On local uniformization in arbitrary characteristic I

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23. 10. 1998

Abstract

We prove that every place of an algebraic function field $F|K$ of arbitrary characteristic admits local uniformization in a finite extension $\mathcal{F}$ of F. We show that $\mathcal{F}|F$ can be chosen to be normal. If K is perfect and P is of rank 1, then alternatively, $\mathcal{F}$ can be obtained from F by at most two Galois extensions; if in addition P is zero-dimensional, then we only need one Galois extension. Certain rational places of rank 1 can be uniformized already on F. We introduce the notion of "relative uniformization" for arbitrary finitely generated extensions of valued fields. Our proofs are based solely on valuation theoretical theorems, which are of fundamental importance in positive characteristic.

1 Introduction

In [Z1], Zariski proved the Local Uniformization Theorem for places of algebraic function fields over base fields of characteristic 0. In [Z3], he uses this theorem to prove resolution of singularities for surfaces in characteristic 0. As the resolution of singularities for arbitrary dimensions in positive characteristic is still an open problem, one is interested in generalizations of the Local Uniformization Theorem to positive characteristic (cf. [S]). In this paper, we will prove a weak version of the Local Uniformization Theorem, for function fields of arbitrary characteristic:

Theorem 1.1 *Let $F|K$ be a function field of arbitrary characteristic, and P a place of $F|K$. Then there exist a finite normal extension $\mathcal{F}$ of F, an extension of P from F to $\mathcal{F}$, a finite purely inseparable extension $\mathcal{K}$ of K within $\mathcal{F}$ and a model of $\mathcal{F}|\mathcal{K}$ on which P is centered in a smooth point.*

Throughout this paper, "function field" will always mean "algebraic function field". By a place of $F|K$ we mean a place whose restriction to K is the identity. Talking of an extension $\mathcal{F}$ of F, we will from now on tacitly assume that it is equipped with an extension of P (which is again denoted by P).

Theorem 1.1 follows from the results of J. de Jong [dJ] (who proves resolution of singularities after a finite normal extension of the function field). However, we will give an entirely valuation theoretical proof which will provide important additional information. In particular, we wish to get as close as possible to taking $\mathcal{F}|F$ Galois. Why do we want that $\mathcal{F}|F$ is Galois? Apart from geometrical reasons, it is because the work of

S. Abhyankar seems to indicate that there is a chance to "pull down" local uniformization through Galois extensions. This would give us what we actually want: local uniformization without extending the function field.

On the other hand, for certain applications of Theorem 1.1 (e.g., to the model theory of fields in the spirit of [J–R]; cf. also [K3]), it is important to have a valuation theoretical control on the extension $\mathcal{F}|F$ and the residue field extension $\mathcal{F}P|FP$. (We want to have $\mathcal{F}P$ to be as close to FP as possible, but in positive characteristic we may expect that we have to take a purely inseparable extension into the bargain.) We cannot obtain this control if we insist that $\mathcal{F}|F$ be Galois. Instead, we will show in a subsequent paper [K6] that in case of a perfect base field K, the extension $\mathcal{F}|F$ can always be chosen to be separable and such that $\mathcal{F}P|FP$ is purely inseparable. See also [K7] for background information.

In order to prove Theorem 1.1, we will give a description of the special form in which the Jacobian condition for smoothness can be satisfied. For polynomials $f_1,\ldots,f_n$ in variables $X_1,\ldots,X_n$, we write $f=(f_1,\ldots,f_n)$ and denote by J_f the Jacobian matrix $\left(\frac{\partial f_i}{\partial X_j}\right)_{i,j}$ of their partial derivatives. Take a finitely generated extension $\mathcal{F}|\mathcal{K}$, not necessarily transcendental, and a place P of $\mathcal{F}$, not necessarily the identity on $\mathcal{K}$. We write $\mathcal{O}_\mathcal{F}$ for the valuation ring of P on $\mathcal{F}$, and $\mathcal{O}_\mathcal{K}$ for that of P on $\mathcal{K}$. For an element a, its P-residue will be denoted by aP. We will say that $(\mathcal{F}|\mathcal{K},P)$ **is weakly uniformizable** if there are

- a transcendence basis $T=\{t_1,\ldots,t_s\}\subset\mathcal{O}_\mathcal{F}$ of $\mathcal{F}|\mathcal{K}$ (which may be empty),
- elements $\eta_1,\ldots,\eta_n\in\mathcal{O}_\mathcal{F}$,
- polynomials $f_i(X_1,\ldots,X_n)\in\mathcal{O}_\mathcal{K}[t_1,\ldots,t_s,X_1,\ldots,X_n]$, $1\le i\le n$,

such that $\mathcal{F}=\mathcal{K}(t_1,\ldots,t_s,\eta_1,\ldots,\eta_n)$, and

(U1) for $i<j$, X_j does not occur in f_i,
(U2) $f_i(\eta_1,\ldots,\eta_n)=0$ for $1\le i\le n$,
(U3) $(\det J_f(\eta_1,\ldots,\eta_n))P\neq 0$.

In this case, we will call T a **uniformizing transcendence basis**.

Assertion (U1) implies that J_f is lower triangular. Assertion (U3) says that

$$\det J_{fP}(\eta_1P,\ldots,\eta_nP) \;=\; (\det J_f(\eta_1,\ldots,\eta_n))P\;\neq\;0\;, \tag{1}$$

where $fP=(f_1P,\ldots,f_nP)$ and f_iP denotes the P-reduction of f_i, i.e., the polynomial obtained from f_i through replacing every coefficient by its P-residue. Note that $\det J_f(\eta_1,\ldots,\eta_n)\in\mathcal{O}_\mathcal{F}$ since $t_1,\ldots,t_s,\eta_1,\ldots,\eta_n\in\mathcal{O}_\mathcal{F}$ and the f_i have coefficients in $\mathcal{O}_\mathcal{K}$.

Given elements $\zeta_1,\ldots,\zeta_m\in\mathcal{O}_\mathcal{F}$, we will say that $(\mathcal{F}|\mathcal{K},P)$ **is uniformizable with respect to** $\zeta_1,\ldots,\zeta_m$ if the elements $\eta_1,\ldots,\eta_n$ can be chosen such that $\zeta_1,\ldots,\zeta_m$ appear among them. We say that $(\mathcal{F}|\mathcal{K},P)$ **is uniformizable** if it is uniformizable with respect to *every* choice of finitely many elements in $\mathcal{O}_\mathcal{F}$. This property is transitive:

Theorem 1.2 *If $(\mathcal{F}|\mathcal{L},P)$ and $(\mathcal{L}|\mathcal{K},P)$ are uniformizable, then $(\mathcal{F}|\mathcal{K},P)$ is uniformizable.*

Note the following well-known fact: if all conditions except (U1) for weak uniformizability hold for g_i in the place of f_i, then there are f_i which satisfy all conditions (for the same

T and η's). We only have included condition (U1) since it is a nice natural side-effect of our approach which uses the transitivity.

Now assume that $\mathcal{F}|\mathcal{K}$ is a function field (i.e., $\mathrm{trdeg}\,\mathcal{F}|\mathcal{K} \geq 1$) and that P is the identity on $\mathcal{K}$. Then $\mathcal{O}_{\mathcal{K}} = \mathcal{K}$, and the P-residues of the coefficients are obtained by just replacing t_j by t_jP, for $1 \leq j \leq n$. Hence if we view the polynomials f_i as polynomials in the variables $Z_1, \ldots, Z_s, X_1, \ldots, X_n$ with coefficients in $\mathcal{K}$, then assertion **(U3)** means that the Jacobian matrix at the point $(t_1P, \ldots, t_sP, \eta_1P, \ldots, \eta_nP)$ has maximal rank. This assertion says that on the variety defined over $\mathcal{K}$ by the f_i (having generic point $(t_1, \ldots, t_s, \eta_1, \ldots, \eta_n)$ and function field $\mathcal{F}$), the place P is centered at the smooth point $(t_1P, \ldots, t_sP, \eta_1P, \ldots, \eta_nP)$.

This discussion shows that Theorem 1.1 is a consequence of the following two theorems:

Theorem 1.3 *Let $F|K$ be a function field of arbitrary characteristic and P a place of $F|K$. Take any elements $\zeta_1, \ldots, \zeta_m$ in the valuation ring $\mathcal{O}_F$ of P on F. Then there exist a finite extension $\mathcal{F}$ of F, an extension of P to $\mathcal{F}$, and a finite purely inseparable extension $\mathcal{K}$ of K within $\mathcal{F}$ such that $(\mathcal{F}|\mathcal{K}, P)$ is uniformizable with respect to $\zeta_1, \ldots, \zeta_m$.*

(Here, "$\mathcal{K}.F$" denotes the field compositum of $\mathcal{K}$ and F inside of $\mathcal{F}$, i.e., the smallest subfield of $\mathcal{F}$ containing $\mathcal{K}$ and F.) Note that $\mathcal{K} = K$ if K is perfect.

Theorem 1.4 *Take any subextension $E|K$ of $F|K$ of the same transcendence degree. Then in addition to the assertion of Theorem 1.3, $\mathcal{F}$ can always be chosen to be a normal extension of E and of F.*

By uniformizing with respect to the ζ's, we obtain the following important information: if we have already a model V of $F|K$ with generic point $(z_1, \ldots, z_k)$, where $z_1, \ldots, z_k \in \mathcal{O}_F$, then we can choose our new model $\mathcal{V}$ of $\mathcal{F}|\mathcal{K}$ in such a way that the local ring of the center of P on $\mathcal{V}$ contains the local ring of the center $(z_1P, \ldots, z_kP)$ of P on V. For this, we only have to let $z_1, \ldots, z_k$ appear among the ζ's.

In important special cases, we can show much stronger results. Before we state them, let us introduce some useful notions. Let P be an arbitrary place on a field F. We will call (F, P) a **valued field**, keeping in mind its associated valuation, which we denote by v_P . Its value group is denoted by v_PF, and its residue field by FP. When we write $(F|K, P)$ then we mean an extension of valued fields, that is, P is a place on L, and K is endowed with its restriction (which we will also denote by P). This restriction need not be the identity.

If $F|K$ is a function field and P is a place of $F|K$, then $K \subseteq FP$. By the **dimension of** P we mean the transcendence degree $\mathrm{trdeg}\,FP|K$. Hence, P is called **zero-dimensional** if $FP|K$ is algebraic. We will say that (F, P) has **rank 1** if v_PF is archimedean ordered, that is, embeddable in the ordered additive group of the reals. For the general definition of the **rank**, see Section 2.2.

Theorem 1.5 *Assume that (F, P) has rank 1, and take a subextension $E|K$ of $F|K$ such that $F|E$ is separable-algebraic. Suppose that P is zero-dimensional. Then in addition to the assertion of Theorem 1.3, $\mathcal{F}$ can be chosen such that $\mathcal{F}|\mathcal{K}.E$ and $\mathcal{F}|\mathcal{K}.F$ are Galois extensions.*

For rank 1 places of non-zero dimension, we can still prove:

Theorem 1.6 *Assume that (F,P) has rank 1. Then in addition to the assertion of Theorem 1.3, $\mathcal{F}$ can be obtained from $\mathcal{K}.F$ by at most two Galois extensions.*

Next, we will discuss some important special cases, in particular those where we can uniformize without extending the function field. For every place P of $F|K$, we have the following inequality (we will introduce a more general inequality (4) later):

$$\operatorname{trdeg} F|K \;\geq\; \operatorname{trdeg} FP|K \,+\, \dim_{\mathbb{Q}} \mathbb{Q}\otimes v_P F\,. \tag{2}$$

This is a special case of the Abhyankar inequality. Note that $\dim_{\mathbb{Q}} \mathbb{Q}\otimes v_P F$ is the **rational rank** of the value group $v_P F$, i.e., the maximal number of rationally independent elements in $v_P F$. We call P an **Abhyankar place** if equality holds in (2). An arbitrary place P of a function field $F|K$ is called **rational** if $FP=K$.

Theorem 1.7 *Assume that P is an Abhyankar place of $F|K$ and that (F,P) has rank 1. Then $\mathcal{F}$ can always be obtained from $F.\mathcal{K}$ by a single Galois extension, and the following additional assertions hold:*
a) If $FP|K$ is separable, then we can choose $\mathcal{K}=K$.
b) If P is zero-dimensional, then there is a finite Galois extension $\mathcal{K}'|\mathcal{K}$ such that we can set $\mathcal{F}=F.\mathcal{K}'$, that is, $\mathcal{F}$ can be obtained from F by a normal constant extension.
c) If P is rational, then we can choose $\mathcal{F}=F$ and $\mathcal{K}=K$.

Let us note that if FP can be embedded over K in a trivially valued subfield of F, then we may replace K by the image of FP. In this way, this more general case is subsumed under the case of rational places. For example, if $FP|K$ is a rational function field, then FP can always be embedded in such a way. On the other hand, if $FP|K$ is separable, then FP can always be embedded in the henselization of (F,P): choose a separating transcendence basis, embed the rational function field generated by it in a trivially valued subfield of F, and extend the embedding by Hensel's Lemma. As an algebraic extension of a trivially valued field is again trivially valued, the same will hold for the image of FP in F. Hence if $FP|K$ is a function field, then we only have to take a finite extension of F within its henselization in order to reduce to the case of rational places.

There is yet another interesting particular case. At first sight, it seems to be completely opposed to the case of Abhyankar places; but see Theorem 1.12 below and the subsequent remark. A valued field (F,P) is called **discretely valued** and P is called **discrete** if $v_P F\simeq\mathbb{Z}$.

Theorem 1.8 *Assume that P is a rational discrete place of $F|K$. Then we can choose $\mathcal{F}=F$ and $\mathcal{K}=K$.*

The assertions of this theorem and of part c) of Theorem 1.7 can also be formulated as follows:

Corollary 1.9 *On a function field $F|K$, all rational rank 1 Abhyankar places and all rational discrete places are uniformizable.*

In [K3], we show that the zero-dimensional rank 1 Abhyankar places, as well as the zero-dimensional discrete places, lie dense in the Zariski space of all places of $F|K$, with respect to a "Zariski patch topology". This topology is finer than the Zariski topology (but still compact); its basic open sets are the sets of the form

$$\{P \mid P \text{ a place of } F|K \text{ such that } a_1P \neq 0, \ldots, a_kP \neq 0\,;\, b_1P = 0, \ldots, b_\ell P = 0\}$$

with $a_1, \ldots, a_k, b_1, \ldots, b_\ell \in F \setminus \{0\}$.

