\mathcal{R}$ with $f(b) = a$. By polynomial boundedness there exist $c \in \mathcal{R}$ with $c \neq 0$ and $\lambda \in K$ such that $f(x) = cx^\lambda + o(x^\lambda)$ as $x \to +\infty$ in $\mathcal{R}$. If $\lambda \neq 0$, we use $\mathcal{R} \prec \mathcal{R}\langle a\rangle$ to get $f(b) = cb^\lambda + \epsilon b^\lambda$ with $\epsilon \in \mathcal{R}\langle a\rangle$ and $|\epsilon| < \delta$ for all positive $\delta \in \mathcal{R}$, which implies $w(a) = w(f(b)) = \lambda w(b) + v(c) \notin \Gamma_v$, and we are done. Hence we may assume that $\lambda = 0$. Then $f(x) \to c$ as $x \to +\infty$ in $\mathcal{R}$. But $f(x) \neq c$ for all sufficiently large $x \in \mathcal{R}$, because $f(b) = a \neq c$. Thus there exist a $\mu < 0$

in K and a non-zero $d \in \mathcal{R}$ such that $f(x) = c + dx^\mu + o(x^\mu)$ as $x \to +\infty$ in $\mathcal{R}$, and the same argument as in the case $\lambda \neq 0$ with $f - c,\ \mu,\ d$ instead of $f,\ \lambda,\ c$ gives $w(a-c) \notin \Gamma_v$.

Case 3: $C = (-\infty, \gamma)$ for some $\gamma \in \Gamma_v,\ \gamma \leq 0$. This actually falls under Case 1: take $c \in \mathcal{R},\ c > 0$ with $v(c) = \gamma$ and note that $V < b/c < |\mathcal{R} \setminus V|$.

Case 4: $C = (-\infty, \gamma]$ for some $\gamma \in \Gamma_v,\ \gamma < 0$. This also falls under Case 1: take $c \in \mathcal{R},\ c > 0$ with $v(c) = \gamma$ and note that $V < c/b < |\mathcal{R} \setminus V|$.

Case 5: $C \neq \emptyset$ and C has no supremum in Γ_v. Here the reduction to Case 1 involves changing the valuations. We introduce the convex K-linear subspaces Δ_v and Δ_w of Γ_v and Γ_w respectively:

$$\Delta_v := \{\gamma \in \Gamma_v :\ |\gamma| < \lambda |w(b)| \text{ for all positive } \lambda \in K\},$$
$$\Delta_w := \{\gamma \in \Gamma_w :\ |\gamma| < \lambda |w(b)| \text{ for all positive } \lambda \in K\},$$

so that $\Delta_w \cap \Gamma_v = \Delta_v$. Put $\Gamma_{v_\Delta} := \Gamma_v/\Delta_v$ and $\Gamma_{w_\Delta} := \Gamma_w/\Delta_w$. Then we have $\Gamma_{v_\Delta} \subseteq \Gamma_{w_\Delta}$ as ordered K-linear spaces after the obvious identification. The composition

$$\mathcal{R}^\times \xrightarrow{v} \Gamma_v \longrightarrow \Gamma_{v_\Delta}$$

is a valuation on $\mathcal{R}$ (a "coarsening" of v), which we shall denote by v_Δ. Similarly w gives rise to the valuation $w_\Delta : \mathcal{R}\langle a\rangle^\times \to \Gamma_{w_\Delta}$, and clearly w_Δ extends v_Δ, the valuation rings V_Δ and W_Δ of v_Δ and w_Δ are convex subrings of $\mathcal{R}$ and $\mathcal{R}\langle a\rangle$ with $V_\Delta = W_\Delta \cap \mathcal{R}$. Now observe that $\gamma < w_\Delta(b) < 0$ for all negative $\gamma \in \Gamma_\Delta$, so we are back in Case 1 (with $V_\Delta,\ W_\Delta$ instead of $V,\ W$). Hence there exists an $r \in \mathcal{R}$ such that $w_\Delta(a-r) \notin \Gamma_{v_\Delta}$, and thus $w(a-r) \notin \Gamma_v$. □

Remark. The same proof establishes the Valuation Property in a more general setting: the assumption that $T = \mathrm{Th}\big(\widetilde{\mathbb{R}}\big)$ with polynomially bounded $\widetilde{\mathbb{R}}$ can be replaced by the weaker one that T is a (complete) power bounded o-minimal theory extending the theory of ordered real closed fields; "convex subring" has to be replaced then everywhere by "T-convex subring". In this setting the Valuation Property was stated as an open problem in [6, 7.4], to which we also refer the reader for information about previously known special cases of the Valuation Property and some of its interesting consequences (not needed here). It is also shown there that the Valuation Property fails if T is not power bounded.

Proof of Theorem B. We can now almost copy the proof in [9, Section 4] to establish Theorem B on $\big(\widetilde{\mathbb{R}}, \exp\big)$ from the Introduction. The only changes are the following. The role of T_{an} is taken over by T, and the value groups that occur should now be considered as ordered K-linear spaces instead of ordered $\mathbb{Q}$-linear spaces. The Valuation Property is needed to make Lemma 4.2 there go through here. To make Lemmas 4.3 and 4.4 there go through here, we also use that if $(\mathcal{R}, V)$ and $(\mathcal{R}\langle a\rangle, W)$ are as at the beginning of this section, and $b \in \mathcal{R}\langle a\rangle$ is any element with $w(b) \notin \Gamma_v$, then $\Gamma_w = \Gamma_v \oplus Kw(b)$; this fact is established in [6, Section 5]. □

The first consequence of Theorem B that we mentioned in the Introduction, namely that functions that are 0-definable in $\big(\widetilde{\mathbb{R}}, \exp\big)$ are piecewise given by terms in $L(\exp, \log)$, now follows exactly as in [9, Corollaries 4.7 and 2.15]. (The same holds for *definable* functions and terms in that language *with real parameters*.) The second consequence of Theorem B, the o-minimality of $(\widetilde{\mathbb{R}}, \exp)$, follows as in [9, Section 5]. The third consequence, the exponential boundedness, requires more work and will be done below.

