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A Note on the Place Topology of Projective Planes

Dedicated to Prof. H. Salzmann on the occasion of his 70th birthday

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Abstract. It is shown that the place topology induced by a proper epimorphism of a projective plane Π, which is known to make Π a Lenz-topological plane, makes Π even a topological projective plane, if the extended radical of some underlying ternary field is bounded.

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As shown by Klingenberg in [9], proper epimorphisms $\varphi\colon \Pi \to \Pi'$ between Pappian projective planes can be described algebraically by places of the commutative fields K coordinatizing Π, places in the sense of Krull's valuation theory. It is not difficult to check, that the associated valuation topology on K makes Π a topological projective plane, i.e. it induces topologies on the point and line set of Π such that joining of points and intersecting of lines became continuous (as a general reference to topological planes we name the book of Salzmann *et al.* [12]).

In contrast to this classical setting, epimorphisms $\varphi\colon \Pi \to \Pi'$ between arbitrary projective planes Π and Π' are usually not induced by valuations (see say [10]), but can algebraically be described by places in the sense of André [1]. Following Pickert [11, § 1 p.31], we coordinatize $\Pi = \Pi(K, T)$ with respect to some frame (o, u, v, e) of Π by a Hall ternary field $K(o, u, v, e) = (K, T)$, identify the line joining o and v with $K \cup \{\infty\}$ by writing y for $(0, y)$ and ∞ for v, put $a + b := T(1, a, b), ab := T(a, b, 0), K^* := K \setminus \{0\}$, and take $a - b, -b, a/c$, and $c \setminus a$ to be the elements defined by $(a - b) + b = a, (-b) + b = 0$, $(a/c)c = a$, and $c(c \setminus a) = a$, respectively $(a, b, c \in K, c \neq 0)$. If φ maps the frame (o, u, v, e) onto a frame (o', u', v', e') of Π', then by

$$\varphi(y) = \lambda(y), \quad \text{for all } y \in K,$$

φ corresponds to a *place* λ between the associated ternary fields and vice versa, i.e. to a map $\lambda\colon K(o,u,v,e) \to K'(o',u',v',e') \cup \{\infty\}$ satisfying André's eight axioms [1]

(S1) $\lambda(0) = 0$ and $\lambda(1) = 1$,
(S2) $\lambda(m),\ \lambda(x),\ \lambda(c) \neq \infty \Rightarrow \lambda(T(m,x,c)) = T'(\lambda(m),\ \lambda(x),\ \lambda(c))$,
(S3) $\lambda(x) = \infty,\ \lambda(T(m,x,c)) \neq \infty,\ \lambda(c) \neq \infty \Rightarrow \lambda(m) = 0$,
(S4) $\lambda(m) = \infty,\ \lambda(T(m,x,c)) \neq \infty,\ \lambda(c) \neq \infty \Rightarrow \lambda(x) = 0$,
(S5) $\lambda(c) = \infty,\ \lambda(T(m,x,c)) \neq \infty \Rightarrow \infty \in \{\lambda(m),\ \lambda(x)\}$,
(S6) $y = T(m,x,c) = T(n,x,0),\ \lambda(x) = \lambda(y) = \infty,\ \lambda(c) \neq \infty \Rightarrow \lambda(m) = \lambda(n)$,
(S7) $0 = T(m,x,c),\ \lambda(m) = \lambda(c) = \infty,\ \lambda(T(m,u,c)) \neq \infty \Rightarrow \lambda(x) = \lambda(u)$,
(S8) $y = T(m,x,c) = T(n,x,0),\ 0 = T(m,u,c),\ \lambda(y),\ \lambda(m),\ \lambda(x),\ \lambda(c) = \infty$
$\Rightarrow \infty \in \{\lambda(n),\ \lambda(u)\}$,

a notion, which reduces to the well-known valuation theoretic notion of a place, provided K is a commutative field. As in the classical environment, let

$A_\lambda := \{k \in K \mid \lambda(k) \neq \infty\}$ denote the *place ring* and
$I_\lambda := \{k \in K \mid \lambda(k) = 0\}$ denote the *place ideal* of λ.

