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Orderings, Real Places, and Valuations on Noncommutative Integral Domains

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This paper is a continuation of our recent joint work with K. H. Leung on the real spectrum of a noncommutative ring [15]. The object of the present paper is to develop a theory of real places and valuations to accompany the theory of orderings developed in [15]. Because of the existence of integral domains which are not embeddable in a skew field, it is necessary to deal directly with integral domains. Consequently, the theory of real places and valuations that we obtain is somewhat less precise than the corresponding theory for fields or skew fields.

Archimedean classes in ordered abelian groups were considered already by Hahn [11] and the connection between orderings and real places on fields was worked out already by Baer and Krull in [2, 12]. In the 1970s, inspired by Pfister's earlier work on signatures of quadratic forms [17], the connection between orderings, valuations, and quadratic forms on fields was firmly established by Becker, Bröcker, Brown, Prestel, and others; see [13]. In the 1980s, after the real spectrum of a ring was introduced by Coste and Roy, these ideas were applied to real algebraic geometry and real analytic geometry; see [3, 1].

Abstract real spectra, also called spaces of signs, were introduced just recently in [1, 5, 16] in an attempt to axiomatize parts of real algebraic geometry and real analytic geometry. In [15] it is shown, and perhaps this is a bit surprising (although there is some hint of it already in [14, Chap. 6; 8], for example), that the orderings on a noncommutative ring form an abstract real spectrum, exactly as in the commutative case. In the course of the proof, it is shown that if $\mathfrak{p}$ is a real prime of A, then $A/\mathfrak{p}$ is an integral domain and the orderings on A having support $\mathfrak{p}$ form a space of orderings.

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In the present paper, we define real places on a (not necessarily commutative) integral domain A and examine the relationship between real places on A and support $\{0\}$ orderings on A. We show that a version of the Baer–Krull theorem holds, and that the real places yield a natural P-structure on the space of support $\{0\}$ orderings. (See [16] for the meaning of this terminology.) More generally, we show, for an arbitrary noncommutative ring, that the real places on the various residue domains yield a natural P structure for the real spectrum of the ring. We also define valuations on an integral domain and prove a general version of Bröcker's trivialization theorem for fans; see [4]. Finally, as an example, we study orderings and real places on the twisted polynomial ring $\mathbb{R}[x, y]_t$ where the multiplication is defined by $yx = axy$, $a \in \mathbb{R}$, $a > 0$, $a \neq 1$, and we compute the stability index.

1. TERMINOLOGY AND NOTATION

Let A be a (not necessarily commutative) ring with 1. For any subset $S \subseteq A$, let $A^2(S)$ denote the set of all permuted products of elements $a_1, a_1, \dots, a_n, a_n, s_1, \dots, s_m$, for $a_1, \dots, a_n \in A$, $s_1, \dots, s_m \in S$, $n \geq 0$, $m \geq 0$, and let $\sum A^2(S)$ denote the set of all finite sums of elements of $A^2(S)$. We denote $\sum A^2(\{1\})$ by $\sum A^2$ for short. As in [15], a *preordering* in A is a subset $T \subseteq A$ such that $\sum A^2(T) \subseteq T$. $\sum A^2(S)$ is the smallest preordering in A containing S. In particular, $\sum A^2$ is the unique smallest preordering in A. A preordering T of A is said to be *proper* if $-1 \notin T$.

A subset P of A is called an *ordering* if $P + P \subseteq P$, $PP \subseteq P$, $P^- P = A$, and $P \cap -P$ is a prime ideal of A. The prime ideal $P \cap -P$ is called the *support* of P. Every proper preordering of A is contained in an ordering. In particular, A possesses an ordering iff $-1 \notin \sum A^2$. A prime $\mathfrak{p}$ in A is said to be *real* if it is the support of some ordering of A. A prime $\mathfrak{p}$ is the support of an ordering containing the preordering T iff $(T + \mathfrak{p}) \cap -(T + \mathfrak{p}) = \mathfrak{p}$. In particular, $\mathfrak{p}$ is real iff $(\sum A^2 + \mathfrak{p}) \cap -(\sum A^2 + \mathfrak{p}) = \mathfrak{p}$.

If $\mathfrak{p}$ is a real prime, then the factor ring $A/\mathfrak{p}$ is an integral domain and $\{0\}$ is a real prime of $A/\mathfrak{p}$. Also, the map $P \mapsto P/\mathfrak{p}$ defines a one-to-one correspondence between support $\mathfrak{p}$ orderings of A and support $\{0\}$ orderings of $A/\mathfrak{p}$. Thus, in studying support $\mathfrak{p}$ orderings on A, one is reduced immediately to the case where A is an integral domain and $\mathfrak{p} = \{0\}$.

Throughout the first several sections, we assume that A is a (not necessarily commutative) integral domain, and $\{0\}$ is a real prime of A, i.e., $(\sum A^2) \cap -(\sum A^2) = \{0\}$. A^* denotes the set of all nonzero elements of A, and X denotes the set of all support $\{0\}$ orderings on A. Each $a \in A^*$

gives rise to a mapping a: $X \to \{-1, 1\}$ via

$$\bar{a}(P) = \begin{cases} 1 & \text{if } a \in P \\ -1 & \text{if } a \in -P. \end{cases}$$

Let G denote the set of all such mappings, i.e., $G = \{\bar{a} \mid a \in A^*\}$. G is a subgroup of $\{-1, 1\}^X$, $\bar{a}\bar{b} = \overline{ab}$ for any $a, b \in A^*$, and (X, G) is a space of ordering; see [15]. The topology on X is the weakest such that the mappings $\bar{a} \in G$ are continuous (giving $\{-1, 1\}$ the discrete topology). In other words, $\{U(\bar{a}) \mid \bar{a} \in G\}$ is an open subbasis for X where $U(\bar{a}) = \{P \in X \mid \bar{a}(P) = 1\}$. X is a Boolean space, i.e., X is compact, Hausdorff, and totally disconnected; see [16, Theorem 2.1.5]. Given a preordering T on A with $T \cap -T = \{0\}$, we denote by X_T the set of all support $\{0\}$ orderings on A containing T and $G_T = G|_{X_T}$ the set of all restrictions of elements of G to X_T. (X_T, G_T) is a subspace of (X, G) (so is itself a space of orderings) and every subspace of (X, G) is of this form. Each ordering $P \in X_T$ gives rise to a character χ_P on G_T given by $\bar{a} \mapsto \bar{a}(P)$. The mapping $P \mapsto \chi_P$ identifies X_T with a subset of the character group of G_T and $\chi_P(-1) = -1$ holds for each $P \in X_T$. X_T (more precisely, (X_T, G_T)) is said to be a *fan* if every character χ: $G_T \to \{-1, 1\}$ satisfying $\chi(-1) = -1$ is of the form $\chi = \chi_P$ for some ordering $P \in X_T$. In concrete terms, this just means that any subset P of A satisfying $T \subseteq P$, $PP \subseteq P$, $P \cup -P = A$, $P \cap -P = \{0\}$ is an ordering (i.e., is closed under addition).

2. ORDERINGS, REAL PLACES, AND VALUATIONS

Let P be a support $\{0\}$ ordering on A and let $\leq$ be the associated total ordering on A, i.e., $a \leq b \Leftrightarrow b - a \in P$. For any $a \in A$, define $|a| = a$, if $a \in P$ and $|a| = -a$, if $a \in -P$. Since $(A, +, \leq)$ is an ordered abelian group we have a set valuation v: $A \to \Gamma \cup \{\infty\}$ induced by P. This is defined as follows. $a, b \in A$ are said to be *archimedean equivalent* if $|a| \leq m|b|$ and $|b| \leq n|a|$ for some positive integers m and n. $v(a)$ is just the archimedean equivalence class of a, $\infty = v(0)$, and $\Gamma = \{v(a) \mid a \in A^*\}$. $\Gamma \cup \{\infty\}$ is totally ordered by $v(a) \leq v(b)$ iff $n|a| \geq |b|$ for some positive integer n. All this works for any ordered abelian group $(A, +, \leq)$ [10, 11]. In our situation, because of the multiplication on A, there is a (not necessarily commutative) binary operation $+$ on Γ defined by $v(a) + v(b) = v(ab)$, and $(\Gamma, +)$ is a cancellation semigroup. That is, $+$ is associative, there is an identity element $0 = v(1)$, and if $\alpha + \gamma = \beta + \gamma$ or $\gamma + \alpha = \gamma + \beta$ then $\alpha = \beta$. This semigroup is ordered, i.e., $\leq$ is a total ordering on Γ and $\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma$ and $\gamma + \alpha \leq \gamma + \beta$. Thus v is a valuation of A in the following sense.