From Theorems A and B and the corollary in Example 8.1, we get immediately the following consequence. Define $\Gamma_{>0} : \mathbb{R} \longrightarrow \mathbb{R}$ by $\Gamma_{>0}(x) := \Gamma(x)$ if $x > 0$ and $\Gamma_{>0}(x) := 0$ otherwise.

9.3. Corollary. *The expansion $(\mathbb{R}_{\mathcal{G}}, \exp)$ of the field of real numbers is model complete and o-minimal. In particular, the expansion $(\mathbb{R}, <, 0, 1, +, -, \cdot, \Gamma_{>0})$ of the ordered real field is o-minimal.* □

In a similar way, Theorem B together with some remarks made in the Introduction of [12] imply a corresponding result for the Riemann zeta Function ζ. Define $\zeta_{>1} : \mathbb{R} \longrightarrow \mathbb{R}$ by $\zeta_{>1}(x) := \zeta(x)$ if $x > 1$ and $\zeta(x) := 0$ otherwise.

9.4. Corollary. *The expansion $(\mathbb{R}_{an^*}, \exp)$ of the field of real numbers is model complete and o-minimal. In particular, the expansion $(\mathbb{R}, <, 0, 1, +, -, \cdot, \zeta_{>1})$ of the ordered real field is o-minimal.* □

For the rest of this section we assume that the hypotheses of Theorem B hold, in particular that $\exp|[0, 1]$ is definable in $\widetilde{\mathbb{R}}$.

9.5. Proposition. *For each function $f : \mathbb{R} \to \mathbb{R}$ that is definable in $(\widetilde{\mathbb{R}}, \exp)$ there exists an n such that $|f(x)| \le \exp_n(x)$ for all sufficiently large x.*

Proof. Let H be the Hardy field of germs at $+\infty$ of functions $f : \mathbb{R} \to \mathbb{R}$ that are definable in $\widetilde{\mathbb{R}}$, and let H^* be the Hardy field of germs at $+\infty$ of functions $f : \mathbb{R} \to \mathbb{R}$ definable in $(\widetilde{\mathbb{R}}, \exp)$. Let $\mathrm{LE}(H)$ be the smallest subfield of H^* that contains H and contains $\exp(f)$ whenever it contains f, and $\log(f)$ whenever it contains f and $f > 0$. Then $\mathrm{LE}(H)$ is a Hardy field and is differentially algebraic over H, and the value group of H is archimedean (in fact, isomorphic to K, the field of exponents). Hence by [20] there exists for each $g \in \mathrm{LE}(H)$ an n such that $|g| \le \exp_n(x)$. (Here x denotes the germ at $+\infty$ of the identity function on $\mathbb{R}$.) Therefore it suffices to show that $\mathrm{LE}(H)$ is cofinal in H^*. By Consequence (1) of Theorem B the field H^* consists exactly of the germs at $+\infty$ of the functions $\mathbb{R} \to \mathbb{R}$ defined by some one-variable $L(\exp, \log)$-term with parameters from $\mathbb{R}$. Also, by [9, Section 5] the Hardy field H^* is an $\widetilde{\mathbb{R}}$-field, and as such is an elementary extension of $\widetilde{\mathbb{R}}$. Hence the desired result follows by an elementary chain argument from the two claims below.

Claim 1. *Let F be a Hardy field, $H \subseteq F \subseteq H^*$, and suppose that for all $f \in F$ there is a $g \in \mathrm{LE}(H)$ such that $|f| \le |g|$. Let $a \in F$, $a > 0$. Then for all $f \in F(\exp(a))$ and for all $f \in F(\log(a))$ there exists a $g \in \mathrm{LE}(H)$ such that $|f| \le |g|$.*

Proof. Replacing F by its real closure in H^* if necessary, we may assume that F is real closed. Clearly for all $f \in F[\exp(a)]$ and for all $f \in F[\log(a)]$ there exists a $g \in \mathrm{LE}(H)$ with $|f| \le |g|$. For $f \in H^*$, write $f \sim 0$ if $|f| < |g|$ for all $g \in \mathrm{LE}(H) \setminus \{0\}$. It is now enough to show that for no $f \in F[\exp(a)] \setminus \{0\}$ we have $f \sim 0$, and similarly for no $f \in F[\log(a)] \setminus \{0\}$.

Suppose on the contrary that there exists an $f \in F[\exp(a)] \setminus \{0\}$ with $f \sim 0$. It follows that $\exp(a) \notin F$. Take a polynomial $p(T) \in F[T] \setminus \{0\}$ of minimal degree d such that

$p(\exp(a)) \sim 0$. We now apply the well-known "Hardy trick": $p(0) \neq 0$ since $p(T)$ is of minimal degree, so after dividing by $p(0)$ we may assume that

$$p(T) = 1 + c_1 T + \cdots + c_d T^d \quad \text{with } c_1, \ldots, c_d \in F.$$

Taking the derivative of $p(\exp(a)) \sim 0$ gives

$$(c_1' + c_1 a') \exp(a) + \cdots + (c_d' + d c_d a')(\exp(a))^d \sim 0.$$

Dividing by $\exp(a)$ gives $q(\exp(a)) \sim 0$, where

$$q(T) = (c_1' + c_1 a') + \cdots + (c_d' + d c_d a') T^{d-1} \in F[T].$$

But $q(T)$ is of degree $d-1$ in T: $c_d' + d c_d a' \neq 0$, because otherwise $c_d'/c_d = (\log|c_d|)' = (-da)$, so that

$$-da = \log|c_d| + \text{some real constant},$$

and hence

$$\exp(-da) = (\text{some positive real constant}) \cdot |c_d|,$$

and thus $\exp(a) \in F$, a contradiction. This contradicts the minimality property of d.

For $\log(a)$ we proceed in the same way, except that, instead of the constant term, we make the leading coefficient equal to 1 before taking the derivative. This finishes the proof of Claim 1.