If K is algebraically closed, then every zero-dimensional place of $F|K$ is rational. So Theorem 1.7 shows (and the first assertion also follows from Theorem 1.8):

Corollary 1.10 *If K is algebraically closed, then the uniformizable places of $F|K$ lie dense in the Zariski space of $F|K$, with respect to the Zariski patch topology. If K is perfect, then the same holds for the places of $F|K$ which are uniformizable with respect to given elements of F after a finite Galois constant extension of F.*

Now we turn to places of arbitrary rank.

Theorem 1.11 *Assume that P is an Abhyankar place of rank $r > 1$ of the function field $F|K$. Then we can obtain $\mathcal{F}$ from $F.\mathcal{K}$ by a sequence of at most $r-1$ Galois extensions if P is zerodimensional, or at most r Galois extensions otherwise.*

In [K6] we will show that $(F|K, P)$ is always weakly uniformizable if P is an Abhyankar place of $F|K$ for which $FP|K$ is separable. But we do not obtain that they are uniformizable in general.

For non-Abhyankar places of arbitrary rank, we are not able to prove that one can obtain $\mathcal{F}$ by Galois extensions. The obstruction is, roughly speaking, that we work with extensions in henselizations, but that taking normal hulls of such extensions may lead to inseparable residue field extensions. But we will show in [K6] that $\mathcal{F}$ can be taken such that it differs from a Galois extension of $F.\mathcal{K}$ only by an extension in the henselization.

The construction of places given in [K3] yields Abhyankar places or, if so desired, non-Abhyankar places which are still "very close to" Abhyankar places: they lie in the completion of a subfield on which their restriction is an Abhyankar place. Therefore, it is important to know that also such places are uniformizable. By "completion" we mean the completion with respect to the uniformity induced by the valuation. Note that (F', P) lies in the completion of (F, P) if it is an extension of (F, P) satisfying that for every $a \in F'$ and $\alpha \in v_PF'$ there is some $b \in F$ such that $v_P(a-b) \geq \alpha$.

Theorem 1.12 *If $(F|K, P)$ satisfies the assumptions of Theorem 1.7 or Theorem 1.11, then the assertions of these theorems carry over to every function field $(F'|K, P)$ for which (F', P) lies in the completion of (F, P).*

Remark 1.13 Theorem 1.8 follows directly from this theorem together with part c) of Theorem 1.7. Indeed, if P is a rational discrete place of $F|K$ and we choose $x \in F$ such that v_Px is the smallest positive element in v_PF, then (the restriction of) P is a rational Abhyankar palce of $K(x)|K$ and F lies in the completion of $(K(x), P)$.

In characteristic 0, Theorem 1.3 is obviously weaker than Zariski's original result. On the other hand, our proof will yield an interesting additional assertion. In general, it seems impossible to obtain it without taking into the bargain a finite extension of the function field (see the example given in [K5], [K6]). Let us consider a place P of the function field $E|K$. We set $\rho = \dim_{\mathbb{Q}} \mathbb{Q} \otimes v_P E$ and $\tau = \operatorname{trdeg} EP|K$. We take elements $x_1, \ldots, x_\rho \in E$ such that $v_P x_1, \ldots, v_P x_\rho$ are rationally independent elements in $v_P E$. Further, we take elements $y_1, \ldots, y_\tau \in E$ such that $y_1 P, \ldots, y_\tau P$ are algebraically independent over K. Then $x_1, \ldots, x_\rho, y_1, \ldots, y_\tau$ are algebraically independent over K (cf. Theorem 2.6) and therefore, $\rho + \tau \leq \operatorname{trdeg} E|K$. Every subfield $K(x_1, \ldots, x_\rho, y_1, \ldots, y_\tau)$ of E obtained in this way will be called an **Abhyankar field of** $(E|K, P)$. Note that if P is a place of a function field $F|K$ and $E|K$ is a subextension such that $F|E$ is algebraic, then an Abhyankar field of $(E|K, P)$ will also be an Abhyankar field of $(F|K, P)$.

Theorem 1.14 *Assume the situation as given in Theorems 1.3 and 1.4. Suppose in addition that (E, P) has rank 1, and take any Abhyankar field E_0 of $(E|K, P)$. Then in addition to the assertion of Theorem 1.3 and Theorem 1.4, $t_1, \ldots, t_{\rho+\tau}$ can be chosen algebraic over E_0.*

We do not know whether this theorem can be proved for arbitray rank if one insists in taking normal or Galois extensions. See [K5] and [K6] for other versions.

To describe a necessary condition for $\mathcal{F}$ to be equal to F, we need some further definitions. For an arbitrary valued field (F, P) and a given extension of P from F to its separable-algebraic closure F^{sep}, the **absolute inertia field** is defined to be the inertia field of the normal extension $(F^{\mathrm{sep}}|F, P)$. The decomposition field of $(F^{\mathrm{sep}}|F, P)$ is the **henselization of** (F, P) **in** (F^{sep}, P); we will denote it by (F^h, P). A valued function field $(F|K, P)$ will be called **inertially generated** if it admits a transcendence basis T such that (F, P) lies in the absolute inertia field of $(K(T), P)$ (for some extension of P from $K(T)$ to $K(T)^{\mathrm{sep}}$). If it admits a transcendence basis T such that (F, P) lies in the henselization of $(K(T), P)$, then we call it **henselian generated**.

Theorem 1.15 *Assume that $(\mathcal{F}|\mathcal{K}, P)$ is weakly uniformizable (where P is not necessarily the identity on $\mathcal{K}$). Then $(\mathcal{F}|\mathcal{K}, P)$ is inertially generated. In particular, $\mathcal{F}|\mathcal{K}$ is separable. If in addition $\mathcal{F}P = \mathcal{K}$, then $(\mathcal{F}|\mathcal{K}, P)$ is even henselian generated.*

OPEN PROBLEM: Is every inertially generated valued function field weakly uniformizable?

We will deduce Theorems 1.3 through 1.14 from two main theorems which we proved in [K1] (cf. also [K2]).

The first theorem is a generalization of the "Grauert–Remmert Stability Theorem". To state it, we introduce a fundamental notion. Every finite extension $(L|K, P)$ of valued fields satisfies the **fundamental inequality** (cf. [EN], [R], [Z–S] or [K2]):

$$n \;\geq\; \sum_{i=1}^{\mathrm{g}} \mathrm{e}_i \mathrm{f}_i \tag{3}$$

where $n = [L:K]$ is the degree of the extension, $P_1, \ldots, P_{\rm g}$ are the distinct extensions of P from K to L, ${\rm e}_i = (v_{P_i}L : v_PK)$ are the respective ramification indices and ${\rm f}_i = [LP_i : KP]$ are the respective inertia degrees. Note that ${\rm g} = 1$ if (K,P) is henselian.

A valued field (K,P) is called **defectless** (or **stable**) if equality holds in (3) for every finite extension $L|K$. If $\mathrm{char}\, KP = 0$, then (K,P) is defectless (this is a consequence of the "Lemma of Ostrowski", cf. [EN], [R], [K2]).

Now let $(L|K,P)$ be any extension of valued fields (we do not require that P be the identity on K). Assume that $L|K$ has finite transcendence degree. Then (by Corollary 2.7 below):

$$\mathrm{trdeg}\, L|K \;\geq\; \mathrm{trdeg}\, LP|KP \,+\, \dim_{\mathbb{Q}} \mathbb{Q} \otimes (v_PL/v_PK)\;. \tag{4}$$

Note that (2) is the special version of (4) for the case of a valued function field with trivially valued base field. We will say that $(L|K,P)$ is **without transcendence defect** if equality holds in (4). Now we are able to state the **Stability Theorem**, which deals with an arbitrary valued function field $(F|K,P)$ (where P need not be the identity on K).

Theorem 1.16 *Let $(F|K,P)$ be a valued function field without transcendence defect. If (K,P) is a defectless field, then also (F,P) is a defectless field.*

The second theorem is a structure theorem for immediate function fields. An extension $(L|K,P)$ is called **immediate** if the canonical embedding of v_PK in v_PL and the canonical embedding of KP in LP are surjective (we then write $v_PK = v_PL$ and $KP = LP$). In this paper, we will only need a special case of the theorem. We will state and employ the full theorem in [K6].

Theorem 1.17 *Let K be a separable-algebraically closed field and $(F|K,P)$ an immediate function field of transcendence degree 1. If $F|K$ is separable, then*

$$\text{there is } x \in F \text{ such that } \;(F^h,P) \,=\, (K(x)^h,P)\;, \tag{5}$$

that is, $(F|K,P)$ is henselian generated.

For valued fields of residue characteristic 0, the assertion is a direct consequence of the fact that every such field is defectless (in fact, every $x \in F \setminus K$ will then do the job). In contrast to this, the case of positive residue characteristic requires a much deeper structure theory of immediate algebraic extensions of henselian fields, in order to find suitable elements x.

In Chapter 2, we introduce some further valuation theoretical tools, including a part of Kaplansky's theory of immediate extensions, which will also play a crucial role in our proofs. In Chapter 3, we give a criterion for valued function fields to be inertially generated, and prove Theorem 1.15. In Chapter 4, we prove Theorem 1.2. This transitivity result allows us to build up our function fields by various sorts of algebraic and transcendental extensions which all can be shown separately (in Chapter 5) to be uniformizable. Finally, in Chapters 6, 7 and 8 we put everything together to prove our main theorems.

I would like to thank Peter Roquette, Mark Spivakovsky, Bernard Teissier and Frans Oort for support and inspiring conversations, and the staff of the Fields Institute for their hospitality. Very special thanks to Hans Schoutens for many pleasant and encouraging discussions.

2 Valuation theoretical preliminaries

For basic facts from valuation theory, see [EN], [R], [W], [Z–S], [K2].

We will denote the algebraic closure of a field K by $\tilde{K}$. Whenever we have a place P on K, we will automatically fix an extension of P to the algebraic closure $\tilde{K}$ of K. It does not play a role which extension we choose, except if P is also given on an extension field L of K; in this case, we choose the extension to $\tilde{K}$ to be the restriction of the extension to $\tilde{L}$. We say that P is **trivial** on K if it is an isomorphism of K, which is equivalent to $v_P K = \{0\}$. If P is given on some extension field L of K and is trivial on K, then there is some place P' of L which is equivalent to P (i.e., they have the same valuation ring on L) and whose restriction to K is the identity.

A valued field is **henselian** if it satisfies Hensel's Lemma; see [R], [W], [K2]. Originally, Hensel's Lemma was proved for complete discretely valued fields. But it also holds for complete valued fields of rank 1; i.e., such fields are henselian (cf. [W], Theorem 32.11, or [K2]). The henselization (K^h, P) of a valued field (K, P) (in (K^{sep}, P)) is the minimal henselian extension of (K, P), in the following sense: if (L, P') is a henselian extension field of (K, P), then there is a unique embedding of (K^h, P) in (L, P'). This is the **universal property of the henselization**. We note that every algebraic extension of a henselian field is again henselian. In particular, since the absolute inertia field of an arbitrary valued field contains its henselization, it is henselian.

The following lemma is proved in [K2] (and partially also in [EN]):

Lemma 2.1 *A valued field (K, P) is defectless if and only if its henselization (K^h, P) is.*

Corollary 2.2 *If (K, P) is defectless, then (K^h, P) does not admit proper immediate algebraic extensions.*

Proof: If (K, P) is defectless, then so is (K^h, P), by the foregoing lemma. Suppose that $(L|K^h, P)$ is a finite immediate algebraic extension. Hence, $(v_P L : v_P K^h) = 1 = [LP : K^h P]$. Since (K^h, P) is a henselian field, there is a unique extension of v_P from K^h to L. Since (K^h, P) is defectless, we have that $[L : K^h] = (v_P L : v_P K^h)[LP : K^h P] = 1$, showing that $L = K^h$. As every proper immediate extension would contain a proper finite immediate extension, it follows that (K^h, P) does not admit any proper immediate algebraic extension. $\square$

We also note (see [K2] for the easy proof):

Lemma 2.3 *If K is an arbitrary field and P is a place on K^{sep}, then $v_P K^{\mathrm{sep}}$ is the divisible hull of $v_P K$, and $(KP)^{\mathrm{sep}} \subseteq K^{\mathrm{sep}} P$. If in addition P is non-trivial on K, then $K^{\mathrm{sep}} P$ is the algebraic closure of KP.*

Lemma 2.4 *Let P be a place of $F|K$ and suppose that E is a subfield of F on which P is trivial. Let (F^i, P) denote the absolute inertia field of (F, P). Then $E^{\mathrm{sep}} \subset F^i$. Further, if $FP|EP$ is algebraic, then $(F.E^{\mathrm{sep}})P$ is the separable-algebraic closure of FP.*

Proof: By assumption, P induces an embedding of E in FP. Further, we know by ramification theory ([EN], [K2]) that $F^i P$ is separable-algebraically closed. Thus,

$(EP)^{\mathrm{sep}} \subset F^iP$. Using Hensel's Lemma, one shows that the inverse of the isomorphism $P|_E$ can be extended from EP to an embedding of $(EP)^{\mathrm{sep}}$ in F^i. Its image is separable-algebraically closed and contains E. Hence, $E^{\mathrm{sep}} \subset F^i$. Further, $(F.E^{\mathrm{sep}})P$ contains $E^{\mathrm{sep}}P$, which by Lemma 2.3 contains $(EP)^{\mathrm{sep}}$. As $F.E^{\mathrm{sep}}|F$ is algebraic, so is $(F.E^{\mathrm{sep}})P|FP$. Therefore, if $FP|EP$ is algebraic, then $(F.E^{\mathrm{sep}})P$ is algebraic over $(EP)^{\mathrm{sep}}$ and hence separable-algebraically closed. Since $(F.E^{\mathrm{sep}})P \subset F^iP = (FP)^{\mathrm{sep}}$, it follows that $(F.E^{\mathrm{sep}})P = (FP)^{\mathrm{sep}}$. □

Lemma 2.5 *Take any extension $L|K$ of valued fields and assume that P is a place of L which is trivial on K. If $LP|K$ is separable, then also $L|K$ is separable.*

Proof: If $LP|K$ is separable and P' is equivalent to P, then also $LP'|K$ is separable; thus, we can assume that the restriction of P to K is the identity. Take a finite purely inseparable extension $K'|K$; we have to show that it is linearly disjoint from $L|K$. As P is the identity on K and $K'|K$ is purely inseparable, P is also the identity on K'. Hence, $K' \subset (L.K')P$. It follows that

$$[K':K] \;\geq\; [L.K':L] \;\geq\; [(L.K')P:LP] \;\geq\; [LP.K':LP] \;=\; [K':K]\;,$$

where the last equality holds since $LP|K$ is separable by assumption. Hence, equality must hold everywhere, showing that $[K':K] = [L.K':L]$, i.e., $K'|K$ is linearly disjoint from $L|K$. □

A generalization of this lemma to the case of P not being trivial on K is stated in [K1].