DEFINITION 2.1. A *valuation* on an integral domain A is a map v from A onto $\Gamma \cup \{\infty\}$, an ordered cancellation semigroup $\Gamma = (\Gamma, +, \leq)$ with ∞ adjoined, such that

(1) $v(a) = \infty \Leftrightarrow a = 0$.

(2) $v(ab) = v(a) + v(b)$.

(3) $v(a+b) \geq \min\{v(a), v(b)\}$.

If we drop the requirement that the ordered semigroup $(\Gamma, +, \leq)$ has the cancellation property then v is called a *weak valuation* on A.

Note. For any valuation or weak valuation v on A:

(1) $v(1) = 0$, $v(-1) = 0$, $v(-a) = v(a)$.

(2) If $v(a) \neq v(b)$ then $v(a+b) = \min\{v(a), v(b)\}$.

(3) For a skew field, valuations and weak valuations are the same thing and the value semigroup Γ is a group.

(4) We will need to consider weak valuations later, in Section 4.

LEMMA 2.2. *If $a, b \in P$, $a \neq 0$, $b \neq 0$, and $v(a) = v(b)$, then there exists a unique real number $\mu(a,b) \in (0, \infty)$ such that $\mu(a,b) \in [m/n, (m+1)/n]$ for any positive integers m, n satisfying $mb \leq na \leq (m = 1)b$.*

Proof. See [10]. ∎

By Lemma 2.2, a support $\{0\}$ ordering P on A induces a map $\alpha = \alpha_P$: $(A \times A) \setminus (0,0) \to \mathbb{R} \cup \{\infty\}$ defined by

$$\alpha(a,b) = \begin{cases} \infty & \text{if } v(a) < v(b) \\ \mu(|a|, |b|) & \text{if } v(a) = v(b),\ ab \in P \\ -\mu(|a|, |b|) & \text{if } v(a) = v(b),\ ab \in -P \\ 0 & \text{if } v(a) > v(b). \end{cases}$$

Obviously, α has the following properties.

(i) $\alpha(a,b) = \infty \Leftrightarrow \alpha(b,a) = 0$.

(ii) If $\alpha(a,b)$, $\alpha(b,c) \neq \infty$ then $\alpha(a,b)\alpha(b,c) = \alpha(a,c)$.

(iii) If $\alpha(a,c)$, $\alpha(b,c) \neq \infty$ then $\alpha(a,c) + \alpha(b,c) = \alpha(a+b, c)$.

(iv) $\alpha(a,b) = \alpha(ac, bc) = \alpha(ca, cb)$ for any $c \in A^*$.

If A is a commutative integral domain, properties (i)–(iv) characterize completely a real place on the quotient field of A (considering (a,b), $b \neq 0$ as representing the element a/b in the quotient field). This motivates us to define a real place on a not necessarily commutative integral domain A as follows.

DEFINITION 2.3. A *real place* on A is a map $\alpha\colon (A \times A) \setminus (0,0) \to \mathbb{R} \cup \{\infty\}$ which satisfies properties (i)–(iv).

PROPOSITION 2.4. *Let α be a real place on A. Then*

(1) $\alpha(1,1) = 1$, $\alpha(-1,1) = -1$, *and* $\alpha(0,1) = 0$.

(2) $m, n \in \mathbb{Z}$, $n \neq 0 \Rightarrow \alpha(m,n) = m/n$.

(3) *If* $\alpha(a,b), \alpha(c,d) \neq \infty$ *then* $\alpha(a,b)\alpha(c,d) = \alpha(ac,bd)$ *and* $\alpha(a,b) + \alpha(c,d) = \alpha(ad+bc,bd)$.

(4) *The finite elements in the image of α form a subfield of $\mathbb{R}$.*

Proof. (1) Property (i) ensures that $\alpha(1,1) \neq 0$ or ∞. By property (ii), $\alpha(1,1)\alpha(1,1) = \alpha(1,1)$ so $\alpha(1,1) = 1$. By property (iv), $\alpha(-1,1) = \alpha(1,-1)$ so, by property (i), $\alpha(-1,1) \neq 0$ or ∞. Thus $\alpha(1,-1)\alpha(-1,1) = \alpha(1,1) = 1$ so either $\alpha(-1,1) = 1$ or $\alpha(-1,1) = -1$. If $\alpha(-1,1) = 1$ then $\alpha(0,1) = \alpha(1,1) + \alpha(-1,1) = 2$, contradicting $\alpha(0,1) + \alpha(1,1) = \alpha(1,1)$. Thus $\alpha(-1,1) = -1$ and $\alpha(0,1) = 0$.

(2) By (1) and Property (iii), $\alpha(m,1) = m$. Then $\alpha(1,n)\alpha(n,1) = \alpha(1,1) = 1$ if $n \neq 0$, so $\alpha(1,n) = 1/n$ and $\alpha(m,n) = \alpha(m,1)\alpha(1,n) = m/n$.

(3) $\alpha(a,b)\alpha(c,d) = \alpha(ac,bc)\alpha(bc,bd) = \alpha(ac,bd)$ and, similarly, $\alpha(a,b) + \alpha(c,d) = \alpha(ad,bd) + \alpha(bc,bd) = \alpha(ad+bc,bd)$.

(4) This is clear. ∎

A real place α on A gives rise naturally to a valuation $v_\alpha\colon A \to \Gamma_\alpha \cup \{\infty\}$. Namely, we define $v_\alpha(a) = \{b \in A^* \mid \alpha(a,b) \neq 0,\ \alpha(a,b) \neq \infty\}$, for $a \in A^*$ and $\Gamma_\alpha = \{v_\alpha(a) \mid a \in A^*\}$. Also, we define $v_\alpha(a) \leq v_\alpha(b)$, $a, b \in A^*$ to mean that $\alpha(b,a) \neq \infty$ and we define $+$ on Γ_α by $v_\alpha(a) + v_\alpha(b) = v_\alpha(ab)$. If $v_\alpha(a) < v_\alpha(b)$, $a, b \in A^*$ then $\alpha(b,a) = 0$ so, for any $c \in A^*$, $\alpha(bc,ac) = \alpha(cd,ca) = 0$ so $v_\alpha(a) + v_\alpha(c) < v_\alpha(b) + v_\alpha(c)$ and $v_\alpha(c) + v_\alpha(a) < v_\alpha(c) + v_\alpha(b)$. This shows that Γ_α is an ordered cancellation semigroup.

Of course, if $\alpha = \alpha_P$, the real place induced by some support $\{0\}$ ordering P of A, then v_α is just the valuation induced by P. In the commutative case, every real place is of this type. In the noncommutative case, it is not clear if this is always the case. We have the following result.

THEOREM 2.5. *Let α be a real place on A and let $S_\alpha = \{ab \in A \mid \alpha(a,b) \in (0,\infty)\}$. Then the following are equivalent*:

(1) $a, b \in A^2(S_\alpha) \Rightarrow \alpha(a,b) \in [0,\infty) \cup \{\infty\}$.

(2) *If $a_1, \ldots, a_n$ are non-zero elements of $A^2(S_\alpha)$ then* $v_\alpha(a_1 + \cdots + a_n) = \min\{v_\alpha(a_i) \mid i = 1, \ldots, n\}$.

(3) $(\Sigma A^2(S_\alpha)) \cap -(\Sigma A^2(S_\alpha)) = \{0\}$.

(4) *There exists a support* $\{0\}$ *ordering* P *on* A *such that* $P \supseteq S_\alpha$.

(5) $\alpha = \alpha_P$ *for some support* $\{0\}$ *ordering* P *on* A.

(6) $A^2(S_\alpha) \cap -S_\alpha = \emptyset$.

Proof. (1) ⇒ (2). We can assume $v_\alpha(a_i) \geq v_\alpha(a_1)$ for $i = 1, \ldots, n$. Then $\alpha(a_i, a_1) \geq 0$ and $\alpha(a_1, a_1) = 1 > 0$ so $\alpha(\sum_{i=1}^n a_i, a_1) = \sum_{i=1}^n \alpha(a_i, a_1) > 0$.

(2) ⇒ (3). If not, we have non-zero elements $a_1, \ldots, a_n \in A^2(S_\alpha)$ with $a_1 + \cdots + a_n = 0$. This contradicts (2).

(3) ⇒ (4). It follows from [14, Theorem 17.10] or [15, Theorem 3.2].