Claim 2. *Let $\mathcal{R} \prec \mathcal{R}\langle b\rangle \models T$. Then the field $\mathcal{R}(b)$ is cofinal in $\mathcal{R}\langle b\rangle$.*

Proof. This follows easily from the Valuation Property 9.2, but here is a more elementary argument. Let V be the convex hull of $\mathcal{R}(b)$ in $\mathcal{R}\langle b\rangle$. Then V is a convex valuation ring of $\mathcal{R}\langle b\rangle$. The canonical map $x \mapsto \overline{x} : V \to \overline{V}$ sends $\mathcal{R}(b)$ isomorphically onto the subfield $\overline{\mathcal{R}}\left(\overline{b}\right)$ of $\overline{V}$. Note that $\overline{V}$ is naturally a model of $\widetilde{T}$ with $\overline{\mathcal{R}} \prec \overline{V}$, and $\overline{\mathcal{R}}\left(\overline{b}\right) \subseteq \overline{\mathcal{R}}\left\langle\overline{b}\right\rangle \subseteq \overline{V}$. The elements b and $\overline{b}$ realize cuts in $\mathcal{R}$ and $\overline{\mathcal{R}}$ that correspond under the isomorphism $\mathcal{R} \to \overline{\mathcal{R}}$. Hence this isomorphism extends to an isomorphism $\mathcal{R}\langle b\rangle \to \overline{\mathcal{R}}\left\langle\overline{b}\right\rangle$ (of models of $\widetilde{T}$) that sends b to $\overline{b}$. Since $\mathcal{R}(b)$ is cofinal in V, so is $\overline{\mathcal{R}}\left(\overline{b}\right)$ in $\overline{V}$, and hence in $\overline{\mathcal{R}}\left\langle\overline{b}\right\rangle$. Thus $\mathcal{R}(b)$ is cofinal in $\mathcal{R}\langle b\rangle$. $\square$

10. Power Series Models

There is a natural way to expand the power series field $\mathbb{R}((t^\Gamma))$ to an $L_\mathcal{G}$-structure, for any divisible ordered abelian group Γ. We indicate in this section how to do this. We would like to show that these expansions are elementary extensions of $\mathbb{R}_\mathcal{G}$, as was done with $\mathbb{R}_{\text{an}}$ instead of $\mathbb{R}_\mathcal{G}$ (and L_{an} instead of $L_\mathcal{G}$) in [9, Section 3]. However, the argument in [9] breaks down for $\mathbb{R}_\mathcal{G}$, because we lack a simple axiomatization of its elementary theory. Thus we have only been able to obtain the desired result when Γ is subject to certain restrictions (being archimedean, for instance). We proceed rather indirectly, by first establishing some facts about arbitrary polynomially bounded o-minimal expansions of the real field. These are then used in studying the natural $L_\mathcal{G}$-expansions of the power series fields $\mathbb{R}((t^\Gamma))$.

At the end of this section we show that our partial results on power series models are nevertheless sufficient to extend some theorems from [8] about $(\mathbb{R}_{\text{an}}, \exp)$ to $(\mathbb{R}_\mathcal{G}, \exp)$.

We keep the notations and conventions of the previous section. In particular $\mathcal{R} \prec \mathcal{R}\langle a\rangle \models T$, and we are given convex subrings V and W of $\mathcal{R}$ and $\mathcal{R}\langle a\rangle$ with $W \cap \mathcal{R} = V$. Here is an analogue for residue fields of the Valuation Property: residue field extension is witnessed by an affine transform of the generator a.

10.1. Proposition (Residue Property). *Suppose the residue field $\overline{V}$ is isomorphic to the field of real numbers, and let $\overline{V} \neq \overline{W}$. Then a has an affine transform $a' = ra + s$ with $r, s \in \mathcal{R}$, $r \neq 0$, such that $a' \in W$ and $\overline{a'} \notin \overline{V}$.*

Proof. Note that $\Gamma_v = \Gamma_w$ by [6, Section 5]. Suppose first that we are in the trivial case that $V = \mathcal{R}$. Then $\Gamma_v = \Gamma_w = \{0\}$, so $W = \mathcal{R}\langle a\rangle$ and $a' = a$ has itself the desired property. So from now on we may assume that $V \neq \mathcal{R}$. Since $\overline{V}$ is isomorphic to the real field and $\overline{V} \neq \overline{W}$, there exists a $b \in W$ with $\overline{b} > \overline{V}$, so $V < b < |\mathcal{R} \setminus V|$. In other words, we are now in a similar situation as in Case 1 of the proof of the valuation property. We make the same reductions as in that proof, also introducing f, p, q, $\overline{f}$, λ, and c. We also make the same case distinctions:

Subcase (i): $\lambda \neq 0$. The same argument as in the proof of the Valuation Property gives $\left|f(b) - cb^\lambda\right| \leq \left|cb^\lambda\right|/2$. If $\lambda > 0$, then this implies $f(b) = a \in W$ and $|\overline{a}| > \overline{V}$, so $a' = a$ has the desired property. If $\lambda < 0$, then we also obtain $a \in W$ and $0 < |a| < r'$ for all positive $r' \in \mathcal{R}'$, and again $a' = a$ has the desired property.

Subcase (ii): $\lambda = 0$. We reduce this to Subcase (i) as in the proof of the Valuation Property, after appealing to (9.3) to first exclude the possibility that $\overline{f}(x) = c$ for all sufficiently large $x \in \mathcal{R}'$. ☐

10.2. Lemma. *Let Γ be an ordered K-linear space and let $\widetilde{\mathbb{R}} \preceq \mathcal{R}$ with $R \subseteq \mathbb{R}((t^\Gamma))$, where R denotes the underlying ordered field of $\mathcal{R}$. Then the ordered field $\mathbb{R}((t^\Gamma))$ can be expanded to a model of T in such a way that $\mathcal{R} \preceq \mathbb{R}((t^\Gamma))$.*

Proof. We may assume that $R \neq \mathbb{R}((t^\Gamma))$.

Claim. *There exists a model $\mathcal{S}$ of T such that $\mathcal{S}$ is a proper elementary extension of $\mathcal{R}$ and $R \subseteq S \subseteq \mathbb{R}((t^\Gamma))$, where S denotes the underlying ordered field of $\mathcal{S}$.*

Once this claim is established we are done by an application of Zorn's Lemma. Let $v : \mathbb{R}((t^\Gamma))^\times \to \Gamma$ be the usual valuation. To establish our claim we distinguish two cases.