2.1 Valuation independence

For the easy proof of the following theorem, see [B], Chapter VI, §10.3, Theorem 1, or [K2].

Theorem 2.6 *Let $(L|K,P)$ be an extension of valued fields. Take elements $x_i, y_j \in L$, $i \in I$, $j \in J$, such that the values v_Px_i, $i \in I$, are rationally independent over v_PK, and the residues y_jP, $j \in J$, are algebraically independent over KP. Then the elements x_i, y_j, $i \in I$, $j \in J$, are algebraically independent over K.*

Moreover, if we write

$$f \;=\; \sum_k c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \in K[x_i, y_j \mid i\in I, j\in J]$$

in such a way that for every $k \neq \ell$ there is some i s.t. $\mu_{k,i} \neq \mu_{\ell,i}$ or some j s.t. $\nu_{k,j} \neq \nu_{\ell,j}$, then

$$v_Pf \;=\; \min_k \, v_P\, c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \;=\; \min_k \left(v_P\, c_k \,+\, \sum_{i\in I} \mu_{k,i} v_P x_i \right)\;.$$

That is, the value of the polynomial f is equal to the least of the values of its monomials. In particular, this implies:

$$\begin{aligned} v_P K(x_i, y_j \mid i \in I, j \in J) &= v_P K \oplus \bigoplus_{i\in I} \mathbb{Z} v_P x_i \\ K(x_i, y_j \mid i \in I, j \in J)P &= KP\,(y_j P \mid j \in J)\,. \end{aligned}$$

It also implies that the valuation v_P on $K(x_i, y_j \mid i \in I, j \in J)$ is uniquely determined by its restriction to K, the values $v_P x_i$ and the residues $y_j P$.

Corollary 2.7 *Let $(L|K, P)$ be an extension of valued fields of finite transcendence degree. Then (4) holds. If in addition $L|K$ is a function field, and if equality holds in (4), i.e., $(L|K, P)$ is a valued function field without transcendence defect, then the extensions $v_P L|v_P K$ and $LP|KP$ are finitely generated. In particular, if P is trivial on K, then $v_P L$ is a product of finitely many copies of $\mathbb{Z}$, and LP is again a function field over K.*

Proof: Choose elements $x_1, \ldots, x_\rho, y_1, \ldots, y_\tau \in L$ such that the values $v_P x_1, \ldots, v_P x_\rho$ are rationally independent over $v_P K$ and the residues $y_1 P, \ldots, y_\tau P$ are algebraically independent over KP. Then by the foregoing lemma, $\rho + \tau \leq \operatorname{trdeg} L|K$. This proves that $\operatorname{trdeg} LP|KP$ and the rational rank of $v_P L/v_P K$ are finite. Therefore, we may choose the elements x_i, y_j such that $\tau = \operatorname{trdeg} LP|KP$ and $\rho = \dim_{\mathbb{Q}} \mathbb{Q} \otimes (v_P L/v_P K)$ to obtain inequality (4).

Assume that this is an equality. This means that for $L_0 := K(x_1, \ldots, x_\rho, y_1, \ldots, y_\tau)$, the extension $L|L_0$ is algebraic. Since $L|K$ is finitely generated, it follows that $L|L_0$ is finite. By the fundamental inequality, this yields that $v_P L|v_P L_0$ and $LP|L_0 P$ are finite extensions. Since already $v_P L_0|v_P K$ and $L_0 P|KP$ are finitely generated by the foregoing theorem, it follows that also $v_P L|v_P K$ and $LP|KP$ are finitely generated. □

Lemma 2.8 *Take a function field $F|K$ and a place P of $F|K$. If $P = Q\overline{Q}$, then*

$$\dim_{\mathbb{Q}} \mathbb{Q} \otimes v_P F \;=\; \dim_{\mathbb{Q}} \mathbb{Q} \otimes v_Q F \,+\, \dim_{\mathbb{Q}} \mathbb{Q} \otimes v_{\overline{Q}} FQ\;.$$

Further, P is an Abhyankar place if and only if Q and $\overline{Q}$ are.

Proof: The first assertion is well-known. The second assertion follows from the first, keeping in mind that $FP = (FQ)\overline{Q}$. We leave the straightforward proofs to the reader.

□

2.2 The rank

The **rank of an ordered abelian group** Γ is the order type of the chain of its proper convex subgroups. By a theorem of Hölder, Γ is embeddable in the ordered additive group of the reals if and only if its rank is 1, that is, the only proper convex subgroup is $\{0\}$. This in turn holds if and only if Γ is **archimedean ordered**, i.e., for every two positive elements $\alpha, \beta \in \Gamma$ there is some $n \in \mathbb{N}$ such that $n\alpha \geq \beta$.

The **rank of** a valued field (F,P) is defined to be the rank of its value group v_PF. If this is finite, say n, then P is the composition of n places: $P = P_1P_2\ldots P_n$, where all P_i have value groups of rank 1 (cf. [Z–S], [K2]).

If Γ_1 is a subgroup of Γ, then its divisible hull $\mathbb{Q}\otimes\Gamma_1$ lies in the convex hull of Γ_1 in $\mathbb{Q}\otimes\Gamma$. Hence if Γ_1 is a proper convex subgroup of Γ, then $\mathbb{Q}\otimes\Gamma_1$ is a proper convex subgroup of $\mathbb{Q}\otimes\Gamma$ and thus, $\dim_{\mathbb{Q}}\mathbb{Q}\otimes\Gamma_1 < \dim_{\mathbb{Q}}\mathbb{Q}\otimes\Gamma$. It follows that if $\{0\}=\Gamma_0 \subsetneqq \Gamma_1 \subsetneqq \ldots \subsetneqq \Gamma_n = \Gamma$ is a chain of convex subgroups of Γ, then $\dim_{\mathbb{Q}}\mathbb{Q}\otimes\Gamma \geq n$. In view of (2), this proves that the rank of a place P of a function field $F|K$ cannot exceed $\mathrm{trdeg}\,F|K$ and thus is finite.

If $(F'|F,P)$ is an algebraic extension of valued fields, then v_PF'/v_PF is a torsion group and $F'P|FP$ is algebraic (this is a consequence of Theorem 2.6). In particular, the rank of v_PF' is equal to that of v_PF. Hence, passing to an algebraic extension does not change the rank of a valued field. As the rank of an ordered abelian group does not increase by passing to a subgroup, the rank of a valued field does not increase by passing to a subfield.

2.3 Kaplansky approximation

The material of this section is based on work by Ostrowski and Kaplansky [KA] (cf. also [K2]). The proof of the first lemma is an easy exercise.

Lemma 2.9 *The extension $(L|K,P)$ is immediate if and only if for every $z\in L$, the set $\{v_P(z-a)\mid a\in K\}$ has no maximal element.*

Lemma 2.10 *Let $(K(z)|K,P)$ be an immediate transcendental extension. Assume that (K,P) is a separable-algebraically closed field or that $(K(z),P)$ lies in the completion of (K,P). Take any polynomial $f\in K[X]$. Then the value $v_Pf(a)$ is fixed for all $a\in K$ sufficiently close to z. That is,*

$$\begin{gathered}\forall f\in K[X]\ \exists\alpha\in v_PK\ \exists\beta\in\{v_P(z-b)\mid b\in K\}\ \forall a\in K:\\ v_P(z-a)\geq\beta\ \Rightarrow\ v_Pf(a)=\alpha\ .\end{gathered}\tag{6}$$

Kaplansky proves that if (6) does not hold, then there is a proper immediate algebraic extension of (K,P). If $(K(z),P)$ does not lie in the completion of (K,P), then this can be transformed into a proper immediate separable-algebraic extension ([K1], [K2]; the proof uses a variant of the Theorem on the Continuity of Roots). But such an extension cannot exist if we assume that K be separable-algebraically closed. If on the other hand $(K(z),P)$ lies in the completion of (K,P), then one can show that if f does not satisfy (6), then $v_Pf(z)=\infty$. But this means that $f(z)=0$, contradicting the assumption that $K(z)|K$ is transcendental.

For a polynomial f in one variable over a field of arbitrary characteristic, we denote by $f^{[i]}$ its i-th formal derivative (cf. [KA], [K2]). These polynomials are defined such that the following Taylor expansion holds in arbitrary characteristic:

$$f(z)\ =\ f(a)+\sum_{i=1}^{\deg f} f^{[i]}(a)(z-a)^i\ .$$

Lemma 2.11 *Assume that (6) holds, and take any polynomial $f \in K[z]$. Then there are $a, b \in K$ such that for $\tilde{z} := \frac{z-a}{b} \in K[z]$, we have that $v_P\tilde{z} = 0$ and the values of the non-zero among the elements $f^{[i]}(a)b^i \in K$ are all distinct. With such a and b,*

$$v_P f(z) \;=\; v_P \sum_{i=0}^{\deg f} f^{[i]}(a)(z-a)^i \;=\; v_P \sum_{i=0}^{\deg f} f^{[i]}(a)b^i\tilde{z}^i \;=\; \min_i v_P f^{[i]}(a)b^i \,. \tag{7}$$

If finitely many polynomials in $K[z]$ are given, then a, b can be chosen such that (7) holds simultaneously for all of them.

Proof: Take finitely many polynomials $f_1, \ldots, f_n \in K[z]$. From Lemma 2.10 we know that for all $a \in K$ close enough to z, the values $v_P f_j^{[i]}(a)$ of the non-zero among the polynomials $f_j^{[i]}$, $i, j \in \mathbb{N}$, are fixed. Since by Lemma 2.9 the set $\{v_P(z-a) \mid a \in K\}$ has no maximal element, we can then take a so close to z that for every fixed j, the values of all non-zero elements $f_j^{[i]}(a)(z-a)^i$, $i \in \mathbb{N}$, are distinct. Having picked such an element $a \in K$, we choose an element $b \in K$ such that $v_P b = v_P(z-a)$. Then (7) holds by the ultrametric triangle law. $\square$

2.4 Transcendence bases of separable valued function fields

We will denote the algebraic closure of K by $\tilde{K}$. We assume that K is a field and that P is a place on the rational function field $\tilde{K}(z)$ and infer the following two lemmata from [KH–K]:

Lemma 2.12 *The following assertions are equivalent:*
a) $(\tilde{K}(z)|\tilde{K}, v_P)$ is immediate,
b) $v_P K(z)/v_P K$ is a torsion group and $K(z)P|KP$ is algebraic,
c) $\{v_P(z-c) \mid c \in \tilde{K}\}$ has no maximal element.

Lemma 2.13 *Assume that $\{v_P(z-c) \mid c \in \tilde{K}\}$ has a maximal element and that $c_0 \in \tilde{K}$ is an element of minimal degree over K such that $v_P(z-c_0)$ is such a maximal element. Take f to be the minimal polynomial of c_0 over K.*

1) If $v_P K(z)/v_P K$ is not a torsion group, then $v_P f(z)$ is not a torsion element modulo $v_P K$.
2) If $K(z)P|KP$ is transcendental, then there is some $e \in \mathbb{N}$ and some $d \in K$ such that $(df(z)^e)P$ is transcendental over KP.

From these we deduce:

Lemma 2.14 *Assume that $(\tilde{K}(z)|\tilde{K}, v_P)$ is not immediate. Then there is some $h \in K[X]$ such that $v_P h(z)$ is non-torsion over $v_P K$ or that $h(z)P$ is transcendental over KP, and such that $K(z)|K(h(z))$ is separable.*

Proof: If $v_P z$ is not torsion modulo $v_P K$ or zP is transcendental over KP, then we set $h(X) := X$.

Otherwise, we set $g(X) := f(X)$ if case 1) of the foregoing lemma holds, and $g(X) := df(X)^e$ if case 2) holds (by Theorem 2.6, only one of the two cases can hold at a time). If the polynomial $g(X) - g(z)$ is separable over $K(g(z))$, then we set $h(X) := g(X)$. Otherwise, we proceed as follows. Set $n := \deg g$; this must be divisible by the characteristic p of K. In case 1), $v_P z$ is torsion modulo $v_P K$ by assumption, and it follows that for $h(X) := Xg(X)$, the value $v_P h(z) = v_P z + v_P g(z)$ is still not torsion modulo $v_P K$. As $\deg h(X) = n+1$ is not divisible by p, we find that $h(X) - h(z)$ is separable over $K(h(z))$. In case 2), $v_P z$ is torsion modulo $v_P K$ by (4), and it follows that there is some $b \in K$ such that $v_P bz > 0$. Then for $h(X) := g(X) + bX$ we have that $h(z)P = g(z)P$, and that $h(X) - h(z)$ is separable over $K(h(z))$. □

Now we are able to prove:

Lemma 2.15 *Let $(F|K, P)$ be any valued function field (where P is not necessarily trivial on K). Assume that $F|K$ is separable. Then there is a separating transcendence basis of $F|K$ containing elements $x_1, \ldots, x_\rho, y_1, \ldots, y_\tau$ such that $v_P x_1, \ldots, x_\rho v_P$ is a maximal set of elements in $v_P F$ rationally independent modulo $v_P K$, and $y_1 P, \ldots, y_\tau P$ form a transcendence basis of $FP|KP$.*

Proof: Since $F|K$ is separable, we can choose a separating transcendence basis $z_1, \ldots, z_n$ of $F|K$. We set $K_0 := K$ and $K_i := K(z_1, \ldots, z_i)$. We proceed by induction on i. If the extension $(\tilde{K}_{i-1}(z_i)|\tilde{K}_{i-1}, v_P)$ is not immediate, then we choose $h_i(z_i)$ according to the assertion of the foregoing lemma. Otherwise, we set $h_i(X) := X$. Since every extension $K_{i-1}(z_i)|K_{i-1}(h_i(z_i))$ is separable, we obtain a separating transcendence basis $h_1(z_1), \ldots, h_n(z_n)$ of $F|K$.