(4) ⇒ (5). We want to show $\alpha(a,b) = \alpha_P(a,b)$. We may assume $\alpha(a,b) \neq \infty$ and $b \in P$, $b \neq 0$. For any $m, n \in \mathbb{Z}$, $n \geq 1$ such that $m/n < \alpha(a,b) < (m+1)/n$, we have $0 < \alpha(na - mb, nb)$ and $\alpha(na - (m+1)b, nb) < 0$ so $(na - mb)nb$, $((m+1)b - na)nb \in S_\alpha \subseteq P$. This implies $na - mb$, $(m+1)b - na \in P$ so $m/n \leq \alpha_P(a,b) \leq (m+1)/n$. Therefore, $\alpha(a,b) = \alpha_P(a,b)$.

(5) ⇒ (6). $S_\alpha \subseteq P$ so $A^2(S_\alpha) \cap -S_\alpha \subseteq P \cap -P = \{0\}$. But $0 \notin S_\alpha$.

(6) ⇒ (1). If not then there exist $a, b \in A^2(S_\alpha)$ with $\alpha(a,b) < 0$. Then $-ab \in S_\alpha$ so $ab \in A^2(S_\alpha) \cap -S_\alpha$, a contradiction. ∎

DEFINITION 2.6. A real place satisfying the equivalent conditions (1)–(6) in Theorem 2.5 is said to be *order compatible*.

In the commutative case, $\sum A^2(S_\alpha) = A^2(S_\alpha) = S_\alpha \cup \{0\}$ so every real place is order compatible.

Let α be an order compatible real place on A and let P be a support $\{0\}$ ordering on A. If $\alpha_P = \alpha$, then clearly $S_\alpha \subseteq P$. Conversely, if $S_\alpha \subseteq P$, then $\alpha_P = \alpha$ by the proof of the implication (4) ⇒ (5) of Theorem 2.5. In this situation, we will say P is *compatible* with α. We denote by X_α the set of all support $\{0\}$ orderings of A which are compatible with α and let $G_\alpha = G|_{X_\alpha}$. X_α is precisely the set of support $\{0\}$ orderings containing the preordering $\sum A^2(S_\alpha)$, so (X_α, G_α) is a subspace of (X, G).

LEMMA 2.7. *Let* α *be an order compatible real place on* A, *and let* P *be a subset of* A. *Then* P *is a support* $\{0\}$ *ordering on* A *compatible with* α *if and only if* $PP \subseteq P$, $P \cap -P = \{0\}$, $P \cup -P = A$, *and* $P \supseteq S_\alpha$.

Proof. The "only if" part is clear. To see the "if" part, it suffices to show $P + P \subseteq P$. Suppose $a, b \in P$. We can suppose $a, b \neq 0$. Also, interchanging a, b if necessary, we can assume $v_\alpha(a) \geq v_\alpha(b)$. Then $\alpha(a+b, b) > 0$. For, otherwise, $\alpha(a,b) \leq -1$, so $\alpha(-a, b) > 0$, so $-ab \in S_\alpha \subseteq P$, so $ab \in P \cap -P$, a contradiction. Therefore we have $(a+b)b \in S_\alpha \subseteq P$, which implies $a + b \in P$. ∎

In the commutative case the Baer–Krull theorem (e.g., see [13]) relates X_α with the value group Γ_α. This result carries over to the noncommutative case. We need some notation. Suppose α is a real place on A. We define an equivalence relation $\sim$ on the value semigroup Γ_α as follows. Define $\gamma \sim \delta$, $\gamma, \delta \in \Gamma_\alpha \Leftrightarrow \gamma + t + \delta \in 2\Gamma_\alpha$ for some $t \in 2\Gamma_\alpha$, where $2\Gamma_\alpha$ is the set of all permuted sums of elements $\gamma_1, \gamma_1, \dots, \gamma_n, \gamma_n$ for $\gamma_1, \dots, \gamma_n \in \Gamma_\alpha$, $n \geq 0$. If $\gamma \sim \delta$, say, $\gamma + t + \delta \in 2\Gamma_\alpha$, $t \in 2\Gamma_\alpha$, then $\delta + s + \gamma \in 2\Gamma_\alpha$ where $s = \gamma + t + \delta \in 2\Gamma_\alpha$ so $\delta \sim \gamma$. Moreover, if $\gamma \sim \delta$ and $\delta \sim \eta$, say, $\gamma + t + \delta \in 2\Gamma_\alpha$ and $\delta + s + \eta \in 2\Gamma_\alpha$ for some $t, s \in 2\Gamma_\alpha$, then $\gamma + r + \eta \in 2\Gamma_\alpha$ where $r = t + 2\delta + s \in 2\Gamma_\alpha$ so $\gamma \sim \eta$. This shows that $\sim$ is an equivalence relation on Γ_α. Let $\tilde{\Gamma}_\alpha = \Gamma_\alpha / \sim$. Since, for any $\gamma, \delta \in \Gamma$, $\gamma + \delta \sim \delta + \gamma$ and, for any $\eta \in \Gamma_\alpha$, $\gamma \sim \delta \Rightarrow \eta + \gamma \sim \delta + \eta$, we see that $\tilde{\Gamma}_\alpha$ is a quotient semigroup of Γ_α, which is in fact an abelian group of exponent 2.

THEOREM 2.8. *Let α be an order compatible real place on A. Let X_α denote the set of all support $\{0\}$ orderings on A compatible with α, and $G_\alpha = G|_{X_\alpha}$. Then*

(1) *(X_α, G_α) is a fan.*

(2) *There is a natural (split) short exact sequence $0 \to \{\pm 1\} \to G_\alpha \to \tilde{\Gamma}_\alpha \to 0$.*

(3) *X_α is in non-canonical one-to-one correspondence with the set of characters of $\tilde{\Gamma}_\alpha$.*

(4) *X_α is finite if and only if $\tilde{\Gamma}_\alpha$ is finite and, if this is the case, then $|X_\alpha| = |\tilde{\Gamma}_\alpha|$.*

Proof. Part (1) is immediate from Lemma 2.7. For (2), let $\tilde{v}_\alpha(a) \in \Gamma_\alpha$, $a \in A^*$ denote the equivalence class of $v_\alpha(a)$. We show first that the natural surjection $\bar{a}|_{X_\alpha} \mapsto \tilde{v}_\alpha(a)$ from G_α to $\tilde{\Gamma}_\alpha$ is well-defined. Suppose $\bar{a} = \bar{b}$ on X_α. Then $-ab \notin P$ for each $P \in X_\alpha$. Since $X_\alpha = X_{\sum A^2(S_\alpha)}$, by [14, Lemma 17.8] or [15, Theorem 3.5] we have $sab = t$ for some $s, t \in \sum A^2(S_\alpha)$, $s, t \neq 0$. Using the fact that if $c_i \in \sum A^2(S_\alpha)$, $c_i \neq 0$, $i = 1, \dots, n$ then $v_\alpha(c_1 + \cdots + c_n) = \min\{v_\alpha(c_i) \mid i = 1, \dots, n\}$ and the fact $\tilde{v}_\alpha(c) = 0$ for any non-zero $c \in A^2(S_\alpha)$, we get $\tilde{v}_\alpha(s) = \tilde{v}_\alpha(t) = 0$. Thus, $\tilde{v}_\alpha(a) = \tilde{v}_\alpha(b)$. To see the sequence is exact, it remains to prove that if $\tilde{v}_\alpha(a) = 0$ then $\bar{a} = \pm 1$ on X_α. If not, we have $P, Q \in X_\alpha$ such that $a \in P$, $a \notin Q$. Let $\chi\colon P^* \to \{\pm 1\}$ be defined by

$$\chi(b) = \begin{cases} 1 & \text{if } b \in Q \\ -1 & \text{if } b \in -Q, \end{cases}$$

where $P^* = P \setminus \{0\}$. Clearly, χ is a character on P^*. Suppose $v_\alpha(c) = v_\alpha(d)$, $c, d \in P^*$. Then $cd \in S_\alpha \subseteq Q$ so $\chi(c) = \chi(d)$. Hence, χ induces a character π on Γ_α. Note that $\gamma \sim \delta$, $\gamma, \delta \in \Gamma_\alpha \Rightarrow \pi(\gamma) = \pi(\delta)$, so π induces a character $\tilde{\pi}$ on $\tilde{\Gamma}_\alpha$. Now $\tilde{\pi}(\tilde{v}_\alpha(a)) = \pi(v_\alpha(a)) = \chi(a) = -1$. This is impossible because $\tilde{v}_\alpha(a) = 0$. This proves (2). Part (3) follows from (2). X_α, being a fan, is naturally identified with the set of characters χ on G_α such that $\chi(-1) = -1$ which, in turn, is in non-canonical one-to-one correspondence with the set of characters on $\tilde{\Gamma}_\alpha$. Part (4) is immediate from (3). ▮