Case 1: $v(R^\times) \neq \Gamma$. Then we take some element $\gamma \in \Gamma \setminus v(R^\times)$ and consider an elementary extension $\mathcal{R}\langle a\rangle$ of $\mathcal{R}$ such that a and t^γ realize the same cut in the ordered field R. We now have an isomorphism $i : R(a) \to R(t^\gamma)$ of ordered fields such that i is the identity on R and $i(a) = t^\gamma$. The idea is now to extend i to a valued field embedding $\mathcal{R}\langle a\rangle \to \mathbb{R}((t^\Gamma))$ for a suitable valuation on $\mathcal{R}\langle a\rangle$, and let $\mathcal{S}$ be the "image" of $\mathcal{R}\langle a\rangle$ under this embedding.

Let V be the (convex) valuation ring of $v\big|R^\times$. We first extend $v\big|R^\times$ to a valuation $v_a : R(a)^\times \to v(R^\times) \oplus \mathbb{Z}\gamma \subseteq \Gamma$ by setting $v_a(a) := \gamma$. Then i is also an isomorphism from the valued field $(R(a), v_a)$ onto the valued field $\big(R(t^\gamma), v\big|R(t^\gamma)^\times\big)$. In particular, the valuation ring V_a of v_a is a convex subring of $R(a)$. Let W be the convex hull of V_a in the underlying ordered field $R\langle a\rangle$ of $\mathcal{R}\langle a\rangle$, and let w be a valuation on this underlying field that extends v_a and has valuation ring W. So W is a convex subring of $R\langle a\rangle$ and $W \cap R = V$. Since $w(a) = \gamma \notin v(R^\times)$ we have $w(R\langle a\rangle^\times) = v(R^\times) \oplus K\gamma$ (internal direct sum) and $\overline{W} = \overline{V} \cong \mathbb{R}$ by [6, Section 5]. Hence we have an embedding $j : w(R\langle a\rangle^\times) \to \Gamma$ of ordered K-linear spaces such that j is the identity on the common subspace $v(R^\times)$ and $j(\gamma) = \gamma$. The universal property of the valued field $\mathbb{R}((t^\Gamma))$ implies that i extends to a valued field embedding $h : (R\langle a\rangle, w) \to (\mathbb{R}((t^\Gamma)), v)$. We now expand the ordered subfield $h(R\langle a\rangle)$ of $\mathbb{R}((t^\Gamma))$ to a model $\mathcal{S}$ of T in such a way that h is an isomorphism from $\mathcal{R}\langle a\rangle$ onto $\mathcal{S}$. Then $\mathcal{S}$ has the property of our claim.

Case 2. $v(R^\times) = \Gamma$. Then we take any element $b \in \mathbb{R}((t^\Gamma)) \setminus R$ and take an elementary extension $\mathcal{R}\langle a\rangle$ of $\mathcal{R}$ such that a and b realize the same cut in the ordered field R. As in Case 1 we extend $v\big|R^\times$ to a valuation v_a on $R(a)$ such that we have an ordered and valued field isomorphism $i : (R(a), v_a) \to (R(b), v\big|R(b)^\times)$ with $i\big|R = id_R$ and $i(a) = b$. Since $(\mathbb{R}((t^\Gamma)), v)$ is an immediate extension of $(R, v\big|R^\times)$, so is $(R(b), v\big|R(b)^\times)$, and hence $(R(a), v_a)$ is an immediate extension of $(R, v\big|R^\times)$. As in Case 1 we extend v_a further to a valuation w on the underlying ordered field $R\langle a\rangle$ of $\mathcal{R}\langle a\rangle$ whose valuation ring W is the convex hull of V_a in $R\langle a\rangle$. The previous remark, the Valuation Property and the Residue Property imply that $(R\langle a\rangle, w)$ is an immediate extension of $(R, v\big|R^\times)$. Hence i can be extended to a valued field embedding $h : (R\langle a\rangle, w) \to (\mathbb{R}((t^\Gamma)), v)$. Just as in Case 1 this produces an $\mathcal{S}$ as required. □

Note the following special case of this lemma:

10.3. Corollary. *If Γ is an ordered vector space over K, then the ordered field $\mathbb{R}((t^\Gamma))$ can be expanded to a model of T that contains $\widetilde{\mathbb{R}}$ as elementary submodel.* □

We now turn our attention to the expansion $\mathbb{R}_{\mathcal{G}}$. Below we let Γ denote a divisible ordered abelian group, or equivalently, an ordered vector space over $\mathbb{Q}$. We are going to expand the power series field $K := \mathbb{R}((t^\Gamma))$ to a structure for the language $L_{\mathcal{G}}$. Let μ be the set of infinitesimals of K. For an ordinary formal power series $F(Z) \in \mathbb{R}[\![Z]\!]$, where $Z = (Z_1, \dots, Z_N)$, and for $\epsilon = (\epsilon_1, \dots, \epsilon_N) \in \mu^N$, we define the element $F(\epsilon) \in K$ as usual.

Consider a power series $F(X) \in \mathbb{R}\{X\}_{\mathcal{G},1}$ where $X = (X_1, \dots, X_m)$. As in [9, Section 2] we take F as an m-ary function symbol of the language $L_{\mathcal{G}}$. We expand the ordered field K to an $L_{\mathcal{G}}$-structure $K_{\mathcal{G}} = \mathbb{R}((t^\Gamma))_{\mathcal{G}}$ by interpreting this function symbol F as the following function $\widetilde{F} : K^m \to K$:

Let $x = (x_1, \dots, x_m) \in K^m$. If $0 \le x_i \le 1$ for all i, we take the unique point $a = (a_1, \dots, a_m) \in \mathbb{R}^m$ such that $x = a + \epsilon$ for some infinitesimal $\epsilon \in \mu^m$; then $0 \le a_i \le 1$ for all i, so that we have by Corollary 4.9 a well defined power series

$$F_a(X) := \sum_{\alpha \in \mathbb{N}^m} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} F}{\partial X^\alpha}(a) X^\alpha \in \mathbb{R}\{X\}_{\mathcal{G}},$$

and we put $\widetilde{F}(x) := F_a(\epsilon)$. If $x_i < 0$ or $x_i > 1$ for some i, we put $\widetilde{F}(x) := 0$.

