Divisors on varieties over valuation domains

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Abstract

Let $\mathcal{X}$ be a separated integral normal scheme of finite type over the valuation ring $\mathcal{O}_v$. It is shown that the set $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ of coherent fractionary $\mathcal{O}_{\mathcal{X}}$-ideals $\mathcal{J} \subseteq K(\mathcal{X})$ satisfying the relation $\mathcal{J} = \mathcal{J} := (\mathcal{O}_{\mathcal{X}} : (\mathcal{O}_{\mathcal{X}} : \mathcal{J}))$ - the so called divisorial $\mathcal{O}_{\mathcal{X}}$-ideals - forms a group with the composition law $(\mathcal{I}, \mathcal{J}) \mapsto \mathcal{I} \cdot \mathcal{J}$. This group posesses a natural embedding

$$\mathrm{div} : \mathcal{D}_{\mathrm{coh}}(\mathcal{X}) \to \mathrm{Div}(X) \oplus \underset{\mathrm{v}\in\mathrm{V}}{} \mathrm{v}(K(\mathcal{X})),$$

where $\mathrm{Div}(X)$ denotes the group of Weil divisors of the generic fibre X of $\mathcal{X}|\mathrm{Spec}(\mathcal{O}_v)$, and V is a set of valuations of $K(\mathcal{X})$ determined by a subset of the generic points of the fibres $\mathcal{X} \times_{\mathcal{O}_v} \kappa(\mathcal{P})$, $\mathcal{P} \in \mathrm{Spec}(\mathcal{O}_v) \setminus 0$. The image $\mathrm{Div}(\mathcal{X})$ of div is proved to satisfy $\mathrm{Div}(\mathcal{X}) = \mathrm{Div}(X) \oplus \mathrm{Ver}(\mathcal{X})$ with a subgroup $\mathrm{Ver}(\mathcal{X}) \subseteq \underset{\mathrm{v}\in\mathrm{V}}{} \mathrm{v}(K(\mathcal{X}))$. The structure of $\mathrm{Ver}(\mathcal{X})$ is determined provided that $\mathcal{X}$ satisfies additional conditions - for example if $\mathcal{X}$ is projective over $\mathrm{Spec}(\mathcal{O}_v)$.

These facts are deduced from general results on the semigroup $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ of coherent divisorial $\mathcal{O}_{\mathcal{X}}$-ideals on an integral scheme $\mathcal{X}$: A criterion for $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ to be a group based on the notion of so called Prüfer v-multiplication rings, and a valuation theoretic description of this group using valuations of $K(\mathcal{X})$ naturally associated to $\mathcal{X}$. The considerations leading to these results show that $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$ can be understood as an ideal theoretic generalization of the group of Weil divisors on a normal noetherian scheme.

Following this idea a criterion for $\mathcal{D}_{\mathrm{coh}}(\mathcal{X})$, $\mathcal{X}$ a separated integral normal $\mathcal{O}_v$-scheme of finite type, to be equal to the group of Cartier divisors on $\mathcal{X}$ is given. The criterion is obtained by showing that for any point x on such a scheme the local generalized Weil divisor groups $\mathcal{D}_{\mathrm{coh}}(\mathrm{Spec}(\mathcal{O}_{\mathcal{X},x}))$ exist and by analyzing the structure of these groups.