Due to Hartmann [2], [3], each epimorphism $\varphi\colon \Pi \to \Pi'$ induces a topology on the point and line set of Π, called the *place topology* τ_φ of Π, which is the coarsest topology such that the inverse images of points and lines are open and such that central projections are continuous. Restricted to (the line represented by) $K \cup \{\infty\}$ it yields the sets $\{k \in K \mid \lambda(\pi(k)) = \lambda(\pi(0))\}$ where π ranges over the projectivities on $K \cup \{\infty\}$ as a fundamental system of neighborhoods at 0. In particular, for some special cases of projective planes Π, as for instance desarguesian planes, Moufang planes, and certain generalized Moulton planes, Hartmann was able to show that τ_φ makes Π a topological plane (of course, provided φ is not injective, see [2] and [3]). We also obtain a topological projective plane, if $\varphi\colon \Pi \to \Pi'$ happens to be induced by a non-trivial *uniform valuation* v of the underlying ternary field (K,T), i.e. by a mapping $v\colon K \to \Gamma \cup \{0\}$ from K into an ordered loop $(\Gamma,\cdot)$ united with a least element 0 fulfilling:

(V1) $v(a) = 0 \Leftrightarrow a = 0$,
(V2) $v(ab) = v(a) \cdot v(b)$,
(V3) $v(a-b) \leqslant \max\{v(a), v(b)\}$,
(V4) $v(r) = 1$ for all $r \in R(K)$.

Herein, $R = R(K)$ denotes the *radical* of K, a normal subloop of $(K^*,\cdot)$ which measures the 'weakness' of (K,T). It is generated by those elements $r \in K^*$ for which there are $a,b,c,d,m,n,x,y \in K$ with $a \neq b$, $n \neq m$, $y \neq x$, and $T(m,y,c) =$

$T(n, y, d)$, such that at least one of the following equations holds

$$T(m, x, a) - T(m, x, b) = r \cdot (a - b)$$
$$T(n, x, d) - T(m, x, c) = r \cdot ((n - m) \cdot (x - y)).$$

Clearly, $v(K^*) \subset \Gamma$ is an abelian group, if and only if the *extended radical* $R_a = R_a(K)$ lies in $U_v := \{k \in K \mid v(k) = 1\}$, i.e. the normal subloop R_a of K^* generated by R and by those $r \in K^*$ for which there exist $x, y, z \in K^*$ such that

$$x(yz) = r \cdot ((xy)z) \quad \text{or} \quad xy = r \cdot (yx).$$

K is a commutative field, if and only if $R_a(K) = \{1\}$. In [4] we have shown that each uniform valuation v of (K, T) gives rise to a place $\lambda: K \to A_v/I_v \cup \{\infty\}$ with *valuation ring* $A_v := \{k \in K \mid v(k) \leqslant 1\} = A_\lambda$ and *valuation ideal* $I_v := \{k \in K \mid v(k) < 1\} = I_\lambda$, and thus yields an epimorphism φ_v from Π onto $\Pi' = \Pi(A_v/I_v)$. If additionally $R_a \subset U_v$, then such an epimorphism (eventually concatenated with an isomorphism of Π') is called a *friendly epimorphism*, cf. [8]. As shown in [7], given a non-trivial uniform valuation v of K (i.e. $A_v \neq K$), taking the sets

$$A_v \cdot g = \{k \in K \mid v(k) \leqslant v(g)\} \qquad \text{with } g \in K^*$$

or equivalently the sets

$$I_v \cdot g = \{k \in K \mid v(k) < v(g)\} \qquad \text{with } g \in K^*$$

as a fundamental system of neighborhoods at 0 we obtain a topological ternary field (K, T) that makes Π a totally disconnected, topological projective plane. The place topology τ_φ associated to φ_v meets this *valuation topology* on Π (cf. [7, 2.5b] or (7) below with $\lambda = \mu$) and therefore yields a topological projective plane.

However, in general it is an open question, whether or not the place topology induced by some proper (i.e. non-injective) epimorphism of an arbitrary projective plane Π makes Π a topological projective plane. It is even uncertain, if the place topology is always not discrete. In [2], [3] Hartmann was only able to show, that the place topology makes Π a (probably discrete) Lenz-topological projective plane. The goal of this note is it to prove the following.