Note that the $L_{\mathcal{G}}$-structure $\mathbb{R}((t^\Gamma))_{\mathcal{G}}$ contains $\mathbb{R}_{\mathcal{G}}$ as a substructure.

10.4. Proposition. *Suppose Γ is archimedean. Then $\mathbb{R}_{\mathcal{G}} \preceq \mathbb{R}((t^\Gamma))_{\mathcal{G}}$.*

Proof. By the previous lemma we already know that $K := \mathbb{R}((t^\Gamma))$ can be expanded to an $L_{\mathcal{G}}$-structure $K^\sharp$ such that $\mathbb{R}_{\mathcal{G}} \preceq K^\sharp$. We claim that then $K^\sharp = K_{\mathcal{G}}$. Let $F(X) \in \mathbb{R}\{X\}_{\mathcal{G},1}$, considered as an m-ary function symbol of $L_{\mathcal{G}}$, interpreted as the function $F^\sharp$ in $K^\sharp$ and as the function $\widetilde{F}$ in $K_{\mathcal{G}}$. We only have to show that then $F^\sharp = \widetilde{F}$ as functions on K^m.

Let us just verify that $F^\sharp(\epsilon) = \widetilde{F}(\epsilon)$ for $\epsilon \in \mu^m$ with each $\epsilon_i \geq 0$. (The general case is quite similar but notationally awkward to spell out.) For each $n \in \mathbb{N}$ we can write

$$F(X) = P_n(X) + \sum_{|\alpha|=n+1} X^\alpha H_\alpha(X),$$

where $P_n(X)$ is a real polynomial in X of degree $\leq n$, and $H_\alpha(X) \in \mathbb{R}\{X\}_{\mathcal{G}}$ for $\alpha \in \mathbb{N}^m$ with $|\alpha| = n+1$. Take $\gamma > 0$ in Γ such that $\gamma \leq v(\epsilon_i)$ for all $i = 1, \dots, m$. Then $F^\sharp(\epsilon) = P_n(\epsilon) + a_n$ and $\widetilde{F}(\epsilon) = P_n(\epsilon) + b_n$, where $v(a_n) > n\gamma$ and $v(b_n) > n\gamma$. Hence the sequence $(P_n(\epsilon))_n$ converges in the valuation topology both to $F^\sharp(\epsilon)$ and to $\widetilde{F}(\epsilon)$. Therefore $F^\sharp(\epsilon) = \widetilde{F}(\epsilon)$. $\square$

A similar argument will give the following more general result:

10.5. Proposition. *Suppose Δ is a convex $\mathbb{Q}$-linear subspace of Γ and $\mathbb{R}_{\mathcal{G}} \preceq \mathbb{R}((t^\Delta))_{\mathcal{G}}$, and Γ/Δ is archimedean. Then $\mathbb{R}_{\mathcal{G}} \preceq \mathbb{R}((t^\Gamma))_{\mathcal{G}}$.*

Proof. Put $\mathbf{k} := \mathbb{R}((t^\Delta))$ and $K = \mathbb{R}((t^\Gamma))$. Since $\mathbb{R}_{\mathcal{G}} \preceq \mathbf{k}_{\mathcal{G}}$ by assumption, we know from Lemma 10.2 that K can be expanded to an $L_{\mathcal{G}}$-structure $K^\sharp$ such that $\mathbf{k}_{\mathcal{G}} \preceq K^\sharp$. We claim that then $K^\sharp = K_{\mathcal{G}}$. As in the last proof we consider a power series $F(X) \in \mathbb{R}\{X\}_{\mathcal{G},1}$ and view it as an m-ary function symbol interpreted as the function $F^\sharp$ in $K^\sharp$ and as the function $\widetilde{F}$ in $K_{\mathcal{G}}$. Let $a \in K^m$ with $0 \leq a_i \leq 1$ for $i = 1, \dots, m$. We have to show that then $F^\sharp(a) = \widetilde{F}(a)$. To simplify notation we assume that each a_i is infinitesimal. (The argument in this case works essentially also in general, after an initial reduction.) Write $a_i = b_i + \epsilon_i$ with $b_i \in \mathbf{k}$ and ϵ_i a $\mathbf{k}$-infinitesimal, and put $b = (b_1, \dots, b_m)$ and $\epsilon = (\epsilon_1, \dots, \epsilon_m)$. Note that $0 \leq b_i \leq 1$, but it can happen that $\epsilon_i < 0$ if $b_i > 0$. Take some $\gamma \in \Gamma$ such that $\gamma > \Delta$ and $v(\epsilon_i) \geq \gamma$ for all i. Let $U = (U_1, \dots, U_m)$ be a tuple of m distinct indeterminates (not among the X_i). For each n we have a formal power series identity in $\mathbb{R}[\![X, U]\!]$:

$$F(X+U) = \sum_{|\alpha| \leq n} \frac{1}{\alpha!} F_\alpha(X) U^\alpha + \text{ terms of degree } > n \text{ in } U,$$

where $F_\alpha(X) = \partial^{|\alpha|} F / \partial X^\alpha$. Let $P_n(X, U)$ be the finite sum on the right hand side of the identity. Then $\widetilde{F}(a) = \widetilde{F}(b + \epsilon) = P_n(b, \epsilon) + c_n$ with $c_n \in K$ and $v(c_n) > n\gamma$. Since Γ/Δ is archimedean, the sequence $(n\gamma)_{n \in \mathbb{N}}$ is cofinal in Γ. Hence the sequence $(P_n(b, \epsilon))_n$ converges to $\widetilde{F}(a)$ in the valuation topology of K.