We set $\rho := \dim_{\mathbb{Q}} \mathbb{Q} \otimes (v_P F / v_P K)$ and $\tau := \operatorname{trdeg} FP|KP$. As

$$\rho \;=\; \sum_{i=0}^{n-1} \dim_{\mathbb{Q}} \mathbb{Q} \otimes (v_P K_{i+1} / v_P K_i) \quad \text{ and } \quad \tau \;=\; \sum_{i=0}^{n-1} \operatorname{trdeg} K_{i+1}P|K_iP \;,$$

and in view of the fact that

$$\dim_{\mathbb{Q}} \mathbb{Q} \otimes (v_P K_{i+1} / v_P K_i) \;+\; \operatorname{trdeg} K_{i+1}P|K_iP \;\leq\; \operatorname{trdeg} K_{i+1}|K_i \;=\; 1 \;,$$

we find that for precisely ρ many values of i, $v_P h_i(z_i)$ will be rationally independent modulo $v_P K_{i-1}$. Collecting all of these $h_i(z_i)$ and calling them $x_1, \ldots, x_\rho$ we thus obtain that $v_P x_1, \ldots, v_P x_\rho$ is a maximal set of elements in $v_P F$ rationally independent modulo $v_P K$. Similarly, we find that for precisely τ many values of i, the residues $h_i(z_i)P$ will be transcendental over $K_{i-1}P$. Collecting all of these $h_i(z_i)$ and calling them $y_1, \ldots, y_\tau$ we thus obtain that $y_1 P, \ldots, y_\tau P$ form a transcendence basis of $FP|KP$. □

Remark 2.16 We do not know whether in addition to the assertion of the lemma, the y_i can be chosen such that $y_1 P, \ldots, y_\tau P$ form a separating transcendence basis of $FP|KP$.

3 Inertially and henselian generated function fields

Theorem 3.1 *Assume that $F|K$ is a function field and P an Abhyankar place of $F|K$ such that $FP|K$ is a separable extension. Then $(F|K,P)$ is inertially generated. If in addition $FP = K$ or $FP|K$ is a rational function field, then $(F|K,P)$ is henselian generated. If $v_P F = \mathbb{Z}v_P x_1 \oplus \ldots \oplus \mathbb{Z}v_P x_\rho$ and $y_1P,\ldots,y_\tau P$ is a separating transcendence basis of $FP|K$, then $T = \{x_1,\ldots,x_\rho,y_1,\ldots,y_\tau\}$ is a generating transcendence basis, that is, (F,P) lies in the absolute inertia field of $(K(T),P)$, and if $FP = K(y_1P,\ldots,y_\tau P)$, then $F \subseteq K(T)^h$.*

Proof: By Corollary 2.7, value group and residue field of (F,P) are finitely generated. We choose $x_1,\ldots,x_\rho \in F$ such that $v_P F = \mathbb{Z}v_P x_1 \oplus \ldots \oplus \mathbb{Z}v_P x_\rho$. Since $FP|K$ is a finitely generated separable extension, it is separably generated. Therefore, we can choose $y_1,\ldots,y_\tau \in F$ such that $FP|K(y_1P,\ldots,y_\tau P)$ is separable-algebraic ($\tau = \operatorname{trdeg} FP|K$). Now we can choose some $a \in FP$ such that $FP = K(y_1P,\ldots,y_\tau P,a)$. Since a is separable-algebraic over $K(y_1P,\ldots,y_\tau P)$, by Hensel's Lemma there exists an element η in the henselization of (F,P) such that $\eta P = a$ and that the reduction of the minimal polynomial of η over $F_0 := K(x_1,\ldots,x_\rho,y_1,\ldots,y_\tau)$ is the minimal polynomial of a over $KP(y_1P,\ldots,y_\tau P)$. Then η lies in the absolute inertia field of F_0. Now the field $F_0(\eta)$ has the same value group and residue field as F, and it is contained in the henselization F^h of F. As henselizations are immediate extensions and the henselization $F_0(\eta)^h$ of $F_0(\eta)$ can be chosen inside of F^h, we obtain an immediate algebraic extension $(F^h|F_0(\eta)^h,P)$. On the other hand, we observe that (K,P) is a defectless field since P is trivial on K. By construction, $(F_0|K,P)$ is without transcendence defect, and the same is true for $(F_0(\eta)|K,P)$ since this property is preserved by algebraic extensions. Hence we know from Theorem 1.16 that $(F_0(\eta),P)$ is a defectless field. Now Corollary 2.2 shows that the extension $F^h|F_0(\eta)^h$ must be trivial. Therefore, F is contained in $F_0(\eta)^h$, which in turn is a subfield of the absolute inertia field of F_0. This shows that $(F|K,P)$ is inertially generated.

If $FP = K$, then we do not need the elements y_j and a. If $FP|K$ is a rational function field, then we can choose $y_1,\ldots,y_\tau \in F$ such that $FP = K(y_1P,\ldots,y_\tau P)$, and we do not need a. In both cases, we find that $F^h = F_0^h$, which yields that $(F|K,P)$ is henselian generated. $\square$

- **Proof of Theorem 1.15**

Assume that $(\mathcal{F}|\mathcal{K},P)$ is weakly uniformizable (where P is not necessarily trivial on $\mathcal{K}$). Denote by (L,P) the absolute inertia field of $(\mathcal{K}(t_1,\ldots,t_s),P)$.

First, $\det J_{fP}(\eta_1P,\ldots,\eta_nP) \neq 0$ and the fact that the f_iP are polynomials over $\mathcal{K}(t_1,\ldots,t_s)P$ imply that $\eta_1P,\ldots,\eta_nP$ are separable algebraic over $\mathcal{K}(t_1,\ldots,t_s)P$ (cf. [L], Chapter X, §7, Proposition 8). On the other hand, LP is the separable-algebraic closure of $\mathcal{K}(t_1,\ldots,t_s)P$. Therefore, there are elements $\eta'_1,\ldots,\eta'_n$ in L such that $\eta'_iP = \eta_iP$. Since (L,P) is henselian, the multidimensional Hensel's Lemma (cf. [K2], [K7]) now shows the existence of a common root $(\eta''_1,\ldots,\eta''_n) \in L^n$ of the f_i such that $\eta''_iP = \eta'_iP = \eta_iP$. But by the uniqueness assertion of the multidimensional Hensel's Lemma (which also holds in the algebraic closure $\tilde{L}$ of $\mathcal{K}(t_1,\dots,t_s)$), we find that $(\eta_1'',\dots,\eta_n'')=(\eta_1,\dots,\eta_n)$. Hence, the η_i are elements of L, which proves that $(\mathcal{F}|\mathcal{K},P)$ is inertially generated.

If we have in addition that P is a rational place, then $\eta_1 P,\dots,\eta_n P\in\mathcal{K}$. In this case, we can choose $\eta_1',\dots,\eta_n'$ and $\eta_1'',\dots,\eta_n''$ already in the henselization of $(\mathcal{K}(t_1,\dots,t_s),P)$, which implies that also $\eta_1,\dots,\eta_n$ lie in this henselization. □

4 Basic properties of relative uniformization

Recall the definition for "$(\mathcal{F}|\mathcal{K},P)$ is uniformizable" given preceding to Theorem 1.3. If P is not trivial on $\mathcal{K}$, one may think of this as "relative uniformization". For the case of P a place of $F|K$, relative uniformization will help us to prove Theorem 1.3, as we will build up $\mathcal{F}$ by a tower of uniformizable finitely generated extensions of valued fields, starting from $\mathcal{K}$. In the sections below, we will consider the different types of extensions involved in the build-up. Beforehand, we need some easy observations. First, we observe going-up and going-down of uniformizability through constant extensions of the function field.

Lemma 4.1 *Let $(L|K,P)$ be an extension of valued fields and $F|K$ a function field such that $F\subset L$. Take $\zeta_1,\dots,\zeta_m\in\mathcal{O}_F$.*

a) Suppose that $(F|K,P)$ is uniformizable with respect to $\zeta_1,\dots,\zeta_m$. Take an arbitrary subextension $L'|K$ of $L|K$ such that $\operatorname{trdeg} F.L'|L'=\operatorname{trdeg} F|K$. Then with the same t_i,η_i,f_i as for $(F|K,P)$, also $(F.L'|L',P)$ is uniformizable with respect to $\zeta_1,\dots,\zeta_m$.

b) Suppose that $L'|K$ is a subextension of $L|K$ such that $(F.L'|L',P)$ is uniformizable with respect to $\zeta_1,\dots,\zeta_m$. Then there is a finitely generated subextension $L_0|K$ of $L'|K$ such that with the same t_i,η_i,f_i as for $(F.L'|L',P)$, also $(F.L_0|L_0,P)$ is uniformizable with respect to $\zeta_1,\dots,\zeta_m$.

c) Suppose that $(F.K^{\mathrm{sep}}|K^{\mathrm{sep}},P)$ is uniformizable with respect to $\zeta_1,\dots,\zeta_m$. Then there is a finite Galois extension $K'|K$ such that with the same t_i,η_i,f_i as for $(F.K^{\mathrm{sep}}|K^{\mathrm{sep}},P)$, also $(F.K'|K',P)$ is uniformizable with respect to $\zeta_1,\dots,\zeta_m$.

Assertion a) follows directly from the definition since the condition "$\operatorname{trdeg} F.L'|L'=\operatorname{trdeg} F|K$" guarantees that $\{t_1,\dots,t_s\}$ remains a transcendence basis of $F.L'|L'$. Note that in the case of $\operatorname{trdeg} F.L'|L'<\operatorname{trdeg} F|K$ it remains valid if there is a transcendence basis $T\subset\mathcal{O}_{K(t_1,\dots,t_s)}$ of $F.L'|L'$ such that every t_i is contained in $\mathcal{O}_K[T]$. For the proof of assertion b), we just have to do the following. We collect the finitely many coefficients $c_1,\dots,c_\ell\in L'$ of all polynomials $f_i\in\mathcal{O}_{L'}[t_1,\dots,t_s,X_1,\dots,X_n]$. Further, we choose a finitely generated subextension $L_0'|K$ of $L'|K$ so large that $F.L_0'=L_0'(t_1,\dots,t_s,\eta_1,\dots,\eta_n)$ holds. Then we set $L_0:=L_0'(c_1,\dots,c_\ell)$. Part c) is a direct consequence of a) and b).

Now we turn to the **transitivity of relative uniformization**. Take a finitely generated extension $F|K$ and a finitely generated subextension $F_0|K$. Further, take elements $\zeta_1,\dots,\zeta_m\in\mathcal{O}_F$ and assume that $(F|F_0,P)$ is uniformizable with respect to $\zeta_1,\dots,\zeta_m$. So there are elements $t_1,\dots,t_{\tilde{s}}\in\mathcal{O}_F$, algebraically independent over F_0 $(\tilde{s}\geq 0)$, elements $\tilde{\eta}_1,\dots,\tilde{\eta}_{\tilde{n}}\in\mathcal{O}_F$, with $\zeta_1,\dots,\zeta_m$ among them, and polynomials $\tilde{f}_i(X_1,\dots,X_{\tilde{n}})\in\mathcal{O}_{F_0}[t_1,\dots,t_{\tilde{s}},X_1,\dots,X_{\tilde{n}}]$, $1\leq i\leq\tilde{n}$, such that

- $F = F_0(t_1, \ldots, t_{\tilde{s}}, \tilde{\eta}_1, \ldots, \tilde{\eta}_{\tilde{n}})$,
- for $i < j$, X_j does not occur in $\tilde{f}_i$,
- $\tilde{f}_i(\tilde{\eta}_1, \ldots, \tilde{\eta}_{\tilde{n}}) = 0$ for $1 \leq i \leq \tilde{n}$, and
- $(\det J_{(\tilde{f}_1,\ldots,\tilde{f}_{\tilde{n}})}(\tilde{\eta}_1, \ldots, \tilde{\eta}_{\tilde{n}}))P \neq 0$.

Now we collect the coefficients of all polynomials $\tilde{f}_i \in \mathcal{O}_{F_0}[t_1, \ldots, t_{\tilde{s}}, X_1, \ldots, X_{\tilde{n}}]$ and call them the **uniformization coefficients of** $F, \zeta_1, \ldots, \zeta_m$ **in** F_0. Then we extend P to $\tilde{F}$ and take any elements $\zeta'_1, \ldots, \zeta'_{m'} \in \mathcal{O}_{\tilde{F}_0}$ which include these uniformization coefficients. Assume that $\mathcal{F}_0$ is an algebraic extension of $F_0(\zeta'_1, \ldots, \zeta'_{m'})$ such that $(\mathcal{F}_0|K, P)$ is uniformizable with respect to $\zeta'_1, \ldots, \zeta'_{m'}$. So there are elements $t_{\tilde{s}+1}, \ldots, t_s \in \mathcal{O}_{\mathcal{F}_0}$, algebraically independent over K $(s \geq \tilde{s})$, elements $\eta_1, \ldots, \eta_{n'} \in \mathcal{O}_{\mathcal{F}_0}$, with the elements $\zeta'_1, \ldots, \zeta'_{m'}$ among them, and polynomials $f_i(X_1, \ldots, X_{n'}) \in \mathcal{O}_K[t_{\tilde{s}+1}, \ldots, t_s, X_1, \ldots, X_{n'}]$, $1 \leq i \leq n'$, such that

- $\mathcal{F}_0 = K(t_{\tilde{s}+1}, \ldots, t_s, \eta_1, \ldots, \eta_{n'})$,
- for $i < j$, X_j does not occur in f_i ,
- $f_i(\eta_1, \ldots, \eta_{n'}) = 0$ for $1 \leq i \leq n'$, and
- $(\det J_{(f_1,\ldots,f_{n'})}(\eta_1, \ldots, \eta_{n'}))P \neq 0$.