EXAMPLE 2.9. Consider the semigroup ring $\mathbb{R}[\Gamma]$, where $(\Gamma, +, \le)$ is some ordered cancellation semigroup. By definition, $\mathbb{R}[\Gamma]$ consists of all formal finite sums $\sum_{\gamma \in \Gamma} c_\gamma x^\gamma$, $c_\gamma \in \mathbb{R}$ with componentwise addition and with multiplication given by $x^\gamma x^\delta = x^{\gamma+\delta}$. x is just a symbol. There is a natural valuation $v\colon \mathbb{R}[\Gamma] \to \Gamma \cup \{\infty\}$ defined by $v(\sum_{\gamma \in \Gamma} c_\gamma x^\gamma) = \gamma_0$ where $\gamma_0 = \min\{\gamma \in \Gamma \mid c_\gamma \neq 0\}$ if $\sum_{\gamma \in \Gamma} c_\gamma x^\gamma \neq 0$, and $v(0) = \infty$. Note that $v = v_\alpha$ where α is the real place on $\mathbb{R}[\Gamma]$ given by

$$\alpha(f, g) = \begin{cases} \infty & \text{if } v(f) < v(g) \\ c_\gamma / d_\delta & \text{if } v(f) = v(g) \\ 0 & \text{if } v(f) > v(g), \end{cases}$$

where c_γ, d_δ are coefficients of the terms in f, g, respectively, with smallest value. The real place α is order compatible. The support $\{0\}$ orderings on $\mathbb{R}[\Gamma]$ compatible with α can be computed using Theorem 2.8.

3. THE SPACE OF REAL PLACES

In the rest of the paper, the term real place will always refer to an order compatible real place. We denote by M the set of all such real places on A and define $\lambda\colon X \to M$ by $\lambda(P) = \alpha_P$ for all $P \in X$. In the field case, the mapping λ is considered, for example, by Dubois [9], Brown [6], Brown and Marshall [7], and Lam [13]. Various results can be generalized to noncommutative domains. First of all, we need to generalize the real holomorphy ring as defined in [13, 16]. Given $\alpha \in M$, let $B_\alpha = \{(a, b) \in (A \times A) \setminus (0,0) \mid \alpha(a, b) \neq \infty\}$, and define

$$H := \bigcap_{\alpha \in M} B_\alpha.$$

Then H is a generalization of the real holomorphy ring in the field case to noncommutative domains, although it does not have ring structure any more. The following result shows that H is "big enough."

LEMMA 3.1. *$(ab, a^2+b^2), (a^2, a^2+b^2) \in H$, for all $(a,b) \in (A \times A) \smallsetminus (0,0)$.*

Proof. For any $\alpha \in M$, $v_\alpha(a^2+b^2) = \min\{v_\alpha(a^2), v_\alpha(b^2)\} \le v_\alpha(ab), v_\alpha(a^2)$. ∎

Each pair $(a,b) \in (A \times A) \smallsetminus (0,0)$ defines a mapping a/b: $M \to \mathbb{R} \cup \{\infty\}$ given by $(a/b)(\alpha) = \alpha(a,b)$. The topology on M is defined to be the weakest such that the mappings $a/b, (a,b) \in (A \times A) \smallsetminus (0,0)$, are continuous, where $\mathbb{R} \cup \{\infty\}$ is given the topology of the real projective line. Let $\text{Cont}(M, \mathbb{R})$ denote the ring of all continuous functions from M to $\mathbb{R}$ and let $\tilde{H} = \{a/b \mid (a,b) \in H\}$, so $\tilde{H} \subseteq \text{cont}(M, \mathbb{R})$.

THEOREM 3.2. *With the set-up as above, then*

(1) *M is Hausdorff.*

(2) *The natural surjection λ: $X \to M$ is continuous.*

(3) *M is compact.*

(4) *The topology on M is the quotient topology.*

(5) *$\tilde{H}$ is a subring of* $\text{Cont}(M, \mathbb{R})$.

(6) *$\tilde{H}$ is dense in* $\text{Cont}(M, \mathbb{R})$ *in the sup norm.*

(7) *Every fiber $\lambda^{-1}(\alpha)$, $\alpha \in M$, is a fan.*

(8) *Every fan $Y \subseteq X$ intersects at most two fibers of λ.*

Proof. (1) Suppose $\alpha, \beta \in M$, $\alpha \neq \beta$. Thus there exists $(a,b) \in (A \times A) \smallsetminus (0,0)$ such that $\alpha(a,b) \neq \beta(a,b)$. If one of $\alpha(a,b)$ and $\beta(a,b)$ is infinite, say $\alpha(a,b) = \infty$ and $\beta(a,b) < \infty$ then $\alpha(b,a) = 0$ and $\alpha(a+rb, a) = \alpha(a,a) + \alpha(r,1)\alpha(b,a) = 1$ for any integer r, so $\alpha(rb, a+rb) = \alpha(a+rb, a+rb) - \alpha(a, a+rb) = 1 - 1 = 0$. This implies $\alpha(b, a+rb) = 0$. Since $\beta(a+rb, b) = \beta(a,b) + r$, $\beta(b, a+rb) \neq \infty$ and $\beta(b, a+rb) \neq 0$ if $\beta(a,b) \neq -r$. Thus, replacing (a,b) by $(b, a+rb)$ for some appropriate integer r if necessary, we can assume $\alpha(a,b) \neq \infty$, $\beta(a,b) \neq \infty$. Finally, replace (a,b) by $(\pm qa + pb, qb)$ for some appropriate integers p, q, we can assume $\alpha(a,b) < 0 < \beta(a,b)$. This proves M is Hausdorff.

(2) To prove the continuity of λ we note that the inverse image under λ of the subbasic open set $\{\alpha \in M \mid \alpha(a,b) > 0,\ \alpha(a,b) \neq \infty\}$ in M, where $(a,b) \in (A \times A) \smallsetminus (0,0)$, is the union of the open sets $U(\overline{(na-b)a}) \cap U(\overline{(nb-a)b}) \cap U(\overline{ab})$ in X, n running through the positive integers. If $\lambda(P) = \alpha$, $\alpha(a,b) > 0$, $\alpha(a,b) \neq \infty$, then $ab \in P$, and if we pick $n \ge 1$ large enough so that $n > \alpha(a,b), \alpha(b,a)$, then $\alpha(nb-a, b) > 0, \alpha(na-b, a) > 0$ so $(nb-a)b, (na-b)a \in P$. Conversely, if $P \in X$ and $ab, (na-b)a, (nb-a)b \in P$ for some $n \ge 1$ and

$\alpha = \lambda(P)$ then, replacing a, b by $-a, -b$ respectively, if necessary, we can assume $a, b \in P$. Then $na - b, nb - a \in P$ which implies $v_P(a) = v_P(b)$, so $\alpha(a, b) > 0$, $\alpha(a, b) \neq \infty$.

(3) and (4). These follow from (1) and (2) since X is compact.

(5) Given $(a_1, b_1), (a_2, b_2) \in H$, $((a_1/b_1) + (a_2/b_2))(\alpha) = \alpha(a_1, b_1) + \alpha(a_2, b_2) = \alpha(a_1 b_2 + b_1 a_2, b_1 b_2) = ((a_1 b_2 + b_1 a_2)/(b_1 b_2))(\alpha)$ for all $\alpha \in M$, so $(a_1/b_1) + (a_2/b_2) = (a_1 b_2 + b_1 a_2)/(b_1 b_2)$. Similarly, we have $(a_1/b_1)(a_2/b_2) = (a_1 a_2)/(b_1 b_2)$. From this it is clear that $\tilde{H}$ is closed under addition and multiplication. Clearly $\tilde{H}$ contains every rational constant, so this proves $\tilde{H}$ is a subring of $\mathrm{Cont}(M, \mathbb{R})$.

(6) By the Stone-Weierstrass theorem, applied to the closure of $\tilde{H}$, it suffices to show $\tilde{H}$ separates points in M. Let $\alpha, \beta \in M$, $\alpha \neq \beta$. By the proof of (1), we have $(a, b) \in (A \times A)$ such that $\alpha(a, b), \beta(a, b) \neq \infty$ and $\alpha(a, b) < 0 < \beta(a, b)$. Replacing (a, b) by $(ab, a^2 + b^2)$ (using Lemma 3.1), we can assume $(a, b) \in H$.