For $F^\sharp(a)$ we argue as follows: the (unique) function $f \in \mathcal{G}_{m,0;1}$ with $\widehat{f} = F(X)$ is C^∞ on I^m, and its partials f^α for $\alpha \in \mathbb{N}^m$ satisfy $\widehat{f^\alpha} = F_\alpha(X) \in \mathbb{R}\{X\}_{\mathcal{G},1}$. Hence by "Taylor expansion with remainder" there is for each $n \in \mathbb{N}$ a positive real constant C_n such that

$$|F(x+u) - P_n(x, u)| \leq C_n \cdot |u|^{n+1}$$

for all $x \in [0, 1/2]^m \subset \mathbb{R}^m$ and all $u \in \mathbb{R}^m$ with $|u_i| \leq x_i$ for each $i = 1, \dots, m$. Since $\mathbb{R}_{\mathcal{G}} \preceq K^\sharp$, this implies that $F^\sharp(a) = F^\sharp(b+\epsilon) = P_n(b, \epsilon) + d_n$, where $d_n \in K$ with $v(d_n) > n\gamma$. Hence the sequence $(P_n(b, \epsilon))_n$ converges also to $F^\sharp(a)$ in the valuation topology of K. Thus $F^\sharp(a) = \widetilde{F}(a)$. $\square$

Repeated use of the last proposition gives the following result.

10.6. Corollary. *If* $\dim_{\mathbb{Q}}(\Gamma) < \infty$, *then* $\mathbb{R}_{\mathcal{G}} \preceq \mathbb{R}((t^\Gamma))_{\mathcal{G}}$. $\square$

For arbitrary Γ we define $\mathbb{R}((t^\Gamma))_{\mathrm{fd}}$ to be the set of elements of $\mathbb{R}((t^\Gamma))$ whose support is contained in some finite-dimensional $\mathbb{Q}$-linear subspace of Γ. (The subscript "fd" stands here for "finite-dimensional".) Note that $\mathbb{R}((t^\Gamma))_{\mathrm{fd}}$ is a subfield of $\mathbb{R}((t^\Gamma))$; in fact, it is the underlying field of an $L_{\mathcal{G}}$-substructure of $\mathbb{R}((t^\Gamma))_{\mathcal{G}}$ which we shall denote by $\mathbb{R}((t^\Gamma))_{\mathrm{fd},\mathcal{G}}$. Then the corollary above is easily seen to imply the following.

10.7. Corollary. $\mathbb{R}_{\mathcal{G}} \preceq \mathbb{R}((t^\Gamma))_{\mathrm{fd},\mathcal{G}}$. $\square$

It would be nice to know if $\mathbb{R}((t^\Gamma))_{\mathcal{G}}$ is always an elementary extension of $\mathbb{R}_{\mathcal{G}}$. We know of one more result in this direction, which is as follows.

10.8. Proposition. *Let β be an ordinal, $\beta \neq 0$, and $(\Gamma_\alpha)_{\alpha<\beta}$ a family of $\mathbb{Q}$-linear subspaces of Γ such that $\mathbb{R}_{\mathcal{G}} \preceq \mathbb{R}((t^{\Gamma_\alpha}))_{\mathcal{G}}$ for all $\alpha < \beta$ and Γ_{α_1} is a convex subspace of Γ_{α_2} whenever $\alpha_1 < \alpha_2 < \beta$. Assume also that $\Gamma = \bigcup_\alpha \Gamma_\alpha$. Then $\mathbb{R}_{\mathcal{G}} \preceq \mathbb{R}((t^\Gamma))_{\mathcal{G}}$.*

Proof. Note that $\mathbb{R}_{\mathcal{G}} \preceq \bigcup_\alpha \mathbb{R}((t^{\Gamma_\alpha}))_{\mathcal{G}}$. By Lemma 10.2 we know that then $K = \mathbb{R}((t^\Gamma))$ can be expanded to an $L_{\mathcal{G}}$-structure $K^\sharp$ that elementarily extends $\bigcup_\alpha \mathbb{R}((t^{\Gamma_\alpha}))_{\mathcal{G}}$. It suffices to show that that then $K^\sharp = K_{\mathcal{G}}$. This follows along the lines of the prooofs of the two propositions above, using the fact that $\bigcup_\alpha \mathbb{R}((t^{\Gamma_\alpha}))$ is dense in K in the valuation topology. $\square$

We now apply these results to the field $\mathbb{R}((t))^{\mathrm{LE}}$ of **logarithmic-exponential** series. The construction of this ordered exponential field (see [8, Section 2] for details) consists of two parts, which we now briefly review.

In the first part one obtains the subfield $\mathbb{R}((t))^{\mathrm{E}}$ of $\mathbb{R}((t))^{\mathrm{LE}}$ consisting of the **exponential** series as a union $\bigcup_n K_n$, where $K_n = \mathbb{R}((t^{\Gamma_n}))$ and $(\Gamma_n)_{n\in\mathbb{N}}$ is a certain increasing sequence of divisible ordered abelian groups. Each Γ_n is actually also an additive subgroup of the ordered field K_{n-1} (with the induced ordering) and contains Γ_{n-1} as a convex subgroup. For $n = 0$ we make this true by setting $K_{-1} := \mathbb{R}$ (the *field* of real numbers), $\Gamma_0 := \mathbb{R}$ (the ordered *additive group* of real numbers), and $\Gamma_{-1} = 0$. Assuming inductively that Γ_n is an additive subgroup of K_{n-1} and Γ_{n-1} is a convex subgroup of Γ_n, we obtain Γ_{n+1} as follows: put $J_n = \{z \in K_n : \operatorname{supp}(z) < \Gamma_{n-1}\}$, an additive subgroup of K_n, and put $\Gamma_{n+1} := J_n \oplus \Gamma_n$, ordered as an additive subgroup of K_n . (This makes sense, since $\Gamma_n \subseteq K_{n-1} \subseteq K_n$ with $J_n \cap \Gamma_n = 0$. It is also easy to see that Γ_n is convex in Γ_{n+1}.) We refer to [8, Section 2] for the definition of the maps $E_n : K_n \longrightarrow K_{n+1}$, which give rise to the exponential map $E = \bigcup_n E_n$ on $\mathbb{R}((t))^{\mathrm{E}} = \bigcup_n K_n$.