We observe that the elements $t_1, \ldots, t_s$ are algebraically independent over K. We set $n := n' + \tilde{n}$. Trivially, the polynomials f_i , $1 \leq i \leq n'$, can be viewed as polynomials in $\mathcal{O}_K[t_1, \ldots, t_s, X_1, \ldots, X_n]$. Note that by our choice of $\zeta'_1, \ldots, \zeta'_{m'}$, all $\mathcal{O}_{F_0}$-coefficients of the polynomials $\tilde{f}_i \in \mathcal{O}_{F_0}[t_1, \ldots, t_{\tilde{s}}, X_1, \ldots, X_{\tilde{n}}]$ appear as some η_j , with $j \in \{1, \ldots, n'\}$; we may assume that all η_j are distinct. For $1 \leq i \leq \tilde{n}$, we obtain the polynomial $f_{n'+i} \in \mathcal{O}_K[t_1, \ldots, t_s, X_1, \ldots, X_n]$ from the polynomial $\tilde{f}_i$ as follows:

i) for $1 \leq j \leq \tilde{n}$, we replace X_j by $X_{n'+j}$,
ii) if an $\mathcal{O}_{F_0}$-coefficient of $\tilde{f}_i$ is equal to η_j , $1 \leq j \leq n'$, then we replace it by X_j .

Accordingly, we set $\eta_{n'+i} := \tilde{\eta}_i$. Then (U1) and (U2) hold. For $1 \leq i \leq \tilde{n}$, we have that

$$\frac{\partial f_{n'+i}}{\partial X_{n'+i}}(\eta_1, \ldots, \eta_n) = \frac{\partial \tilde{f}_i}{\partial X_i}(\tilde{\eta}_1, \ldots, \tilde{\eta}_{\tilde{n}}) .$$

This shows that the diagonal elements of the lower triangular matrix $J_{(f_1,\ldots,f_n)}(\eta_1, \ldots, \eta_n)$ are precisely the diagonal elements of the lower triangular matrices $J_{(f_1,\ldots,f_{n'})}(\eta_1, \ldots, \eta_{n'})$ and $J_{(\tilde{f}_1,\ldots,\tilde{f}_{\tilde{n}})}(\tilde{\eta}_1, \ldots, \tilde{\eta}_{\tilde{n}})$. Consequently, $\det J_{(f_1,\ldots,f_n)}(\eta_1, \ldots, \eta_n)P \neq 0$. That is, for

$$\begin{aligned} F.\mathcal{F}_0 &= K(t_{\tilde{s}+1}, \ldots, t_s, \eta_1, \ldots, \eta_{n'}, t_1, \ldots, t_{\tilde{s}}, \tilde{\eta}_1, \ldots, \tilde{\eta}_{\tilde{n}}) \\ &= K(t_1, \ldots, t_s, \eta_1, \ldots, \eta_n) \end{aligned}$$

we have that $(F.\mathcal{F}_0|K, P)$ is uniformizable with respect to $\zeta_1, \ldots, \zeta_m$. We have proved:

Lemma 4.2 *Take a finitely generated extension $F|K$, a finitely generated subextension $F_0|K$, and $\zeta_1, \ldots, \zeta_m \in \mathcal{O}_F$. Assume that*

1) $(F|F_0, P)$ is uniformizable with respect to $\zeta_1, \ldots, \zeta_m$, with uniformizing transcendence basis T_1 , and

2) there is a finite extension $\mathcal{F}_0|F_0$ such that $(\mathcal{F}_0|K, P)$ is uniformizable with respect to the uniformization coefficients of $F, \zeta_1, \ldots, \zeta_m$ in F_0 , with uniformizing transcendence basis T_2 .

Then $(F.\mathcal{F}_0|K,P)$ is uniformizable with respect to $\zeta_1,\ldots,\zeta_m$, with uniformizing transcendence basis $T_1\cup T_2$.

Consequently, if $(F|F_0,P)$ and $(F_0|K,P)$ are uniformizable, then so is $(F|K,P)$.

This lemma proves Theorem 1.2. It is the basic form of transitivity, from which we will also derive the transitivity of the following two properties. Let $(L|K,P)$ be an arbitrary extension of valued fields, and E any subfield of L. Then we will say that $(L|K,P)$ has **(relative) Galois-uniformization over** E if $E.K|K$ is a function field and for every choice of elements $\zeta_1,\ldots,\zeta_m\in\mathcal{O}_L$ there is a finite Galois extension $\mathcal{E}|E$ such that $\zeta_1,\ldots,\zeta_m\in\mathcal{E}.K$ and $(\mathcal{E}.K|K,P)$ is uniformizable with respect to $\zeta_1,\ldots,\zeta_m$ (observe that $\mathcal{E}.K|K$ is a function field by the assumptions on E and $\mathcal{E}|E$). Note that this property implies that $L|E.K$ is a separable-algebraic extension since otherwise, there is some $\zeta\in\mathcal{O}_L$ which is not contained in $\mathcal{E}.K$ for any Galois extension $\mathcal{E}$ of E. Similarly, we will say that $(L|K,P)$ has **(relative) normal-uniformization over** E if for every choice of elements $\zeta_1,\ldots,\zeta_m\in\mathcal{O}_L$ there is a finite Galois extension $\mathcal{E}|E$ and a purely inseparable subextension $\mathcal{K}|K$ of $\mathcal{E}.K|K$ such that $\zeta_1,\ldots,\zeta_m\in\mathcal{E}.K$ and $(\mathcal{E}.K|\mathcal{K},P)$ is uniformizable with respect to $\zeta_1,\ldots,\zeta_m$. This implies that $L|E.K$ is algebraic.

Lemma 4.3 *Assume that $(M|L,P)$ and $(L|K,P)$ are extensions of valued fields. Take any subfield E of M such that $E.K|K$ is a function field. If $(M|L,P)$ has Galois-uniformization over E and $(L|K,P)$ has Galois-uniformization over some common subfield E_0 of E and L, then $(M|K,P)$ has Galois-uniformization over E. An analogous assertion holds for normal-uniformization over E.*

Proof: We only prove the first assertion; the proof of the second assertion is similar. Take $\zeta_1,\ldots,\zeta_m\in\mathcal{O}_M$, and choose a finite Galois extension $\mathcal{E}'$ of E such that $\zeta_1,\ldots,\zeta_m\in\mathcal{E}'.L$ and that $(\mathcal{E}'.L|L,P)$ is uniformizable with respect to the ζ's.

By our assumption on $(L|K,P)$, L is algebraic over $E_0.K$. Hence by part a) of Lemma 4.1, there is a finite subextension $L_0|E_0.K$ of $L|E_0.K$ such that $(\mathcal{E}'.L_0|L_0,P)$ is uniformizable with respect to the ζ's. Take $\zeta'_1,\ldots,\zeta'_{m'}\in\mathcal{O}_L$ to include generators of L_0 over $E_0.K$ and the uniformization coefficients of $\mathcal{E}'.L_0,\zeta_1,\ldots,\zeta_m$ in L_0.

By hypothesis, there is a finite Galois extension $\mathcal{E}_0$ of E_0 such that $\zeta'_1,\ldots,\zeta'_{m'}\in\mathcal{E}_0.K$ and $(\mathcal{E}_0.K|K,P)$ is uniformizable with respect to $\zeta'_1,\ldots,\zeta'_{m'}$. Then $\mathcal{E}:=\mathcal{E}'.\mathcal{E}_0$ is a finite Galois extension of E. By construction, we have that $\mathcal{E}_0.K$ is a finite extension of L_0 and that $(\mathcal{E}'.L_0).(\mathcal{E}_0.K)=\mathcal{E}.K$. Hence by Lemma 4.2, $(\mathcal{E}.K|K,P)$ is uniformizable with respect to $\zeta_1,\ldots,\zeta_m$. □

We leave it as an exercise to the reader to prove the following easy lemma:

Lemma 4.4 *Let $E|K$ be a finitely generated field extension and P a trivial place on E. Then $(\tilde{E}|K,P)$ has normal-uniformization over E. If in addition $E|K$ is separable, then $(E|K,P)$ is uniformizable and $(E^{\mathrm{sep}}|K,P)$ has Galois-uniformization over E.*

5 Uniformizable valued field extensions

In this section, we will present various finitely generated valued field extensions which are uniformizable.

5.1 Rational function fields with Abhyankar places

The following lemma was proved (but not explicitly stated) by Zariski in [Z1] for subgroups of $\mathbb{R}$, using the algorithm of Perron. We leave it as an easy exercise to the reader to prove the general case by induction on the rank of the ordered abelian group. However, an instant proof of the lemma can also be found in [EL] (Theorem 2.2).

Lemma 5.1 *Let Γ be a finitely generated ordered abelian group. Take any non-negative elements $\alpha_1, \ldots, \alpha_\ell \in \Gamma$. Then there exist positive elements $\gamma_1, \ldots, \gamma_\rho \in \Gamma$ such that $\Gamma = \mathbb{Z}\gamma_1 \oplus \ldots \oplus \mathbb{Z}\gamma_\rho$ and every α_i can be written as a sum $\sum_j n_{ij}\gamma_j$ with non-negative integers n_{ij}.*

The foregoing lemma and Theorem 2.6 are the main ingredients in the proof of the next proposition. We consider a function field $F|K$ and a place P of F such that $v_P K$ is a convex subgroup of $v_P F$. The latter always holds if P is trivial on K since then, $v_P K = \{0\}$. We take elements $x_1, \ldots, x_\rho$ in F such that $v_P x_1, \ldots, v_P x_\rho$ form a maximal set of rationally independent elements in $v_P F$ modulo $v_P K$. Further, we take elements $y_1, \ldots, y_\tau$ in F such that $y_1 P, \ldots, y_\tau P$ form a transcendence basis of FP over K.

Proposition 5.2 *In the situation described above, $(K(x_1, \ldots, x_\rho, y_1, \ldots, y_\tau)|K, P)$ is uniformizable. More precisely, the transcendence basis $T = \{t_1, \ldots, t_s\}$ can be chosen of the form $\{x'_1, \ldots, x'_\rho, y_1, \ldots, y_\tau\}$, where $x'_1, \ldots, x'_\rho \in \mathcal{O}_{K(x_1,\ldots,x_\rho)}$ and for some $c \in \mathcal{O}_K$ (with $c = 1$ if P is trivial on K), the elements $cx'_1, \ldots, cx'_\rho$ generate the same multiplicative subgroup of $K(x_1, \ldots, x_\rho)^\times$ as $x_1, \ldots, x_\rho$. If $0 \neq c' \in \mathcal{O}_K$ such that $vc' \geq vc$, then c can be replaced by c'.*

Proof: For the proof, we set $F = K(x_1, \ldots, x_\rho, y_1, \ldots, y_\tau)$. By Theorem 2.6, we know that

$$v_P F \;=\; \mathbb{Z}v_P x_1 \oplus \ldots \oplus \mathbb{Z}v_P x_\rho \oplus v_P K\;.$$

Thus,

$$v_P F/v_P K \ni \mu_1(v_P x_1 + v_P K) + \ldots + \mu_\rho(v_P x_\rho + v_P K) \mapsto x_1^{\mu_1} \cdot \ldots \cdot x_\rho^{\mu_\rho} \qquad (\mu_1, \ldots, \mu_\rho \in \mathbb{Z})$$

is an isomorphism from $v_P F/v_P K$ onto the multiplicative subgroup of $F^\times$ generated by $x_1, \ldots, x_\rho$. We denote this group by $\mathcal{X}$.

Now let $\zeta_1, \ldots, \zeta_m \in \mathcal{O}_F$. Take ζ to be any of these elements and write $\zeta = f/g$ with polynomials $f, g \in K[x_1, \ldots, x_\rho, y_1, \ldots, y_\tau]$. Write

$$f \;=\; \sum_{i=1}^{d} c_i\, x_1^{\mu_{1,i}} \cdot \ldots \cdot x_\rho^{\mu_{\rho,i}} \cdot y_1^{\nu_{1,i}} \cdot \ldots \cdot y_\tau^{\nu_{\tau,i}} \quad \text{and} \quad g \;=\; \sum_{i=1}^{d'} c'_i\, x_1^{\mu'_{1,i}} \cdot \ldots \cdot x_\rho^{\mu'_{\rho,i}} \cdot y_1^{\nu'_{1,i}} \cdot \ldots \cdot y_\tau^{\nu'_{\tau,i}}$$

as sums of monomials (in such a way that in either polynomial, two different monomials differ in at least one exponent). Then by Theorem 2.6, the value of f is equal to the least of the values of its monomials, say, to the one of the first. Similarly for g. So we can write

$$\zeta \;=\; \frac{\sum_i \frac{c_i}{c'_1}\, x_1^{\mu_{1,i}-\mu'_{1,1}} \cdot \ldots \cdot x_\rho^{\mu_{\rho,i}-\mu'_{\rho,1}} \cdot y_1^{\nu_{1,i}} \cdot \ldots \cdot y_\tau^{\nu_{\tau,i}}}{\sum_i \frac{c'_i}{c'_1}\, x_1^{\mu'_{1,i}-\mu'_{1,1}} \cdot \ldots \cdot x_\rho^{\mu'_{\rho,i}-\mu'_{\rho,1}} \cdot y_1^{\nu'_{1,i}} \cdot \ldots \cdot y_\tau^{\nu'_{\tau,i}}} \tag{8}$$

where the denominator has value 0 and the summands appearing in it all have value ≥ 0. Since $\zeta \in \mathcal{O}_F$, also the numerator has value ≥ 0, and the same must thus be true for all its summands. The only obstruction is that some of the x_i's may appear with negative exponents in some summands, and that, if P is non-trivial on K, some of the c_i/c'_1 or c'_i/c'_1 may have negative value.