(7) $\lambda^{-1}(\alpha) = X_\alpha$ so this is just Theorem 2.8 (1).

(8) Suppose, to the contrary, that Y is a fan in X which intersects three or more fibers of λ. Thus we have $P_1, P_2, P_3 \in Y$ with $\alpha_i = \lambda(P_i)$, $i = 1, 2, 3$, distinct. Since Y is a fan, we have $P_4 \in Y$ such that $\prod_{i=1}^4 \bar{a}(P_i) = 1$ for all $a \in A^*$. Let $\alpha_4 = \lambda(P_4)$. Reindexing, we can assume that either $\alpha_1, \ldots, \alpha_4$ are all distinct or $\alpha_3 = \alpha_4$. Using the Tietze extension theorem, we get a continuous function $f\colon M \to \mathbb{R}$ with $f(\alpha_1) = -1$, $f(\alpha_i) = 1$, $i = 2, 3, 4$. Using (6), we get $(a, b) \in H$ with $\alpha_1(a, b) < 0$, $\alpha_i(a, b) > 0$, $i = 2, 3, 4$. Then $ab \neq 0$ and $ab \in -P_1$ and $ab \in P_i$, $i = 2, 3, 4$, so $\prod_{i=1}^4 \overline{ab}(P_i) = -1$, a contradiction. ∎

As in [16], a P-*structure* on a space of orderings (X, G) is defined to be a surjection λ from X to a set M such that the following hold:

(1) For each $\alpha \in M$, $\lambda^{-1}(\alpha)$ is a fan in X.

(2) For each fan $Y \subseteq X$, there exists $\alpha, \beta \in M$ with $Y \subseteq \lambda^{-1}(\alpha) \cup \lambda^{-1}(\beta)$.

Also, a P-structure $\lambda\colon X \to M$ on (X, G) is said to be *Hausdorff* if M, with the induced quotient topology, is Hausdorff.

COROLLARY 3.3. *The map $\lambda\colon X \to M$ defines a Hausdorff P-structure on the space of orderings (X, G).*

4. TRIVIALIZATION THEOREM FOR FANS

In this section, we develop a version of Bröcker's trivialization theorem for fans [4] which holds for noncommutative domains.

DEFINITION 4.1. Given two valuations v, w on A, we say w is *coarser* than v (or v is *finer* than w) if $v(a) \geq v(b) \Rightarrow w(a) \geq w(b)$ $\forall a, b \in A$.

LEMMA 4.2. *Suppose w_1, w_2 are coarser than v. Then either w_1 is coarser than w_2 or w_2 is coarser than w_1.*

Proof. Otherwise, there are $a_1, b_1, a_2, b_2 \in A$ such that $w_1(a_1) \geq w_1(b_1)$, $w_2(a_1) < w_2(b_1)$ and $w_2(a_2) \geq w_2(b_2)$, $w_1(a_2) < w_1(b_2)$. Then $w_1(a_1 b_2) > w_1(b_1 a_2)$ and $w_2(a_1 b_2) < w_2(b_1 a_2)$, which implies $v(a_1 b_2) > v(b_1 a_2)$ and $v(a_1 b_2) < v(b_1 a_2)$. This is impossible. ∎

Thus the family of valuations coarser than a given valuation forms a chain. Note however that our proof of Lemma 4.2 breaks down if w_1 and w_2 are only assumed to be valuations in the weak sense. Moreover, valuations in the weak sense are precisely the sort of valuation we obtain in the trivialization theorem (Theorem 4.7) below. This is indeed unfortunate. But anyway, because of this, we need to allow valuations in the weak sense in the remainder of this section.

DEFINITION 4.3. Let v be a valuation on A in the weak sense, and let P be a support $\{0\}$ ordering on A. We say P is *compatible* with v if $\forall a, b \in A$, $v(a) > v(b) \Rightarrow (a+b)b \in P$.

Note. (1) P is compatible with $v_{\lambda(P)}$. (2) If P is compatible with v, then v is coarser than $v_\lambda(P)$. The first assertion is clear. For the second, suppose $v(a) > v(b)$. Then, for any integer $n \geq 1$, $v(-na) > v(b)$ so $(b - na)b \in P$. We may assume $a, b \in P$. Then $b - na \in P$ for any integer $n \geq 1$, so $v_{\lambda(P)}(a) > v_{\lambda(P)}(b)$. ∎

LEMMA 4.4. *Let P be a support $\{0\}$ ordering on A compatible with a valuation v on A. Suppose Q is a subset of A such that $QQ \subseteq Q$, $Q \cup -Q = A$, $Q \cap -Q = \{0\}$, and $P \cap \{ab \mid v(a) = v(b)\} = Q \cap \{ab \mid v(a) = v(b)\}$. Then Q is a support $\{0\}$ ordering on A and Q is compatible with v.*

Proof. To see Q is an ordering, it suffices to show $Q + Q \subseteq Q$. Suppose $0 \neq c, d \in Q$. If $v(c) \neq v(d)$, say, $v(c) > v(d)$, then $c + d \neq 0$, $(c + d)d \in P$ and $v(c+d) = v(d)$. Then $(c+d)d \in Q$ which implies $c + d \in Q$. Now suppose that $v(c) = v(d)$. Since $cd \in Q$, this implies $cd \in P$ so either $c, d \in P$ or $c, d \in -P$. In either case, we have $(c+d)d \in P$. We claim that $v(c+d) = v(d)$. For, otherwise, $v(c+d) > v(d) = v(-d)$. Then $-cd = -(-d + c + d)d \in P$ and $v(c) = v(-d)$, so $-cd \in Q$ so $cd \in Q \cap -Q$, a contradiction. Thus, $v(c+d) = v(d)$ so $(c+d)d \in Q$

which forces $c + d \in Q$. To see Q is compatible with v, let $v(a) > v(b)$, $a,b \in A$. Then $v(a+b) = v(b)$ and P is compatible with v implies $(a+b)b \in P$, so $(a+b)b \in Q$. This proves that Q is compatible with v. ∎

Given a valuation v on A, let X_v denote the set of all support $\{0\}$ orderings on A which are compatible with v. Suppose $X_v \neq \emptyset$ and let $G_v = G|_{X_v}$. Also, denote by S_v the set of all products $(a+b)b$, $a, b \in A$, $v(a) > v(b)$. Clearly a support $\{0\}$ ordering P on A belongs to X_v iff $P \supseteq S_v$. Thus $X_v = X_{\sum A^2(S_v)}$, so (X_v, G_v) is a subspace of (X, G).

LEMMA 4.5. *If* $a_1, \dots, a_n \in A^2(S_v)$ *then* $v(a_1 + \cdots + a_n) = \min\{v(a_i) \mid i = 1, \dots, n\}$.

Proof. Suppose $v(a_1 + \cdots + a_n) > \min\{v(a_i) \mid i = 1, \dots, n\}$. We can assume $v(a_i) \geq v(a_1)$, $i = 1, \dots, n$ and that n is chosen minimal. Then $(a_1 - (a_1 + \cdots + a_n))a_1 = -(a_2 + \cdots + a_n)a_1 \in S_v$ so $(a_2 + \cdots + a_n)a_1 \in (\sum A^2(S_v)) \cap -(\sum A^2(S_v)) = \{0\}$, a contradiction. ∎

Let $K_v = \{\overline{ab} \mid a, b \in A^*,\ v(a) = v(b)\}$, and let $\tilde{G}_v = K_v|_{X_v}$. K_v is a subgroup of G and $\tilde{G}_v$ is a subgroup of G_v. Note that $-1 \in \tilde{G}_v$ since $-1 \in K_v$. Define an equivalence relation $\sim$ on X_v by $P \sim Q \Leftrightarrow \bar{c}(P) = \bar{c}(Q)$ or all $\bar{c} \in K_v$, and let $\tilde{X}_v = X_v/\sim$. We refer to $(\tilde{X}_v, \tilde{G}_v)$ as the *residue space* of (X, G) at v. In the field case, $(\tilde{X}_v, \tilde{G}_v)$ is the space of orderings of the residue field of v, but in the general case considered here, the residue field may not even be defined.