In the second part of the construction of $\mathbb{R}((t))^{\mathrm{LE}}$ we "close under logarithms": this consists of building an increasing chain $L_0 \subset L_1 \subset L_2 \subset \cdots$ of ordered exponential fields, where each L_i comes equipped with a specific isomorphism $\eta_i : L_i \longrightarrow \mathbb{R}((t))^{\mathrm{E}}$ of ordered exponential fields, and where η_0 is the identity on $L_0 = \mathbb{R}((t))^{\mathrm{E}}$. The details of how this is done (see [8, Section 2]) guarantee that the positive elements of L_i have a logarithm in L_{i+1}. It turns out that then the ordered exponential field $\mathbb{R}((t))^{\mathrm{LE}} = \bigcup_{i\in\mathbb{N}} L_i$ is an elementary extension of the ordered exponential field of real numbers.

We now adapt this to our situation. It is not enough to restrict to the exponential series of "finite-dimensional support", that is, to replace $\mathbb{R}((t^{\Gamma_n}))$ in the construction above by

$\mathbb{R}((t^{\Gamma_n}))_{\mathrm{fd}}$. It will be necessary to impose the stronger restriction of "hereditarily finite-dimensional support". (This does not affect any potential applications we can think of.)

The subfield $\mathbb{R}((t))^{\mathrm{E,f}}$ of $\mathbb{R}((t))^{\mathrm{E}}$ consisting of the **exponential series of hereditarily finite-dimensional support** is again obtained as a union $\bigcup_n K_n^{\mathrm{f}}$ where $K_n^{\mathrm{f}} := \mathbb{R}((t^{\Gamma_n^{\mathrm{f}}}))_{\mathrm{fd}}$, and $(\Gamma_n^{\mathrm{f}})_{n\in\mathbb{N}}$ is a certain increasing sequence of divisible ordered abelian groups. Each Γ_n^{f} is actually a subgroup of Γ_n (with the induced ordering) as well as an additive subgroup of the ordered field K_{n-1}^{f} (also with the induced ordering) and contains $\Gamma_{n-1}^{\mathrm{f}}$ as a convex subgroup.

For $n = 0$ we make this true by setting $K_{-1}^{\mathrm{f}} := \mathbb{R}$, $\Gamma_0^{\mathrm{f}} := \mathbb{R}$, and $\Gamma_{-1}^{\mathrm{f}} = 0$. Assume inductively that

(1) Γ_n^{f} is an additive subgroup of K_{n-1}^{f},
(2) $\Gamma_n^{\mathrm{f}} \subseteq \Gamma_n$ as ordered groups, and $\Gamma_{n-1}^{\mathrm{f}} = \Gamma_{n-1} \cap \Gamma_n^{\mathrm{f}}$. (In particular, $\Gamma_{n-1}^{\mathrm{f}}$ is a convex subgroup of Γ_n^{f}.)

Then we define $\Gamma_{n+1}^{\mathrm{f}}$ as follows: put $J_n^{\mathrm{f}} = \left\{z \in K_n^{\mathrm{f}} : \operatorname{supp} z < \Gamma_{n-1}^{\mathrm{f}}\right\}$, an additive subgroup of K_n^{f}, and put $\Gamma_{n+1}^{\mathrm{f}} := J_n^{\mathrm{f}} \oplus \Gamma_n^{\mathrm{f}}$, ordered as an additive subgroup of K_n^{f}. (This makes sense, since $\Gamma_n^{\mathrm{f}} \subseteq K_{n-1}^{\mathrm{f}} \subseteq K_n^{\mathrm{f}}$ and $J_n^{\mathrm{f}} \cap \Gamma_n^{\mathrm{f}} = 0$.) It is easy to see that (1) and (2) above remain true when n is replaced by $n+1$. This finishes the inductive construction of the sequences (Γ_n^{f}) and (K_n^{f}). It is also clear from this construction that $E(K_n^{\mathrm{f}}) \subset K_{n+1}^{\mathrm{f}}$ for all n, and we may therefore consider $\mathbb{R}((t))^{\mathrm{E,f}}$ as an ordered exponential subfield of $\mathbb{R}((t))^{\mathrm{E}}$. Moreover,

$$\mathbb{R}((t))_{\mathcal{G}}^{\mathrm{E,f}} := \bigcup_n \left(K_n^{\mathrm{f}}\right)_{\mathcal{G}}$$

is clearly an elementary extension of $\mathbb{R}_{\mathcal{G}}$, by Corollary 10.7.

It is easy to check that $\Phi(\mathbb{R}((t))^{\mathrm{E,f}}) \subseteq \mathbb{R}((t))^{\mathrm{E,f}}$, where $\Phi : \mathbb{R}((t))^{\mathrm{E}} \to \mathbb{R}((t))^{\mathrm{E}}$ is the map defined in [8, 2.2]. Moreover, the proof of [8, Lemma 2.6] shows that each positive element of $\Phi(\mathbb{R}((t))^{\mathrm{E,f}})$ has its logarithm in $\mathbb{R}((t))^{\mathrm{E,f}}$. Because of these two facts we can now close under logarithms as before. Put

$$L_i^{\mathrm{f}} := \eta_i^{-1}\mathbb{R}((t))^{\mathrm{E,f}} .$$

Then L_i^{f} is an ordered exponential subfield of L_i, and each positive element of L_i^{f} has its logarithm in L_{i+1}^{f}. Next we expand the ordered field L_i^{f} to the $L_{\mathcal{G}}$-structure $(L_i^{\mathrm{f}})_{\mathcal{G}}$ so that η_i maps this expansion isomorphically onto the $L_{\mathcal{G}}$-structure $\mathbb{R}((t))_{\mathcal{G}}^{\mathrm{E,f}}$. Finally, let $\mathbb{R}((t))_{\mathcal{G}}^{\mathrm{LE,f}} := \bigcup (L_i^{\mathrm{f}})_{\mathcal{G}}$, an ordered exponential subfield of $\mathbb{R}((t))^{\mathrm{LE}}$ and at the same time an $L_{\mathcal{G}}$-structure. Then we have the following analogue of the corresponding result [8, Corollary 2.8] for $\mathbb{R}_{\mathrm{an}}$.