We collect all summands of the form $h = cx_1^{\mu_1} \cdot \ldots \cdot x_\rho^{\mu_\rho} \cdot y_1^{\nu_1} \cdot \ldots \cdot y_\tau^{\nu_\tau}$ which appear in the numerator or denominator of (8), and all products of the form $\xi = x_1^{\mu_1} \cdot \ldots \cdot x_\rho^{\mu_\rho} \in \mathcal{X}$ which appear in these summands. We do the same for all elements $\zeta_1, \ldots, \zeta_m$. In this way, we obtain finitely many elements $h_1, \ldots, h_k$, all of them having non-negative value, and corresponding elements $\xi_1, \ldots, \xi_k \in \mathcal{X}$. We note that $v_P\xi_j \in v_P h_j + v_P K$, and as $v_P K$ is a convex subgroup of $v_P F$ and $v_P h_j$ is non-negative, it follows that $v_P \xi_j + v_P K$ is non-negative in $v_P F/v_P K$, for the induced order. After adding some suitably chosen elements of the form x_i or x_i^{-1} (depending on whether $v_P x_i$ or $v_P x_i^{-1}$ is positive) to the ξ's if necessary, we obtain elements $\xi_1, \ldots, \xi_\ell$ which also generate $\mathcal{X}$. At this point, we apply Lemma 5.1 to the non-negative values $v_P\xi_1 + v_P K, \ldots, v_P \xi_\ell + v_P K \in v_P F/v_P K$. Pulling the result back through the above isomorphism, we find generators $x''_1, \ldots, x''_\rho$ of $\mathcal{X}$ for which the values $v_P x''_j + v_P K$ in $v_P F/v_P K$ are positive, and such that every of the ξ's can be written as a (unique) product of the x''_j *with non-negative exponents*.

Let $\alpha \in v_P K$ be the minimum of the values of the coefficients $c_1, \ldots, c_k$ appearing in the monomials $h_1, \ldots, h_k$. We take $c \in \mathcal{O}_K$ such that $c \neq 0$ and $v_P c \geq -\min\{0, \alpha\}$. If $\alpha \geq 0$, which in particular is the case if P is trivial on K, then we can take $c := 1$. Now we set $x'_j := c^{-1} x''_j$. It follows that $h_1, \ldots, h_k \in \mathcal{O}_K[x'_1, \ldots, x'_\rho, y_1, \ldots, y_\tau]$. Since the values $v_P x''_j + v_P K$ in $v_P F/v_P K$ were positive, all values $v_P x'_j$ are positive. This remains true if c is replaced by any $c' \in \mathcal{O}_K$ such that $c' \neq 0$ and $vc' \geq vc$.

As we can read off from (8), every ζ_j can be written as ζ'_j/ζ''_j, where ζ'_j, ζ''_j lie in $\mathcal{O}_K[x'_1, \ldots, x'_\rho, y_1, \ldots, y_\tau]$, with $v_P \zeta''_j = 0$. We set $s = \rho + \tau$, $t_i := x'_i$ for $1 \leq i \leq \rho$ and $t_{\rho+i} := y_i$ for $1 \leq i \leq \tau$. Next, we set $n := m$ and put $\eta_j := \zeta_j$ and $f_j(X_1, \ldots, X_n) := \zeta''_j X_j - \zeta'_j$, and we are done. □

5.2 Immediate simple transcendental extensions

Lemma 5.3 *Let $(K(z)|K, P)$ be an immediate transcendental extension. If (6) holds, then $(K(z)|K, P)$ is uniformizable.*

Proof: Let $\zeta_1, \ldots, \zeta_m \in \mathcal{O}_{K(z)}$ and write $\zeta_j = f_j(z)/g_j(z)$ with polynomials $f_j(z)$, $g_j(z) \in K[z]$. We apply Lemma 2.11 to these finitely many polynomials and choose $\tilde{z} = \frac{z-a}{b}$ according to this lemma. Then by (7), for every j we can find i_j, k_j such that $v_P f_j(z) = v_P f_j^{[i_j]}(a)\, b^{i_j} = \min_i v_P f_j^{[i]}(a)\, b^i$ and $v_P g_j(z) = v_P g_j^{[k_j]}(a)\, b^{k_j} = \min_i v_P g_j^{[i]}(a)\, b^i$. Thus, we can write

$$\zeta_j \;=\; \frac{f_j^{[i_j]}(a)\, b^{i_j}}{g_j^{[k_j]}(a)\, b^{k_j}} \cdot \frac{\tilde{f}_j(\tilde{z})}{\tilde{g}_j(\tilde{z})}$$

where $\tilde{f}_j, \tilde{g}_j$ are polynomials with coefficients in $\mathcal{O}_K$ and $v_P \tilde{f}_j(\tilde{z}) = 0 = v_P \tilde{g}_j(\tilde{z})$. Note that also the first fraction is an element of $\mathcal{O}_K$ since its value is equal to $v_P \zeta_j$ and $\zeta_j \in \mathcal{O}_{K(z)}$

by assumption. Now we set $t_1 := \tilde{z}$, $n := m$, $\eta_j := \zeta_j$ and

$$f_j(X_1,\ldots,X_n) \;:=\; \tilde{g}_j(t_1)\,X_j \;-\; \frac{f_j^{[i_j]}(a)\,b^{i_j}}{g_j^{[k_j]}(a)\,b^{k_j}}\cdot\tilde{f}_j(t_1)$$

for $1\leq j\leq m$, and we are done. □

5.3 Extensions within the completion

Lemma 5.4 *Every finite separable-algebraic extension of a valued field within its completion is uniformizable.*

Proof: Take any separable-algebraic extension $(L|K,P)$ such that (L,P) lies in the completion of (K,P). Further, take $\zeta_1,\ldots,\zeta_m\in\mathcal{O}_L$. Let ζ be any of these elements. We extend v_P to the algebraic closure of the completion. Since $L|K$ is separable, we can write the minimal polynomial of ζ in the form $(X-\zeta)(X-\sigma_1\zeta)\ldots(X-\sigma_k\zeta)$ where σ_j are automorphisms in $\mathrm{Aut}(K^{\mathrm{sep}}|K)$ and all $\sigma_j\zeta$ are distinct from ζ. Since ζ lies in the completion of (K,P), it follows that there is some $a\in K$ such that $v_P(\zeta-a)>v_P(\sigma_j\zeta-a)$ for all j, and that $v_P(\zeta-a)\geq 0$. Since $\zeta\in\mathcal{O}_L$, the latter implies that $a\in\mathcal{O}_K$. On the other hand, $v_P(\zeta-a)\in v_PK(\zeta)=v_PK$ and $K(\zeta)P=KP$, so we can choose $b\in\mathcal{O}_K$ such that $v_P(\frac{\zeta-a}{b}-1)>0$. Thus, $v_P\frac{b}{\zeta-a}=0$ and $v_P\sigma_j\frac{b}{\zeta-a}=v_P\frac{b}{\sigma_j\zeta-a}>0$. Therefore, the reduction hP of the minimal polynomial h of $\frac{b}{\zeta-a}$ over K is the polynomial $X^{k+1}-X^k$ which has $1=\frac{b}{\zeta-a}P$ as a simple root. We set $n:=3m$. If ζ was ζ_j then we set $\eta_{3j-2}:=\frac{b}{\zeta-a}$, $f_{3j-2}:=h(X_{3j-2})$, $\eta_{3j-1}:=\frac{\zeta-a}{b}$, $f_{3j-1}:=X_{3j-2}X_{3j-1}-1$, $\eta_{3j}:=\zeta=b\frac{\zeta-a}{b}+a\in\mathcal{O}_K[\frac{\zeta-a}{b}]$, $f_{3j}:=X_{3j}-bX_{3j-1}-a$, and we are done. □

We can drop the condition that the extension be algebraic:

Proposition 5.5 *Every finitely generated separable extension of a valued field within its completion is uniformizable.*

Proof: It suffices to prove the assertion for every finitely generated separable extension $(L|K,P)$ within the completion of (K,P). As $L|K$ is finitely generated and separable, we can choose a transcendence basis $z_1,\ldots,z_n$ such that $L|K(z_1,\ldots,z_n)$ is separable-algebraic. By induction on the transcendence degree, using Lemma 2.10, Lemma 5.3 and transitivity (Lemma 4.2), we find that $(K(z_1,\ldots,z_n)|K,P)$ is uniformizable. By the foregoing lemma, the same holds for $(L|K(z_1,\ldots,z_n),P)$. Now our assertion follows by transitivity. □

5.4 Extensions within the henselization

The henselization of a valued field (K,P) is always a separable-algebraic extension. If (K,P) has rank 1, then moreover, the henselization lies in the completion of (K,P) (since in this case the completion is henselian, cf. [R], [K2]). Therefore, Lemma 5.4 yields:

Corollary 5.6 *Assume that (K,P) has rank 1. Then every finite extension of (K,v) within its henselization is uniformizable.*

We give a typical application:

Corollary 5.7 *If P is a rational Abhyankar place of rank 1 of a function field $F|K$, then $(F|K,P)$ is uniformizable.*

Proof: By Theorem 3.1, $(F|K,P)$ is henselian generated and there are $x_1,\ldots,x_\rho \in F$ as in the assertion of that theorem such that $(F,P) \subset (K(x_1,\ldots,x_\rho),P)^h$. By the foregoing corollary it follows that $(F|K(x_1,\ldots,x_\rho),P)$ is uniformizable. By Proposition 5.2, $(K(x_1,\ldots,x_\rho)|K,P)$ is uniformizable. Now our assertion follows by transitivity. □

To treat the case of a rank higher than 1, we use a well known lemma about composite places (cf. [R] or [K2]).

Lemma 5.8 *Suppose that the place P of K is composite: $P = Q\overline{Q}$. Then (K,P) is henselian if and only if (K,Q) and $(KQ,\overline{Q})$ are. If $(KQ,\overline{Q})$ is henselian, then the henselization of K with respect to P is equal to the henselization of K with respect to Q (as fields).*

If in this situation, Q has rank 1, the henselization of K with respect to Q lies in its completion with respect to Q. Since $P = Q\overline{Q}$, it follows from general valuation theory that the completion of K with respect to Q is equal to the completion of K with respect to P. So the henselization of K with respect to P lies in the completion of K with respect to P. Hence, we obtain the following corollary from Lemma 5.4:

Corollary 5.9 *Assume that P is a place of K and $P = Q\overline{Q}$ such that (K,Q) has rank 1 and $(KQ,\overline{Q})$ is henselian. Then every finite extension of (K,P) within its henselization is uniformizable.*

5.5 Immediate extensions

Proposition 5.10 *Take a separable-algebraically closed field K, a separable function field $F|K$ of transcendence degree 1, and a place P on F of rank 1 such that $(F|K,P)$ is an immediate extension. Then $(F|K,P)$ is uniformizable.*

The assertion also holds if $P = Q\overline{Q}$ such that (F,Q) has rank 1 and $FQ = KQ$.

Proof: By Theorem 1.17, $(F|K,P)$ is henselian generated. That is, there is some $z \in F$ such that $F \subset K(z)^h$. Since K is separable-algebraically closed, Lemma 2.10 shows that condition (6) holds. Therefore, Lemma 5.3 shows that $(K(z)|K,P)$ is uniformizable. By Corollary 5.6, the same holds for $(F|K(z),P)$. Hence by transitivity, $(F|K,P)$ is uniformizable.

If $P = Q\overline{Q}$ such that (F,Q) has rank 1 and $FQ = KQ$, then we employ Corollary 5.9 in the place of Corollary 5.6. This is possible since $FQ = KQ$ implies that $K(z)Q = KQ$ and thus, being equal to the residue field of a separable-algebraically closed field, $K(z)Q$ is itself separable-algebraically closed and hence henselian under every valuation. □

6 Galois- and normal-uniformization

In this section, we will present valued field extensions which admit Galois-uniformization or normal-uniformization.

6.1 Abhyankar places of rank 1

Proposition 6.1 *Take a function field $E|K$ and a zero-dimensional Abhyankar place P of $E|K$ of rank 1. Then $(\tilde{E}|K,P)$ has normal-uniformization over E. If K is perfect, then $(E^{\rm sep}|K,P)$ has Galois-uniformization over E.*

Proof: For given $\zeta_1,\ldots,\zeta_m \in \mathcal{O}_{E^{\rm sep}}$, we take F to be the normal hull of $E(\zeta_1,\ldots,\zeta_m)$ over E; then $F|E$ is a finite Galois extension. Since K is assumed to be perfect and $EP|K$ to be algebraic, $FP|K$ is a separable-algebraic extension. By Lemma 2.4, for $F' := F.K^{\rm sep}$ we have that $F'P = (FP)^{\rm sep} = K^{\rm sep}$. Thus, Corollary 5.7 shows that $(F'|K^{\rm sep},P)$ is uniformizable. This proves that $(E^{\rm sep}|K^{\rm sep},P)$ has Galois-uniformization over E.

On the other hand, P is trivial on $K^{\rm sep}$. Hence, Lemma 4.4 tells us that $(K^{\rm sep}|K,P)$ has Galois-uniformization over K. Now our assertion follows by transitivity.

The proof for normal-uniformization is similar. □

Proposition 6.2 *Take a function field $E|K$ and a place P of E. Assume that $P = Q\overline{Q}$ such that Q is a zero-dimensional Abhyankar place of $E|K$ of rank 1. Then $(\tilde{E}|\tilde{K},P)$ has normal-uniformization over E.*

Proof: For given $\zeta_1,\ldots,\zeta_m \in \mathcal{O}_{\tilde{E}}$, we take F to be the normal hull of $E(\zeta_1,\ldots,\zeta_m)$ over E. We set $F' := F.\tilde{K}$. As Q is zero-dimensional, we obtain that $F'Q = \tilde{K}$. Hence, Q is a rational Abhyankar place of $F'|\tilde{K}$. By Theorem 3.1, $(F'|\tilde{K},Q)$ is henselian generated: there are $x_1^Q,\ldots,x_k^Q \in E$ such that $v_Q x_1^Q,\ldots,v_Q x_k^Q$ is a maximal set of rationally independent values in $v_Q F'$, and (F',Q) is contained in the henselization of (F_0,Q), where $F_0 := \tilde{K}(x_1^Q,\ldots,x_k^Q)$. But $F_0 Q = \tilde{K}$ by Theorem 2.6, and $(\tilde{K},\overline{Q})$ is henselian. Hence by Corollary 5.9, $(F'|F_0,P)$ is uniformizable.