THEOREM 1. *Let $\lambda: K \to K' \cup \{\infty\}$ be a place of ternary fields such that the extended radical of K is bounded with respect to the place topology, that is $A_\lambda \cdot R_a(K) \neq K$. Then the place topology induced by λ makes the projective plane over K a (non-discrete) topological projective plane.*

The idea behind this theorem is to coarsen λ to a place $\mu: K \to K'' \cup \{\infty\}$ which is induced by a non-trivial uniform valuation with an abelian value group, or equivalently to coarsen the epimorphism $\varphi: \Pi \to \Pi'$ defined by λ to a proper, friendly epimorphism $\psi: \Pi \to \Pi''$. Here ψ (or μ) is called *coarser* than φ (or than

λ), if and only if the following, obviously equivalent conditions hold: φ factors through ψ (i.e. there is an epimorphism $\rho: \Pi'' \to \Pi'$ with $\varphi = \rho\psi$), λ factors through μ, $A_\lambda \subset A_\mu$, and $I_\mu \subset I_\lambda$. To prove our theorem, we start with an algebraic description of friendly epimorphisms.

LEMMA 2. *A subset A of a ternary field (K, T) is a valuation ring of a uniform valuation on K inducing a friendly epimorphism, if and only if*

(A′1) $A - 1 \subset A$,
(A′2) $A \cdot A \subset A$,
(A′3) $R_a(K) \subset A$,
(A′4) *for all $x \in K$ we have $x \in A$ or $1/x \in A$.*

Proof. By (S1) up to (S5) and in view of [8, 2.3], any valuation ring of a uniform valuation inducing a friendly epimorphism satisfies the axioms given above. For the reverse, we shall show that a subset A which fulfils (A′1) up to (A′4) is a valuation ring of (K, T) in the sense of [5], i.e. that

(A1) $(A, +)$ is a subloop of $(K, +)$,
(A2) $A \cdot A \subset A$,
(A3) $R(K) \subset A$,
(A4) A is normal in $(K^*, \cdot)$,
(A5) for all $x \in K$ we have $x \in A$ or $1/x \in A$.

Clearly, (A2), (A3), (A5) and (A4) are fulfilled, the latter by the facts that $R_a \cdot A = A$ and that K^*/R_a is an Abelian group, $R_a := R_a(K)$. Hence it remains to check (A1). First note that from (A′1) and $1 \in R_a \subset A$, we get $0 = 1 - 1 \in A$, so $-1 = 0 - 1 \in A$, and thus, making use of [6, 0.1(4)], $(-1)A \subset A = (-1)^2 A \subset (-1)A$, which by [7, 1.3(5)] finally leads to

$$-(x - y)A = (y - x)A = (x - y)A \qquad \text{for all } x, y \in K.$$

Now let $a, a' \in A$ be given. Then we have $a' \setminus a \in A$ or $a \setminus a' \in A$ (or $a = a' = 0$), because $x := a' \setminus a \in K \setminus A$ implies $a \setminus a' = (a'x) \setminus a' \in R_a \cdot 1/x \subset A$ by (A5). If $a' \setminus a \in A$, then we find

$$a - a' \in A(a - a') = Aa'(a' \setminus a - 1) \subset AA(A - 1) \subset A,$$

and if $a \setminus a' \in A$, then we get

$$a - a' \in A(a - a') = A(a' - a) \subset A,$$

too. Hence we have that the solution y of $y + a' = a$ lies in A.

To show that also the solution z of $a' + z = a$ lies in A, using [7, 1.3(2)] we compute

$$a' + z = a = (a - a') + a' \in R_a(a - a') + a' = a' + R_a(a - a'),$$

which means $z \in R_a(a - a')$. So, by the above, we get

$$z \in R_a(a - a') \subset R_a A = A.$$

Finally, making use of [7, 1.3(3)], we obtain

$$A + A = A + (-A) = A - A \subset A. \qquad \square$$

COROLLARY 3. *Any coarsening of a friendly epimorphism of a projective plane* Π *is also a friendly epimorphism of* Π.

COROLLARY 4. *A surjective mapping* $\lambda: K \to K' \cup \{\infty\}$ *between ternary fields is a place inducing a friendly epimorphism, if and only if the following axioms hold:*

(S*1) $\lambda(a),\ \lambda(b) \neq \infty \Rightarrow \lambda(a + b) = \lambda(a) + \lambda(b),\ \lambda(a \cdot b) = \lambda(a) \cdot \lambda(b)$,
(S*2) $\lambda(a) = \infty,\ \lambda(b) \neq \infty \Rightarrow \lambda(a + b) = \lambda(b + a) = \infty$,
$\lambda(a) = \infty,\ \lambda(b) \neq 0 \Rightarrow \lambda(ab) = \lambda(ba) = \infty$,
(S*3) $r \in R_a(K) \Rightarrow \lambda(r) \neq \infty$.