Also, let $\Gamma = \{v(a) \mid a \in A^*\}$ be the value semigroup of v. We define an equivalence relation $\sim$ on Γ exactly as we did for Γ_α in Section 2 and, just as in Section 2, $\tilde{\Gamma} = \Gamma/\sim$ is an abelian group of exponent 2.

THEOREM 4.6. (1) *(X_v, G_v) is a group extension of $(\tilde{X}_v, \tilde{G}_v)$.*

(2) *$(\tilde{X}_v, \tilde{G}_v)$ is a space of orderings.*

(3) *There is a natural short exact sequence* $0 \to \tilde{G}_v \to G_v \to \tilde{\Gamma} \to 0$.

Proof. Part (1) is immediate from Lemma 4.4. Part (2) follows from (1) using standard facts from the theory of spaces of orderings [16, Theorems 4.1.1 and 4.1.3]. For pat (3), for $a \in A^*$, let $\tilde{v}(a)$ denote the equivalence class of $v(a)$. Exactly as in the proof of Theorem 2.8, the surjective map $\bar{a}|_{X_v} \mapsto \tilde{v}(a)$ from G_v to $\tilde{\Gamma}_v$ is well-defined. It remains to show that if $\tilde{v}(a) = 0$ then $\bar{a}|_{X_v} \in \tilde{G}_v$. If this is not the case then, by Lemma 4.4, we have $P, Q \in X_v$ with $P \sim Q$ and $a \in P$, $a \notin Q$. This yields a contradiction exactly as in the proof of Theorem 2.8. ∎

By Theorem 3.2, for any fan Y in X there are two real places $\alpha, \beta \in M$ such that $Y \subseteq \lambda^{-1}(\alpha) \cup \lambda^{-1}(\beta)$. Let v_α, v_β be the valuation corresponding to α, β, respectively. Then every ordering P in Y is either compatible with v_α or with v_β. Thus there is a unique finest valuation v compatible with every $P \in Y$, namely the finest valuation coarser than both of v_α, v_β. We would like to prove that the pushdown of Y to the residue space $(\tilde{X}_v, \tilde{G}_v)$ is a trivial fan. We are not able to prove this. However, we are able to prove the following weak version:

THEOREM 4.7 (Trivialization Theorem for Fans). *If $Y \subseteq X$ is a fan then there exists a valuation v on A in the weak sense such that every ordering in Y is compatible with v and the induced fan in the residue space $(\tilde{X}_v, \tilde{G}_v)$ is trivial.*

Proof. Let $Y \subseteq \lambda^{-1}(\alpha) \cup \lambda^{-1}(\beta)$ for some $\alpha, \beta \in M$. We can suppose $\alpha \neq \beta$ and $Y \cap \lambda^{-1}(\alpha)$, $Y \cap \lambda^{-1}(\beta)$ are both nonempty. For, otherwise, $Y \subseteq \lambda^{-1}(\alpha)$ say, and we can take $v = v_\alpha$. The induced fan in $\tilde{X}_v$ is singleton so it is trivial. We now define a valuation v in the weak sense so that every ordering in Y is compatible with v. For any $a, b \in A^*$, let $a \leq_1 b$ mean $v_\alpha(a) \leq v_\alpha(b)$ and $a \leq_2 b$ mean $v_\beta(a) \leq v_\beta(b)$ and define $a \leq b$ if there is a finite chain of elements in A, $a = a_0, \dots, a_{2n} = b$, such that $a_0 \leq_1 a_1 \leq_2 a_2 \cdots \leq_1 a_{2n-1} \leq_2 a_{2n}$. Define $a \sim b$ to mean $a \leq b$ and $b \leq a$. Obviously, $\leq$ induces a total ordering on the equivalence classes and $a \leq b \Rightarrow ac \leq bc$, $ca \leq cb$ for all $c \in A^*$ and $a + b \geq \min\{a, b\}$. Let $v(a)$ denote the equivalence class of a with respect to $\sim$ and define $v(a) \leq v(b)$ to mean that $a \leq b$. Also define $v(a) + v(b) = v(ab)$. Then it is easy to check that v is a valuation in the weak sense. Since Y is a fan, we have $Y = X_T$ where $T = \bigcap_{P \in Y} P$. Thus (X_T, G_T) is a fan, where $G_T = \{\bar{a}_T = \bar{a}|_{X_T} \mid a \in A^*\}$. By our assumption, $\lambda(X_T) = \{\alpha, \beta\}$. Let $X_1 = \{P \in X_T \mid \lambda(P) = \alpha\}$ and $X_2 = \{P \in X_T \mid \lambda(P) = \beta\}$. By Theorem 3.2, $\tilde{H} = \{a/b \mid (a,b) \in H\}$ is dense in $\mathrm{Cont}(M, \mathbb{R})$ in the sup norm so $\alpha \neq \beta$ implies there exists $(a, b) \in H$ such that $\alpha(a, b) > 0$ and $\beta(a, b) < 0$. Then $\overline{ab} = 1$ on X_1 and $\overline{ab} = -1$ on X_2. Let $G^* = \{c \in A^* \mid \bar{c}_T \in \{1, -1, \overline{ab}_T, -\overline{ab}_T\}\}$.

Claim 1. If $\bar{c} = 1$ on X_1 or X_2 then $c \in G^*$. For, say, $\bar{c} = 1$ on X_1, then $\bar{c}_T \in D\langle 1, \overline{ab}_T \rangle$ which, by Theorem 3.1.2 in [16], is $\{1, \overline{ab}_T\}$ since Y is a fan.

Claim 2. If $c, d \in A^*$ and $cd \notin G^*$, then $v_\alpha(c) \neq v_\alpha(d)$ and $v_\beta(c) \neq v_\beta(D)$. Suppose $v_\alpha(c) = v_\alpha(d)$. Then either $\overline{cd} = 1$ on X_1 or $\overline{cd} = -1$ on X_1 so, by Claim 1, either $cd \in G^*$ or $-cd \in G^*$ which contradicts our assumption. Similarly, $v_\beta(c) \neq v_\beta(d)$.

Claim 3. If $c, d \in A^*$ and $cd \notin G^*$, then either $v_\alpha(c) > v_\alpha(d)$, $v_\beta(c) > v_\beta(d)$ or $v_\alpha(c) < v_\alpha(d)$, $v_\beta(c) < v_\beta(d)$. Otherwise, say, $v_\alpha(c) > v_\alpha(d)$, $v_\beta(c) < v_\beta(d)$. Then $\overline{c+d}_T \in D\langle \bar{c}_T, \bar{d}_T \rangle = \{\bar{c}_T, \bar{d}_T\}$ since Y is a fan. Note $v_\alpha(c) > v_\alpha(d)$ implies $\alpha(c, d) = 0$ so $\alpha(c + d, d) = 1$ so $\overline{c+d} = \bar{d}$ on X_1.

Similarly, $\overline{c+d} = \bar{c}$ on X_2. Now either $\overline{c+d}_T = \bar{c}_T$ so $\bar{c} = \bar{d}$ on X_1 so $\overline{cd} = 1$ on X_1, a contradiction by Claim 1, or $\overline{c+d}_T = \bar{d}_T$ so $\bar{c} = \bar{d}$ on X_2 so $\overline{cd} = 1$ on X_2, also a contradiction by Claim 1.

Claim 4. If $c, d \in A^*$ and $cd \notin G^*$ then $v(c) \neq v(d)$. If $v(c) = v(d)$, then there exists a finite chain $c = a_0, \dots, a_{2n} = d$ such that $a_0 \leq_1 a_1 \leq_2 a_2 \cdots \leq_1 a_{2n-1} \leq_2 a_{2n}$. Since $\overline{cd} = \prod_{i=1}^{2n} \overline{a_{i-1} a_i}$, $a_{i-1} a_i \notin G^*$ for some $1 \leq i \leq 2n$. By Claim 3, $v_\alpha(a_{i-1}) < v_\alpha(a_i)$, $v_\beta(a_{i-1}) < v_\beta(a_i)$. Then $a_{i-1} \leq_1 a_i$, $a_{i-1} \leq_2 a_i$, so we can shorten the chain by eliminating a_{i-1} and a_i. We may repeat this process until we get a short chain, $c = a_0 \leq_1 a_1 \leq_2 a_2 = d$. Then either $ca_1 \notin G^*$ or $a_1 d \notin G^*$. In either case, we have $v_\alpha(c) < v_\alpha(d)$, $v_\beta(c) < v_\beta(d)$. Similarly, there is a finite chain $b_0, \dots, b_{2m}$ with $c = b_0 \geq_2 b_1 \geq_1 b_2 \cdots \geq_2 b_{2m-1} \geq_1 b_{2m} = d$ so, as before, $v_\alpha(c) > v_\alpha(d)$, $v_\beta(c) > v_\beta(d)$, a contradiction.