Since Q is trivial on $\tilde{K}$ and since $F'Q = \tilde{K}$, we know that $v_P\tilde{K} = v_{\overline{Q}}\tilde{K} = v_{\overline{Q}}(F'Q)$ is a convex subgroup of $v_P F'$, and that $v_Q F' = v_P F'/v_P\tilde{K}$. Consequently, our choice of the x_i^Q's yields that $v_P x_1^Q,\ldots,v_P x_k^Q$ form a maximal set of rationally independent elements in $v_P F'$ modulo $v_P\tilde{K}$. Hence by Proposition 5.2, $(F_0|\tilde{K},P)$ is uniformizable. By transitivity, the same holds for $(F'|\tilde{K},P)$. This proves our assertion. □

Proposition 6.3 *Take a function field $E|K$ and an Abhyankar place P of $E|K$ of rank 1. Then $(\tilde{E}|K,P)$ has normal-uniformization over E.*

Proof: We choose $y_1,\ldots,y_\tau \in E$ such that $y_1P,\ldots,y_\tau P$ is a transcendence basis of $EP|K$. We take K' to be the algebraic closure of $K(y_1,\ldots,y_\tau)$. We extend P to $\tilde{E}$. Then

P induces an isomorphism on K'. Passing to an equivalent place if necessary, we can assume that $\tilde{E}P = K'$. Hence by Proposition 6.1, $(\tilde{E}|K', P)$ has normal-uniformization over E. By Lemma 4.4, $(K'|K, P)$ has normal-uniformization over $K(y_1, \ldots, y_\tau)$. Now our assertion follows by transitivity. $\square$

6.2 Immediate extensions

Proposition 6.4 *Take a function field $E|K$ and a place P on $\tilde{E}$ of rank 1 such that $v_P E/v_P K$ is a torsion group and $EP|KP$ is algebraic. Then the immediate extension $(\tilde{E}|\tilde{K}, P)$ has normal-uniformization over E, and if $E|K$ is separable, then the immediate extension $(E^{\mathrm{sep}}|K^{\mathrm{sep}}, P)$ has Galois-uniformization over E.*

These assertions also hold if $P = Q\overline{Q}$ such that (E, Q) has rank 1, $v_Q E/v_Q K$ is a torsion group and $EQ|KQ$ is algebraic.

Proof: We give the proof for Galois-uniformization. We proceed by induction on the transcendence degree. The case of transcendence degree 1 is covered by Proposition 5.10: For given $\zeta_1, \ldots, \zeta_m \in \mathcal{O}_{E^{\mathrm{sep}}}$, we take F to be the normal hull of $E(\zeta_1, \ldots, \zeta_m)$ over E, which is a finite Galois extension of E. Then we apply Proposition 5.10 to $(F.K^{\mathrm{sep}}|K^{\mathrm{sep}}, P)$. We observe that since $v_P E/v_P K$ is a torsion group and $EP|KP$ is algebraic by hypothesis, Lemma 2.3 implies that the extension $(E^{\mathrm{sep}}|K^{\mathrm{sep}}, P)$ and hence also its subextension $(F.K^{\mathrm{sep}}|K^{\mathrm{sep}}, P)$ are immediate. For the case of $P = Q\overline{Q}$ with Q non-trivial, it also implies that $K^{\mathrm{sep}}Q = \widetilde{KQ}$ and $E^{\mathrm{sep}}Q = \widetilde{EQ}$. Hence, our assumption that $EQ|KQ$ is algebraic implies that $E^{\mathrm{sep}}Q = K^{\mathrm{sep}}Q$.

So let us now assume that $\operatorname{trdeg} E|K = n > 1$ and that our assertion is true for transcendence degree $< n$. We take a separating transcendence basis T of $E|K$. Then we pick a subset $T_0 \subset T$ such that $\operatorname{trdeg} E|E_0 = 1$ for $E_0 := K(T_0) \subset E$. It follows that $E.E_0^{\mathrm{sep}}|E_0^{\mathrm{sep}}$ is a separable function field of transcendence degree 1 and that $E_0|K$ is a separable function field of transcendence degree $n-1$. As $(E_0|K, P)$ is a subextension of $(E|K, P)$, $v_P E_0/v_P K$ is a torsion group, $E_0 P|KP$ is algebraic, and (E_0, P) will have rank at most 1 if (E, P) has rank 1. But the fact that $v_P E/v_P K$ is a torsion group implies that (K, P) has the same rank as (E, P). This shows that (E_0, P) has rank 1 if (E, P) has rank 1. Similarly, if $P = Q\overline{Q}$ with $EQ = KQ$ and (E, Q) has rank 1, then $E_0 Q = KQ$ and (E_0, Q) will have rank at most 1. But the fact that $v_P E/v_P K$ is a torsion group also implies that (K, Q) has the same rank as (E, Q). Hence in this case, (E_0, Q) has rank 1 if (E, Q) has rank 1. We have shown that also $(E_0|K, P)$ satisfies the assumptions of our proposition. As $(E.E_0^{\mathrm{sep}})^{\mathrm{sep}} = E^{\mathrm{sep}}$, our induction hypothesis yields that $(E^{\mathrm{sep}}|E_0^{\mathrm{sep}}, P)$ has Galois-uniformization over E and that $(E_0^{\mathrm{sep}}|K^{\mathrm{sep}}, P)$ has Galois-uniformization over E_0. Hence by transitivity, $(E^{\mathrm{sep}}|K^{\mathrm{sep}}, P)$ has Galois-uniformization over E.

The proof for normal-uniformization is similar: instead of separable-algebraic closures we use algebraic closures. $\square$

6.3 Places of rank 1

Proposition 6.5 *Take a function field $E|K$ and a zero-dimensional place P of $E|K$ of rank 1. If K is perfect, then $(E^{\rm sep}|K,P)$ has Galois-uniformization over E.*

Proof: By Lemma 2.15, we can choose a separating transcendence basis of $E|K$ which contains elements $x_1,\ldots,x_\rho\in E$ such that $v_Px_1,\ldots,v_Px_\rho$ is a maximal set of rationally independent elements in v_PE. We set $E_0:=K(x_1,\ldots,x_\rho)$. Then $E|E_0$ is separable. Further, v_PE/v_PE_0 is a torsion group by Theorem 2.6. By the same theorem, $E_0P=K$. By assumption, EP is algebraic over $K=E_0P$. Hence by Proposition 6.4, the extension $(E^{\rm sep}|E_0^{\rm sep},P)$ has Galois-uniformization over E. By Proposition 6.1, $(E_0^{\rm sep}|K,P)$ has Galois-uniformization over E_0. Now our assertion follows by transitivity. □

6.4 Places of arbitrary rank

Proposition 6.6 *Take a function field $E|K$ and a place P on E. Assume that $P=Q\overline{Q}$ such that Q is a place of $E|K$ of rank 1. Then $(\tilde{E}|\tilde{K},P)$ has normal-uniformization over E.*

Proof: We choose $x_1^Q,\ldots,x_k^Q\in E$ such that $v_Qx_1^Q,\ldots,v_Qx_k^Q$ is a maximal set of rationally independent values in v_QE. We set $L:=K(x_1^Q,\ldots,x_k^Q)\subseteq E$. Then by Theorem 2.6, $LQ=KQ=K$ and v_QE/v_QL is a torsion group. Hence by Proposition 6.4, $(\tilde{E}|\tilde{L},P)$ has normal-uniformization over E.

By construction, $(L|K,P)$ and Q satisfy the assumptions of Proposition 6.2. This yields that $(\tilde{L}|\tilde{K},P)$ has normal-uniformization over L. Now our assertion follows by transitivity. □

Proposition 6.7 *Take a function field $E|K$ and a place P of $E|K$. Then $(\tilde{E}|K,P)$ has normal-uniformization over E.*

Proof: Since the rank of (E,P) is finite (cf. Section 2.2), we can proceed by induction on this rank. Assume that $P=Q\overline{Q}$ such that Q is a place of $E|K$ of rank 1, with $\overline{Q}$ possibly trivial. We take $y_1^Q,\ldots,y_\ell^Q\in E$ such that $y_1^QQ,\ldots,y_\ell^QQ$ is a transcendence basis of $EQ|K$. Then we set $E_1:=K(y_1^Q,\ldots,y_\ell^Q)\subseteq E$ and $E':=E.\tilde{E}_1\subset\tilde{E}$. Since $E'|E$ is algebraic, so is $E'Q|EQ$. On the other hand, EQ is algebraic over E_1Q by construction, and therefore, $\tilde{E}_1Q=\widetilde{E_1Q}$ is equal to the algebraic closure of EQ. As $\tilde{E}_1Q\subseteq E'Q$, this shows that $E'Q=\tilde{E}_1Q$. Since Q is the identity on K and $y_1^QQ,\ldots,y_\ell^QQ$ are algebraically independent over K, it induces an isomorphism on E_1 and hence also on $\tilde{E}_1$. Passing to an equivalent place if necessary, we can assume that Q is a place of $E'|\tilde{E}_1$.

Since $\widetilde{E'}=\tilde{E}$, Proposition 6.6 now shows that $(\tilde{E}|\tilde{E}_1,P)$ has normal-uniformization over E', and hence also over E. As the rank of (E_1,P) is equal to the rank of $(EQ,\overline{Q})$ and thus smaller than the rank of (E,P), our induction hypothesis (or Lemma 4.4, if $\overline{Q}$ is trivial) yields that $(\tilde{E}_1|K,P)$ has normal-uniformization over E_1. Now our assertion follows by transitivity. □

7 Proof of the main theorems for rank 1

- **Proof of Theorems 1.3, 1.4 for rank 1, and of Theorem 1.14**

Theorems 1.3 and 1.4 can be proved by a direct application of Proposition 6.7. But we will use a different approach which at the same time proves Theorem 1.14.

Let $F|K$ be a function field and $E|K$ a subextension of the same transcendence degree; consequently, $F|E$ is finite. Further, take a rank 1 place P of $F|K$ and any elements $\zeta_1,\ldots,\zeta_m \in \mathcal{O}_F$. After extending this list if necessary, we can assume that it includes generators of $F|E$. Finally, take E_0 to be any Abhyankar field of $(E|K,P)$.

By Proposition 6.4, $(\tilde{E}|\tilde{E}_0,P)$ has normal-uniformization over E. Therefore, there is a finite normal extension $\mathcal{F}'$ of E such that $\mathcal{F}'.\tilde{E}_0$ contains the ζ's and $(\mathcal{F}'.\tilde{E}_0|\tilde{E}_0,P)$ is uniformizable with respect to the ζ's. By part b) of Lemma 4.1, there is a finite extension $\mathcal{E}_0|E_0$ such that $(\mathcal{F}'.\mathcal{E}_0|\mathcal{E}_0,P)$ is uniformizable with respect to the ζ's.

We choose $\zeta'_1,\ldots,\zeta'_{m'} \in \mathcal{O}_{\mathcal{E}_0}$ to consist of generators of $\mathcal{E}_0$ over E_0 and of the uniformization coefficients of $\mathcal{F}'.\mathcal{E}_0,\zeta_1,\ldots,\zeta_m$ in $\mathcal{E}_0$. By Proposition 6.3, $(\tilde{E}_0|K,P)$ has normal-uniformization over E_0. Hence there is a finite normal extension $\mathcal{F}_0$ of E_0 and a purely inseparable subextension $\mathcal{K}|K$ of $\mathcal{F}_0|K$ such that $\mathcal{F}_0$ contains $\zeta'_1,\ldots,\zeta'_{m'}$ (and hence also $\mathcal{E}_0$) and $(\mathcal{F}_0|\mathcal{K},P)$ is uniformizable with respect to $\zeta'_1,\ldots,\zeta'_{m'}$, the uniformizing transcendence basis being a transcendence basis of $\mathcal{F}_0|K$. We have that $(\mathcal{F}'.\mathcal{E}_0).\mathcal{F}_0 = \mathcal{F}'.\mathcal{F}_0$.

Now by Lemma 4.2, $(\mathcal{F}'.\mathcal{F}_0|\mathcal{K},P)$ is uniformizable with respect to $\zeta_1,\ldots,\zeta_m$, with uniformizing transcendence basis containing a transcendence basis of $\mathcal{F}_0|K$. With ρ and τ as defined preceding to Theorem 1.14, the latter transcendence basis has $\rho+\tau$ many elements, and they are algebraic over E_0. We set $\mathcal{F} := \mathcal{F}'.\mathcal{F}_0$. As $\mathcal{F}'|E$ is finite and normal and $\mathcal{F}_0|E_0$ is finite and normal, $\mathcal{F}|E$ is also finite and normal. By our additional assumption on the ζ's, $F\subset\mathcal{F}$. Hence also $\mathcal{F}|F$ is finite and normal. This proves Theorems 1.3 and 1.4 in the rank 1 case, and Theorem 1.14.

- **Proof of Theorem 1.5**

If K is not perfect, we will have to pass to its perfect hull to apply Propositions 6.5. This perfect hull will be denoted by K^{1/p^∞}.