Proof. Plainly, by (S1) up to (S5) and in view of [8, 2.3], any surjective place inducing a friendly epimorphism has to fulfil the given axioms. For the reverse, apply the lemma above to show that $A := \{k \in K \mid \lambda(k) \neq \infty\}$ is the valuation ring of some uniform valuation v of K inducing a friendly epimorphism. As in [5, 1.5] one checks, that λ and the place $\lambda_v: K \to A_v/I_v \cup \{\infty\}$ defined by v are equivalent, i.e. that K' and A_v/I_v are isomorphic and that a suitable isomorphism α fulfils $\lambda = \alpha\lambda_v$ with $\alpha(\infty) := \infty$. $\square$

PROPOSITION 5. *Let* $\lambda: K \to K' \cup \{\infty\}$ *be a place of ternary fields. The finest coarsening of* λ *that induces a friendly epimorphism is given by the place ring* $A := A_\lambda \cdot R_a(K)$.

Proof. We shall show that A satisfies the axioms presented in the lemma above. Since, by (S1) up to (S5), A_λ fulfils (A′2) and (A′4), also A does. Further, A trivially fulfils (A′3). So it remains to check (A′1).

Therefore let $ar \in A$ with $a \in A_\lambda$ and $r \in R_a := R_a(K)$ be given. If $r \in A_\lambda$, then we have

$$ar - 1 \in A_\lambda - 1 \subset A_\lambda \subset A.$$

If $r \notin A_\lambda$, then we have $1/r \in A_\lambda$, and using [6, 0.1(2)] we find

$$ar - 1 \in R_a(ar - 1) = R_a(a - 1/r)r \subset R_a A_\lambda r = A. \qquad \square$$

Of course, the place ring $A_\lambda R_a$ considered above may be trivial, i.e. equal to K. But if not, then the fact that the epimorphism φ of Π can be coarsened to a proper friendly one has some impact also on φ, as the following proposition shows.

LEMMA 6. *Let λ be a non-injective place of the ternary field (K, T), and let τ_φ be the place topology of $\Pi = \Pi(K, T)$ induced by the epimorphism $\varphi\colon \Pi \to \Pi'$ associated to λ. If τ is a topology on Π making Π a topological projective plane such that I_λ is open in the restriction $\tau_{|K}$, then τ is finer than τ_φ.*

Proof. Since central projections are continuous in each topology making Π a topological projective plane, we only have to check that inverse images of points (and dually of lines) are open with respect to τ. Therefore let p be any point of Π. Coordinatizing Π by a frame (o, u, v, e) which is mapped onto a frame by φ and which fulfils $o = p$ we obtain a second ternary field (K', T') of Π, where φ corresponds to a place λ' of K', where the point p has the coordinates $(0', 0')$ and where the inverse image $\varphi^{-1}(\varphi(p))$ equals $I_{\lambda'} \times I_{\lambda'}$ with respect to the coordinates associated to K'.

There exists a projectivity π from the line L representing $K \cup \{\infty\}$ onto the line L' representing $K' \cup \{\infty'\}$ with $\pi(0) = 0'$ and with

$$\varphi(x) = \varphi(y) \Leftrightarrow \varphi(\pi(x)) = \varphi(\pi(y)) \qquad \text{for all points } x, y \in K \cup \{\infty\} = L.$$

Namely, in the case $\varphi(0) \notin \varphi(L')$ and $\varphi(0') \notin \varphi(L)$, choose any point z on the line joining 0 and $0'$ with $\varphi(z) \neq \varphi(0), \varphi(0')$ and simply take π to be the perspectivity from L onto L' with center z; in all other cases, choose an additional line L'' and an additional point $0''$ on L'' with $\varphi(0)$, $\varphi(0') \notin \varphi(L'')$ and $\varphi(0'') \notin \varphi(L)$, $\varphi(L')$, apply the construction above twice, and take the concatenation $L \to L'' \to L'$ of the two perspectivities found. Now we compute

$$\pi(I_\lambda) = \{\pi(x) \mid \varphi(x) = \varphi(0)\} = \{\pi(x) \mid \varphi(\pi(x)) = \varphi(\pi(0))\}$$

$$= \{x' \mid \varphi(x') = \varphi(0')\} = I_{\lambda'}.$$

Since π is a homeomorphism with respect to the restrictions of τ to L and to L', we find that also I'_λ is open in $\tau_{|K'}$. Since τ locally meets the product topology of two affine lines, the inverse image $\varphi^{-1}(\varphi(p)) = I_{\lambda'} \times I_{\lambda'}$ (with respect to K') is open in τ. □