Let $\tilde{Y}$ denote the pushdown of Y to $\tilde{X}_v$. By Claim 4, for any $\overline{cd}|_{X_v} \in \tilde{G}_v$, $cd \in G^*$ so $\tilde{G}_v|_{\tilde{Y}} = \{1, -1, \overline{ab}_T, -\overline{ab}_T\}$. Hence $\tilde{Y}$ consists of two elements so it is a trivial fan. ∎

5. REAL PLACES ON A NONCOMMUTATIVE RING

We now return to the general set-up considered at the beginning of Section 1. Namely, we fix a (not necessarily commutative) ring A with 1 and assume $-1 \notin \sum A^2$. We denote by (X, G) the real spectrum of A, i.e., $X =$ the set of all orderings on A and $G = \{\bar{a} \mid a \in A\}$, where $\bar{a}$: $X \to \{-1, 0, 1\}$ is defined by

$$\bar{a}(P) = \begin{cases} 1 & \text{if } a \in P \setminus -P \\ 0 & \text{if } a \in P \cap -P \\ -1 & \text{if } a \in -P \setminus P. \end{cases}$$

X is a spectral space [15, 16]. The sets $U(\bar{a}) = \{P \in X \mid \bar{a}(P) = 1\}$, $\bar{a} \in G$ are a subbasis of open sets. The associated patch topology on X is the weakest such that the mappings $\bar{a} \in G$ are continuous (giving $\{-1, 0, 1\}$ the discrete topology). For a real prime $\mathfrak{p} \subseteq A$, we denote by $(X_\mathfrak{p}, G_\mathfrak{p})$ the space of support $\mathfrak{p}$ orderings on A. This is identified with the space of support $\{0\}$ orderings on the integral domain $A/\mathfrak{p}$. Let $M_\mathfrak{p}$ be the set of all (order compatible) real places on $A/\mathfrak{p}$. By Corollary 3.3, the natural surjection $\lambda_\mathfrak{p}$: $X_\mathfrak{p} \to M_\mathfrak{p}$ defines a Hausdorff P-structure on $(X_\mathfrak{p}, G_\mathfrak{p})$. Let $M = \bigcup_\mathfrak{p} M_\mathfrak{p}$ (disjoint union), $\mathfrak{p}$ running through all real primes of A, and define λ: $X \to M$ by $\lambda(P) = \lambda_\mathfrak{p}(P)$, if $P \in X_\mathfrak{p}$. To show the map λ defines a P-structure on the real spectrum (X, G), terminology as in [16, Sect. 8.6], it only remains to show that λ respects specialization.

Recall: If P, Q are orderings on A, we say Q *specializes* P (or P *generalizes* Q) if $P \subseteq Q$. Note that if Q specializes P and $\mathfrak{p}$ and $\mathfrak{q}$ are the supports of P and Q respectively then $\mathfrak{p} \subseteq \mathfrak{q}$ and $Q = P + \mathfrak{q} = P \cup \mathfrak{q}$.

To simplify notation, it is convenient to identify a real place $\overline{\alpha}$ on $A/\mathfrak{p}$ with the map $\alpha: (A \times A) \setminus (\mathfrak{p} \times \mathfrak{p}) \to \mathbb{R} \cup \{\infty\}$ obtained by composing $\overline{\alpha}$ with the natural map $A \times A \to A/\mathfrak{p} \times A/\mathfrak{p}$. Thus if P has support $\mathfrak{p}$, then $\alpha_P = \lambda(P)$ can be viewed as a mapping from $(A \times A) \setminus (\mathfrak{p} \times \mathfrak{p})$ to $\mathbb{R} \cup \{\infty\}$.

LEMMA 5.1. *Let P be an ordering on A with support $\mathfrak{p}$ and let Q be a specialization of P with support $\mathfrak{q}$. Then*

(1) *α_Q is just the restriction of α_P from $(A \times A) \setminus (\mathfrak{p} \times \mathfrak{p})$ to $(A \times A) \setminus (\mathfrak{q} \times \mathfrak{q})$.*

(2) *If P' is an ordering on A with support $\mathfrak{p}$ such that $\alpha_P = \alpha_{P'}$ then $Q' = P' + \mathfrak{q}$ is an ordering on A with support $\mathfrak{q}$ and $\alpha_Q = \alpha_{Q'}$.*

Proof. (1) This is easy to check. (2) In view of (1), it suffices to show that Q' is an ordering with support $\mathfrak{q}$. Obviously, $Q' + Q' \subseteq Q'$, $Q'Q' \subseteq Q'$ and $Q' \cup -Q' = A$. To see $Q' \cap -Q' = \mathfrak{q}$, we need only to show $Q' \cap -Q' \subseteq \mathfrak{q}$. Let $a = t_1 + s_1 = -t_2 + s_2 \in Q' \cap -Q'$, $t_1, t_2 \in P'$, $s_1, s_2 \in \mathfrak{q}$. Then $t_1 + t_2 \in \mathfrak{q}$. there are four cases:

Case 1. $t_1, t_2 \in P$. Since $\mathfrak{q}$ is compatible with P, Corollary 3.3 in [15] implies $t_1, t_2 \in \mathfrak{q}$.

Case 2. $t_1, t_2 \in -P$. Then $-t_1, -t_2 \in P$ and $-t_1 - t_2 \in \mathfrak{q}$ so $-t_1$, $-t_2 \in \mathfrak{q}$ so also $t_1, t_2 \in \mathfrak{q}$.

Case 3. $t_1 \in P$, $t_2 \in -P$. If $t_1, T_2 \notin \mathfrak{q}$ then either $\alpha_P(t_1, t_2) = 0$ or $\alpha_P(t_2, t_1) = 0$. For, otherwise, $\alpha_{P'}(t_1, t_2) = \alpha_P(t_1, t_2 0 < 0$, so $t_1 t_2 \in -P'$, contradicting $t_1, t_2 \in P'$. Suppose $\alpha_P(t_1, t_2) = 0$. Then $\alpha_P(2t_1 + t_2, t_2) = 1$ so $2t_1 + t_2 \in -P$ so $t_1 \in -(P + \mathfrak{q}) = -Q$. But $t_1 \in P \subseteq Q$. Hence $t_1 \in Q \cap -Q = \mathfrak{q}$, so $t_2 \in \mathfrak{q}$.

Case 4. $t_1 \in -P, t_2 \in P$. By the same sort of argument used in Case 3, we have $t_1, t_2 \in \mathfrak{q}$. ∎

THEOREM 5.2. *The mapping λ: $X \to M$ defines a Hausdorff P-structure on (X, G).*

Proof. Combining Lemma 5.1 with Corollary 3.3, we see that λ: $X \to M$ defines a P-structure on the real spectrum (X, G). It remains to show that the patch topology on X induces a Hausdorff quotient topology on M. Suppose $(\mathfrak{p}_1, \alpha_1), (\mathfrak{p}_2, \alpha_2) \in M$, $(\mathfrak{p}_1, \alpha_1) \neq (\mathfrak{p}_2, \alpha_2)$. If $\mathfrak{p}_1 \neq \mathfrak{p}_2$, say $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_2$, then there exists $a \in \mathfrak{p}_1$, $a \notin \mathfrak{p}_2$. Let $S = \{(\mathfrak{p}, \alpha) \in M \mid a \in \mathfrak{p}\}$, then S is clopen in M, since $\lambda^{-1}(S) = \{P \in X \mid a \in P \cap -P\}$ is clopen in X

in the patch topology. Also, $(\mathfrak{p}_1, \alpha_1) \in S$, $(\mathfrak{p}_2, \alpha_2) \in M \setminus S$. This leaves the case $\mathfrak{p}_1 = \mathfrak{p}_2$ (so $\alpha_1 \neq \alpha_2$). Let $\mathfrak{p}_1 = \mathfrak{p}_2 = \mathfrak{p}$. As in the proof of Theorem 3.2, we get $a, b \in A \setminus \mathfrak{p}$, with $\alpha_1(a,b) < 0 < \alpha_2(a,b)$, $\alpha_i(a,b) \neq \infty$, $i = 1,2$. Define $S_{a,b} = \{(\mathfrak{q}, \beta) \in M \mid b \notin \mathfrak{q}, 0 < \beta(a,b) < \infty\}$. Again as in the proof of Theorem 3.2, one checks that $\lambda^{-1}(S_{a,b}) = \{P \in X \mid \exists n \geq 1$ such that $(na - b)a \notin -P$, $(nb - a)b \notin -P$, $ab \notin -P\}$, which is open in X. Thus $S_{a,b}$ is open in M. Since $(\mathfrak{p}, \alpha_1) \in S_{-a,b}$, $(\mathfrak{p}, \alpha_2) \in S_{a,b}$ and $S_{-a,b} \cap S_{a,b} = \emptyset$. This completes the proof. ∎

6. AN EXAMPLE

We consider the twisted polynomial ring $\mathbb{R}[x,y]_t$ with multiplication given by $yx = axy$ where $a \in \mathbb{R}$, $a > 0$, $a \neq 1$. We compute the real places and orderings on $\mathbb{R}[x,y]_t$.