For the proof of Theorem 1.5, we assume in addition to the above assumptions that P is zero-dimensional and that $F|E$ is separable-algebraic, i.e., $F\subset E^{\mathrm{sep}}$. We set $E_1 := E.K^{1/p^\infty}$; then $E_1^{\mathrm{sep}} = E^{\mathrm{sep}}.K^{1/p^\infty}$. As P is still a zero-dimensional place of $E_1^{\mathrm{sep}}|K^{1/p^\infty}$, Proposition 6.5 shows that $(E_1^{\mathrm{sep}}|K^{1/p^\infty},P)$ has Galois-uniformization over E_1. This gives us a finite Galois extension $\mathcal{F}''$ of E_1 such that $(\mathcal{F}''|K^{1/p^\infty},P)$ is uniformizable with respect to the ζ's. Now we take $\mathcal{F}'$ to be the maximal separable subextension of $\mathcal{F}''|E$. Then $\mathcal{F}'|E$ is a Galois extension and $\mathcal{F}'' = \mathcal{F}'.K^{1/p^\infty}$. Thus, $(\mathcal{F}'.K^{1/p^\infty}|K^{1/p^\infty},P)$ is uniformizable with respect to the ζ's, and by part b) of Lemma 4.1, there is a finite subextension $\mathcal{K}|K$ of $K^{1/p^\infty}|K$ such that $(\mathcal{F}'.\mathcal{K}|\mathcal{K},P)$ is uniformizable with respect to the ζ's. We set $\mathcal{F} := \mathcal{F}'.\mathcal{K}$. As $\mathcal{F}'|E$ is a Galois extension, so is $\mathcal{F}|\mathcal{K}.E$. By our additional assumption on the ζ's, $F\subset\mathcal{F}$. Hence also $\mathcal{F}|\mathcal{K}.F$ is a Galois extension. This proves Theorem 1.5. □

- **Proof of Theorem 1.6**

Let F be as in the hypothesis of Theorem 1.3, with P of rank 1. We set $F_1 := F.K^{1/p^\infty}$. By Lemma 2.15, we can choose a separating transcendence basis of $F_1|K^{1/p^\infty}$ which contains elements $x_1, \dots, x_\rho$ such that $v_P x_1, \dots, v_P x_\rho$ is a maximal set of rationally independent elements in $v_P F_1$, and $y_1, \dots, y_\tau$ such that $y_1 P, \dots, y_\tau P$ is a transcendence basis of $F_1 P | K^{1/p^\infty}$. We set $F_0 := K^{1/p^\infty}(x_1, \dots, x_\rho, y_1, \dots, y_\tau) \subseteq F_1$. Then $F_1 | F_0$ is separable, $v_P F_1 / v_P F_0$ is a torsion group and $F_1 P | F_0 P$ is algebraic. Hence by Proposition 6.4, the extension $(F_1^{\mathrm{sep}} | F_0^{\mathrm{sep}}, P)$ has Galois-uniformization over F_1. That is, there is a finite Galois extension $F_2 | F_1$ such that $(F_2.F_0^{\mathrm{sep}} | F_0^{\mathrm{sep}}, P)$ is uniformizable with respect to the ζ's. By part c) of Lemma 4.1 there is a finite Galois extension F_0' of F_0 such that $(F_2.F_0' | F_0', P)$ is uniformizable with respect to $\zeta_1, \dots, \zeta_m$. Then $F' := F_2.F_0'$ is a finite Galois extension of F_1. By construction, P is an Abhyankar place on F_0 and F_0'.

Now we choose $y_1', \dots, y_\tau' \in F'$ such that $y_1' P, \dots, y_\tau' P$ is a separating transcendence basis of $F_0' P | K^{1/p^\infty}$. We take K' to be the separable-algebraic closure of $K^{1/p^\infty}(y_1', \dots, y_\tau')$. Then by Lemma 2.4, the place P of $F_0'.K' | K'$ is rational. Hence by Corollary 5.7, $(F_0'.K' | K', P)$ is uniformizable. It follows by part a) of Lemma 4.1 and transitivity that $(F'.K' | K', P)$ is uniformizable with respect to the ζ's. Hence by part c) of Lemma 4.1 and transitivity, there is a finite Galois extension K'' of $K^{1/p^\infty}(y_1', \dots, y_\tau')$ such that $(F'.K'' | K'', P)$ is uniformizable with respect to the ζ's. Further, P is trivial on K''. Hence by Lemma 4.4, $(K'' | K^{1/p^\infty}, P)$ is uniformizable. By transitivity, $(F'.K'' | K^{1/p^\infty}, P)$ is uniformizable with respect to the ζ's. Observe that $F'.K'' | K'$ is a finite Galois extension.

We denote by $\mathcal{F}'$ the maximal separable subextension of F in F', and by $\mathcal{F}''$ the maximal separable subextension of $\mathcal{F}'$ in $F'.K''$. Then $\mathcal{F}' | F$ and $\mathcal{F}'' | \mathcal{F}'$ are Galois extensions, and we have that $F'.K'' = \mathcal{F}''.K^{1/p^\infty}$. By part b) of Lemma 4.1 there is a finite purely inseparable extension $\mathcal{K}$ of K such that $(\mathcal{F}''.\mathcal{K} | \mathcal{K}, P)$ is uniformizable with respect to $\zeta_1, \dots, \zeta_m$. As $\mathcal{F} := \mathcal{F}''.\mathcal{K}$ is a Galois extension of the Galois extension $\mathcal{F}'.\mathcal{K}$ of $F.\mathcal{K}$, this proves our theorem. □

Remark: If we could choose $y_1', \dots, y_\tau'$ in F_1, then $F'.K'' | F$ would be a Galois extension. But the algebraic extension $F_0' P | F_0 P$ may well be inseparable.

- **Proof of Theorem 1.7**

Assume that P is an Abhyankar place of $F|K$ and that (F, P) has rank 1. Let us first note that part c) of Theorem 1.7 follows directly from Corollary 5.7. For the remaining cases, we proceed as follows.

We set $K_1 := K$ if $FP|K$ is separable, and $K_1 := K^{1/p^\infty}$ otherwise. Then we set $F_1 := F.K_1$. It follows that $F_1 P | K_1$ is separable. We choose $y_1, \dots, y_\tau \in F_1$ such that $y_1 P, \dots, y_\tau P$ is a separating transcendence basis of $F_1 P | K_1$. Then we take K' to be the separable-algebraic closure of $K_1(y_1, \dots, y_\tau)$ and set $F' := F.K' = F_1.K'$. Since P is trivial on K, it is trivial on K_1 and Theorem 2.6 yields that it is also trivial on $K_1(y_1, \dots, y_\tau)$ and thus on K'. By Lemma 2.4, $F'P = K'$. Therefore, Corollary 5.7 shows that $(F' | K', P)$ is uniformizable. Hence by part c) of Lemma 4.1, for given $\zeta_1, \dots, \zeta_m \in \mathcal{O}_F$ there is a finite Galois extension $\mathcal{F}'$ of $K_1(y_1, \dots, y_\tau)$ such that $(F.\mathcal{F}' | \mathcal{F}', P)$ is uniformizable with respect to the ζ's.

Since P is trivial on $\mathcal{F}'$ and $\mathcal{F}' | K_1$ is separable, Lemma 4.4 shows that $(\mathcal{F}' | K_1, P)$ is uniformizable. By transitivity, $(F.\mathcal{F}' | K_1, P)$ is uniformizable with respect to the ζ's.

If $FP|K$ is separable and hence $K_1 = K$ by definition, then $K_1(y_1, \dots, y_\tau) \subset F$ and thus, $\mathcal{F} := F.\mathcal{F}'$ is a finite Galois extension of F. This proves part a) of our theorem.

For the remaining cases, we proceed as follows. Since $F.K_1$ is a purely inseparable extension of F, there is some $\nu \in \mathbb{N}$ such that $F_0 := K(y_1^{p^\nu}, \dots, y_\tau^{p^\nu}) \subseteq F$. The extension

$\mathcal{F}'|F_0$ is algebraic. Its maximal separable subextension $\mathcal{F}''|F_0$ is a finite Galois extension, and $F.\mathcal{F}' = F.\mathcal{F}''.K_1$. Hence, $(F.\mathcal{F}''.K_1|K_1, P)$ is uniformizable with respect to the ζ's. By part b) of Lemma 4.1, there is a finite subextension $\mathcal{K}|K$ of $K_1|K$ such that $(F.\mathcal{F}''.\mathcal{K}|\mathcal{K}, P)$ is uniformizable with respect to the ζ's. Further, $\mathcal{F} := F.\mathcal{F}''.\mathcal{K}$ is a finite Galois extension of $F.\mathcal{K}$. This proves the general assertion of the theorem. If in addition P is zero-dimensional, then there are no y's and we can take $F_0 = K$. In this case, $\mathcal{F}''$ is a finite Galois extension of K, which proves part b) of the theorem. $\square$

- **Proof of Theorem 1.8**

Assume that P is a rational discrete place of $F|K$. We write $v_P F = \mathbb{Z}$. We choose a set of generators of F over K such that each of these generators has v_P-value 1. By Lemma 2, $F|K$ is separable and thus, we can choose a separating transcendence basis $t_1, \ldots, t_s$ for $F|K$ from these generators (cf. [L], Ch. X, §6, Prop. 5). Then $(F|K(t_1), P)$ is immediate. Since $v_P F = \mathbb{Z}$, this implies that (F, P) lies in the completion of $(K(t_1), P)$. Hence by Proposition 5.5, the separable extension $(F|K(t_1), P)$ is uniformizable. By Proposition 5.2, also $(K(t_1)|K, P)$ is uniformizable. Hence by transitivity, $(F|K, P)$ is uniformizable. This proves Theorem 1.8. $\square$

8 Proof of the main theorems for arbitrary rank

- **Proof of Theorems 1.3 and 1.4 for arbitrary rank**

The proof is a direct application of Proposition 6.7.

- **Proof of Theorem 1.11**

Assume that P is an Abhyankar place of rank $r > 1$ of the function field $F|K$. Since the rank of $(F|K, P)$ is finite (cf. Section 2.2), we can proceed by induction on the rank. We take a maximal proper convex subgroup H of $v_P K$. Then $v_P K/H$ is archimedean ordered. We write $P = Q\overline{Q}$, where Q is a place of $F|K$ of rank 1 with value group $v_P K/H$, and $\overline{Q}$ is a place on FQ with value group H. By Lemma 2.8, Q and $\overline{Q}$ are Abhyankar places. Hence by Corollary 2.7, $v_P K$ and H are finitely generated. Now we choose $x_1, \ldots, x_k$ such that $v_P K/H = \mathbb{Z}(v_P x_1 + H) \oplus \ldots \oplus \mathbb{Z}(v_P x_k + H)$, and $x_{k+1}, \ldots, x_\rho$ such that $H = \mathbb{Z} v_P x_{k+1} \oplus \ldots \oplus \mathbb{Z} v_P x_\rho$. Then $v_P K = \mathbb{Z} v_P x_1 \oplus \ldots \oplus \mathbb{Z} v_P x_\rho$. So if we choose the y's as in Theorem 3.1, then we obtain that $F \subset K(x_1, \ldots, x_\rho, y_1, \ldots, y_\tau)^h$. We set $K' := K(x_{k+1}, \ldots, x_\rho, y_1, \ldots, y_\tau)^{\mathrm{sep}}$. Then it follows that $F' := F.K' \subset K(x_1, \ldots, x_\rho, y_1, \ldots, y_\tau)^h.K' = K'(x_1, \ldots, x_k)^h$. (Here, the last equality is seen as follows: $K'(x_1, \ldots, x_k)^h$ contains K', $K(x_1, \ldots, x_\rho, y_1, \ldots, y_\tau)$ and, by the universal property of henselizations, also its henselization; hence, "$\subseteq$" holds. The converse follows from the universal property since the left hand side is henselian, being an algebraic extension of a henselian field.)

We can extend Q to $K'(x_1, \ldots, x_k)^h$ in such a way that it remains the identity on K'. By Lemma 2.6,

$$K'(x_1, \ldots, x_k)Q = K' ,$$

As K' is separable-algebraically closed, $(K', \overline{Q})$ is henselian. Therefore, we can deduce from Lemma 5.8 and the fact that the henselization is an immediate extension that $K'(x_1, \ldots, x_k)^h Q = K'$. Since $K' \subseteq F' \subseteq K'(x_1, \ldots, x_k)^h$, it follows that $F'Q = K'$.

Hence by Corollary 5.9, $(F'|K'(x_1,\ldots,x_k),P)$ is uniformizable. By Proposition 5.2, the same holds for $(K'(x_1,\ldots,x_k)|K',P)$. By transitivity, $(F'|K',P)$ is uniformizable.

Pick $\zeta_1,\ldots,\zeta_m \in \mathcal{O}_F$. By what we have proved, $(F'|K',P)$ is uniformizable with respect to $\zeta_1,\ldots,\zeta_m$. By part c) of Lemma 4.1, there is a finite Galois extension $\mathcal{K}'$ of $K(x_{k+1},\ldots,x_\rho,y_1,\ldots,y_\tau)$ such that $(F.\mathcal{K}'|\mathcal{K}',P)$ is uniformizable with respect to the ζ's. We take $\zeta'_1,\ldots,\zeta'_{m'} \in \mathcal{O}_{\mathcal{K}'}$ to be the uniformization coefficients of $F.\mathcal{K}',\zeta_1,\ldots,\zeta_m$ in $\mathcal{K}'$. Since P coincides with $\overline{Q}$ on $\mathcal{K}'$, we know that P is an Abhyankar place of $\mathcal{K}'|K$ of rank $r-1$. Now we have to distinguish two cases.

Suppose first that $(\mathcal{K}',P)$ has rank >1. Then by induction hypothesis, there is some $\mathcal{K}''$ obtained by at most $r-2$ (resp. $r-1$) Galois extensions from $\mathcal{K}'$, such that $(\mathcal{K}''|K,P)$ is uniformizable with respect to $\zeta'_1,\ldots,\zeta'_{m'}$. We set $\mathcal{F} := F.\mathcal{K}'' = F.\mathcal{K}'.\mathcal{K}''$. Then by part a) of Lemma 4.1 and the transitivity, $(\mathcal{F}|K,P)$ is uniformizable with respect to the ζ's. Further, $\mathcal{F}$ is obtained from F by at most $r-1$ (resp. r) Galois extensions, as required.

Now suppose that $(\mathcal{K}',P)$ has rank 1, i.e., $r=2$. By Theorem 1.7 there is a finite Galois extension $\mathcal{K}''$ of $\mathcal{K}'$ such that $(\mathcal{K}''|K,P)$ is uniformizable with respect to $\zeta'_1,\ldots,\zeta'_{m'}$. As before, we set $\mathcal{F} := F.\mathcal{K}''$ and it follows that $(\mathcal{F}|K,P)$ is uniformizable with respect to the ζ's. Now $\mathcal{F}$ is obtained from F by two Galois extensions, as required.

If in addition P is zero-dimensional, then we know from Theorem 1.7 that there is a finite Galois extension K' of K such that we can take $\mathcal{K}'' = \mathcal{K}'.K'$. In this case, $\mathcal{F} = F.\mathcal{K}'' = F.\mathcal{K}'.K'$ is a Galois extension of F. That is, $\mathcal{F}$ is obtained from F by one Galois extension.

- **Proof of Theorem 1.12**

Take function fields $F|K$ and $F'|K$ and a place P of $F'|K$ such that (F',P) lies in the completion of (F,P). Then by Proposition 5.5, $(F'|F,P)$ is uniformizable. Hence, the assertion of Theorem 1.12 follows from the corresponding assertions of Theorem 1.7 and Theorem 1.11 by use of part a) of Lemma 4.1 and transitivity.

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