10.9. Proposition. *The* $(L_{\mathcal{G}}, \exp)$*-structure* $\mathbb{R}((t))_{\mathcal{G}}^{\mathrm{LE,f}}$ *is a model of* $\mathrm{Th}(\mathbb{R}_{\mathcal{G}}, \exp)$, *hence an elementary extension of* $(\mathbb{R}_{\mathcal{G}}, \exp)$.

Proof. It is routine to check that $\mathbb{R}((t))^{\mathrm{LE,f}}$ as an ordered exponential field satisfies the Ressayre axioms as given in the Introduction. The Proposition is then an immediate consequence of Theorem B and earlier results in this section. $\square$

This result provides an explicit example of a proper elementary extension of $(\mathbb{R}_{\mathcal{G}}, \exp)$. The Hardy field $\mathrm{H}(\mathbb{R}_{\mathcal{G}}, \exp)$ of the germs at $+\infty$ of the functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ definable in $(\mathbb{R}_{\mathcal{G}}, \exp)$ is also naturally a proper elementary extension of $(\mathbb{R}_{\mathcal{G}}, \exp)$; in fact, it is generated as such over the standard model $(\mathbb{R}_{\mathcal{G}}, \exp)$ by the germ of the identity function, see [9, Section

5]. Hence there is a unique elementary embedding $\mathrm{H}(\mathbb{R}_{\mathcal{G}}, \exp) \to \mathbb{R}((t))^{\mathrm{LE,f}}_{\mathcal{G}}$ of $(L_{\mathcal{G}}, \exp)$-structures that is the identity on $\mathbb{R}$ and sends the germ of the identity function to $x := 1/t$. One can view this embedding as providing a logarithmic-exponential series expansion at $+\infty$ for the one-variable functions definable in $(\mathbb{R}_{\mathcal{G}}, \exp)$.

Sections 3 and 5 (up to Corollary 5.5) of [8] now go through with the following changes. Replace $\mathbb{R}((t))^{\mathrm{E}}$ by $\mathbb{R}((t))^{\mathrm{E,f}}$ and $\mathbb{R}((t))^{\mathrm{LE}}$ by $\mathbb{R}((t))^{\mathrm{LE,f}}$ (and make all other corresponding changes as indicated above). Put $\mathcal{F}_{\mathcal{G}} := \bigcup_{m\in\mathbb{N}} \mathbb{R}\{X_1, \dots, X_m\}_{\mathcal{G}}$, and let $H_{\mathcal{G},\exp}$ be the smallest $\mathcal{F}_{\mathcal{G}}$-closed subfield of $\mathbb{R}((t))^{\mathrm{LE,f}}$ containing $\mathbb{R}(t)$ and closed under log and exp. Finally, in 5.1–5.5 of [8], replace "$\sum a_n X^n$ converges near 0" by "$\sum a_n X^n$ belongs to $\mathbb{R}\{X\}_{\mathcal{G}}$". (Note that Lemma 5.1 there goes through by Corollary 7.4 here.) Combining this with Lemma 8.5, we get the following.

10.10. Corollary. *Suppose $f : (0, \epsilon) \longrightarrow \mathbb{R}$ is definable in $(\mathbb{R}_{\mathcal{G}}, \exp)$ and has an asymptotic expansion $\sum a_n T^n$ at 0. Then $\sum a_n T^n$ belongs to $\mathbb{R}\{T\}_{\mathcal{G}}$. In particular, if $a_n \geq 0$ for all n, then $\sum a_n T^n$ is convergent.* □

Thus the series $\sum n! T^n$ is not the Taylor series at 0 of any C^∞ function $[0, \epsilon) \longrightarrow \mathbb{R}$, $\epsilon > 0$, that is definable in $(\mathbb{R}_{\mathcal{G}}, \exp)$.

Concluding Remarks

1. It is well known that $\mathbb{R}_{\mathrm{an}}$, the field of real numbers with restricted analytic functions, admits elimination of quantifiers in its natural language augmented by a symbol for the function $x \mapsto x^{-1}$ (with $0^{-1} := 0$ by convention); see [9]. We do not know if this is the case for $\mathbb{R}_{\mathcal{G}}$. We suspect it is not, even though $\mathbb{R}_{\mathcal{G}}$ is in many respects quite similar to $\mathbb{R}_{\mathrm{an}}$.

2. At about the same time (summer 1996) as we proved Theorem B, Wilkie [23] established a completely different method for obtaining the o-minimality of certain expansions of the real field. His result (together with the fact that $\mathbb{R}_{\mathcal{G}}$ is o-minimal and generated by total C^∞-primitives) implies for example that the expansion of $\mathbb{R}_{\mathcal{G}}$ by (suitably defined) Pfaffian functions is o-minimal. This gives in particular the o-minimality of $(\mathbb{R}_{\mathcal{G}}, \exp)$, which is also among our conclusions. However, the method of [23] does not (to our knowledge) provide the other results concerning $(\mathbb{R}_{\mathcal{G}}, \exp)$ in our paper: the quantifier elimination and axiomatization relative to $\mathbb{R}_{\mathcal{G}}$ and its various consequences such as exponential boundedness.

3. It seems likely that one can also obtain "analytic cell decomposition" (as defined in [11], see also [10]) for $\mathbb{R}_{\mathcal{G}}$ and $(\mathbb{R}_{\mathcal{G}}, \exp)$ along the lines of this paper.

4. Theorem A implies that any function $f : \mathbb{R} \longrightarrow \mathbb{R}$ definable in $\mathbb{R}_{\mathcal{G}}$ is analytic on $\mathbb{R} \setminus F$ for some finite set $F \subseteq \mathbb{R}$. We do not know whether this conclusion can be strengthened as follows: given a C^∞ function $f : \mathbb{R} \longrightarrow \mathbb{R}$ definable in $\mathbb{R}_{\mathcal{G}}$, is f necessarily analytic? An affirmative answer to this question would imply by [1] the following extension to several variables: if a C^∞ function $f : U \longrightarrow \mathbb{R}$, with $U \subseteq \mathbb{R}^n$ open, is definable in $\mathbb{R}_{\mathcal{G}}$, then f is analytic.

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