Each element $f \in \mathbb{R}[x,y]_t$ is expressible uniquely as $f = \sum_{i,j} b_{ij} x^i y^j$ with $b_i D_j \in \mathbb{R}$, equivalently, as $f = \sum_{ij} c_{ij} y^j x^i$ with $c_{ij} \in \mathbb{R}$. The two expression are related by the equations $c_{ij} = a^{-ij} b_{ij}$.

Proposition 6.1. (1) *$\mathbb{R}[x,y]_t$ is an integral domain.*

(2) *Any non-zero ideal in $\mathbb{R}[x,y]_t$ contains a monomial.*

Proof. (1) Suppose f, g are nonzero elements of $\mathbb{R}[x,y]_t$. Expanding f and g as polynomials in y with coefficients in $\mathbb{R}[x]$, we see that the lowest degree term in the product fg is not zero, so $fg \neq 0$..

(2) Suppose $f \neq 0$, and let $f_1 = a^k xf - fx$, and $f_2 = yf - a^l fy$, where y^k is the highest power of y dividing f and x^l is the highest power of x dividing f. Then f_1 and f_2 have fewer monomial terms than f. Moreover, if f is itself not a monomial, then either $f_1 \neq 0$ or $f_2 \neq 0$. The result follows from this, by induction. ∎

Denote by (X, G) the real spectrum of $\mathbb{R}[x,y]_t$. Also, for a real prime $\mathfrak{p}$ of $\mathbb{R}[x,y]_t$, denote by $(X_\mathfrak{p}, G_\mathfrak{p})$, the space of support $\mathfrak{p}$ orderings of $\mathbb{R}[x,y]_t$. It follows from Proposition 6.1 that the real primes in $\mathbb{R}[x,y]_t$ are $\{0\}$, (x), (y), $(x, y-r)$, $(x-r, y)$, $r \in \mathbb{R}$. also, $\mathbb{R}[x,y]_t/(x) \cong \mathbb{R}[y]$, $\mathbb{R}[x,y]_t/(y) \cong \mathbb{R}[x]$, and $\mathbb{R}[x,y]_t/(x, y-r) \cong \mathbb{R}[x,y]_t/(x-r, y) \cong \mathbb{R}$. Thus, if $\mathfrak{p} \neq \{0\}$, the structure of the space of orderings $(X_\mathfrak{p}, C_\mathfrak{p})$ is well known. the fact that the prime $\{0\}$ is real will be clear in a minute.

If v is a valuation on $\mathbb{R}[x,y]_t$ which is trivial on $\mathbb{R}$ then $v(y) + v(x) = v(yx) = v(axy) = v(a) + v(x) + v(y) = v(x) + v(y)$. Of course, if $v(x), v(y)$ are independent (i.e., if $v(x^i y^j) = v(x^{i'} y^{j'}) \Rightarrow (i,j) = (i',j')$), then $v(\sum_{ij} b_{ij} x^i y^j) = \min\{v(x^i y^j) \mid i, j \geq 0,\ b_{ij} \neq 0\}$, so v is completely determined by $v(x), v(y)$. In this case, identifying $v(x)$ with $(1,0)$, and $v(y)$ with $(0,1)$, we see that the value semigroup of v is identified with a subsemigroup of the group $\mathbb{Z} \times \mathbb{Z}$. Also, the ordering on the value semigroup extends uniquely to an ordering on the group $\mathbb{Z} \times \mathbb{Z}$. This process is obviously reversible. Moreover, there is a unique real place associated to v. This is given by

$$\alpha(f,g) = \begin{cases} \infty & \text{if } v(f) < v(g) \\ c_{ij}/d_{kl} & \text{if } v(f) = v(g) \\ 0 & \text{if } v(f) > v(g), \end{cases}$$

where c_{ij} and d_{kl} are coefficients of the terms in f and g, respectively, with smallest value. It is easy to see that α is a (support $\{0\}$) real place in the sense of Definition 2.3. Moreover, it is order compatible. In fact, let $S_\alpha = \{fg \in \mathbb{R}[x,y] \mid \alpha(f,g) \in (0,\infty)\}$. Then the coefficient of the term with smallest value in any non-zero element of $A^2 S_\alpha$ is positive, so condition (3) of Theorem 2.5 is satisfied.

Conversely, let α be any (support $\{0\}$) real place on $\mathbb{R}[x,y]_t$. We *claim* that $v_\alpha(x), v_\alpha(y)$ are independent. If not, then, since cancellation holds in the value semigroup of v_α, either $\alpha(x^i, y^j) \neq 0, \infty$ or $\alpha(x^i y^j, 1) \neq 0, \infty$ for some $(i,j) \neq (0,0)$. In the first case this yields $\alpha(x^i, y^j) = \alpha(x^{i+1}, y^j x) = \alpha(x^{i+1}, a^j x y^j) = a^{-j}\alpha(x^i, y^j)$, and similarly, $\alpha(x^i, y^j) = \alpha(yx^i, y^{j+1}) = \alpha(a^i x^i y, y^{j+1}) = a^i \alpha(x^i, y^j)$. Since $a > 0$, $a \neq 1$, this forces $i = j = 0$, a contradiction. The argument in the second case is similar. Thus we have proved part (1) of the following:

THEOREM 6.2. (1) *The set of support* $\{0\}$ *real places on* $\mathbb{R}[x,y]_t$ *is in one-to-one correspondence with the set of orderings on the group* $\mathbb{Z} \times \mathbb{Z}$.

(2) *The space of support* $\{0\}$ *orderings on* $\mathbb{R}[x,y]_t$ *has stability index* 2.

Proof. For each support $\{0\}$ real place α, since $v_\alpha(x), v_\alpha(y)$ are independent and generate the value semigroup Γ_α, we see that $|\tilde{\Gamma}_\alpha| = 4$ so X_α is a 4-element fan. Moreover, if α, β are distinct real places on $\mathbb{R}[x,y]_t$ then $v_\alpha \neq v_\beta$, so there exists $(i,j) \neq (i',j')$ with $v_\alpha(x^i y^j) < v_\alpha(x^{i'} y^{j'})$, $v_\beta(x^i y^j) > v_\beta(x^{i'} y^{j'})$. Thus, if v is any weak valuation finer than both v_α, v_β, then $v(x^i y^j) = v(x^{i'} y^{j'})$ so $v(x), v(y)$ are not independent. Also $v(x), v(y)$ generate the value semigroup Γ_v. This implies $|\tilde{\Gamma}_v| \leq 2$. Thus any fan in X_v which is trivial in the residue space of v can have at most 4 elements. Thus, by Theorem 4.7, any fan in the space of orderings $(X_{\{0\}}, G_{\{0\}})$ has at most 4 elements. ∎

Remark. The spaces of orderings $(X_\mathfrak{p}, G_\mathfrak{p})$, $\mathfrak{p} \neq \{0\}$ obviously have stability index 0 or 1. (If $\mathfrak{p} = (x)$ or (y), the stability index is 1; otherwise it is 0.) Thus, by Theorem 6.2 (2), we are in a position to apply the results in [16, Chap. 7] to get minimal generation results for the real spectrum of

$\mathbb{R}[x, y]_t$ exactly as in the commutative case. In more detail, using [16, Corollary 7.2.4, Theorem 7.4.1, and Theorem 7.7.5], we have the following:

—Any basic open set in X is defined by 2 inequalities $f > 0$, $g > 0$.

—Any basic closed set in X is defined by 3 inequalities $f \geq 0$, $g \geq 0$, $h \geq 0$.

—Any constructible set in X is expressible as a union of 4 basic sets.

—Any constructible set in X has a separating family consisting of 5 elements of $\mathbb{R}[x, y]_t$.

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