PROPOSITION 7. *Let λ be a place of the ternary field (K, T) factoring through a place μ of (K, T) which is induced by a non-trivial uniform valuation, and let φ and ψ be the associated epimorphisms of $\Pi = \Pi(K, T)$. Then the place topology τ_φ equals the valuation topology τ induced by ψ on Π and makes Π a topological projective plane.*

Proof. Since φ factors through ψ, by [2, 2.18], the place topology τ_ψ induced by ψ is finer than the place topology τ_φ given by φ. By virtue of [7, 2.1], the valuation topology τ induced by ψ makes Π a topological projective plane, and in view of the lemma above it is finer than τ_ψ. Hence we have that τ is finer that τ_φ.

To show that τ_φ is finer than τ, in face of [2, 2.13], it suffices to check that the system of neighborhoods at the point 0 with respect to the restriction $\tau_{\varphi|G}$ to the line $G = K \cup \{\infty\}$ is finer than that with respect to $\tau_{|G}$. A fundamental system of neighborhoods at 0 of the valuation topology τ restricted to $G \setminus \{\infty\}$ is given by

$$A_\mu \cdot g \qquad \text{with } g \in K^*.$$

Note, that we have $A_\lambda \subset A_\mu$, because μ is a coarsening of λ. So, for any $g \in K^*$ we find $I_\lambda \cdot g \subset A_\lambda \cdot g \subset A_\mu \cdot g$, which shows that each $A_\mu \cdot g$ is a neighborhood of 0 also with respect to τ_φ. □

Now our theorem is immediate from the two propositions above. We close our note with some corollaries and remarks.

COROLLARY 8. *Let φ be an epimorphism of a projective plane Π which factors through a non-injective, friendly epimorphism ψ of Π. Then φ and ψ define the same place topology on Π making Π a (non-discrete) topological projective plane.*

Proof. We coordinatize Π via a frame which is mapped onto a frame by φ (and then also by ψ), and obtain a ternary field (K, T) in which φ and ψ are described by the places λ and μ respectively. Since ψ is friendly, by virtue of [8, 2.3], up to equivalence the place μ is induced by some uniform valuation of K. The claim is now a direct consequence of the preceding proposition. □

COROLLARY 9. *Let λ be a place of the ternary field (K, T) with $A_\lambda \cdot R_a(K) \neq K$, and let φ be the associated epimorphism of $\Pi = \Pi(K, T)$. Then the restriction of the place topology τ_φ (which makes Π a totally disconnected topological plane) to K has the sets*

$$A_\lambda \cdot g \qquad \textit{with } g \in K^*$$

or equivalently the sets

$$I_\lambda \cdot g \qquad \textit{with } g \in K^*$$

as a fundamental system of neighborhoods at 0.

Proof. By the above, $A_v = A_\lambda R_a$ with $R_a := R_a(K)$ is the valuation ring of a non-trivial uniform valuation v on K inducing the same topology on Π as φ. Hence $\tau_{\varphi|K}$ has the sets

$$A_v \cdot g \qquad \text{with } g \in K^*$$

as a fundamental system of neighborhoods at 0. Clearly, we have $A_\lambda \cdot g \subset A_v \cdot g$ for all $g \in K$. For the reverse, choose some $t \in I_v \setminus \{0\}$. Since A_v is normal with respect to multiplication, we get for all $g \in K^*$

$$A_\nu \cdot (tg) = (A_\nu t) \cdot g \subset I_\nu \cdot g \subset I_\lambda \cdot g \subset A_\lambda \cdot g.$$

Similarly, one shows that also the sets $I_\lambda \cdot g$ with g ranging over K^* form a fundamental system of neighborhoods at 0. □

Remarks. (a) Note that the verification of axiom (A1) in the proof of (2) only requires $R(K) \subset A$ and not $R_a(K) \subset A$. Hence, also within the characterization of the valuation rings of arbitrary uniform valuations in [6], the axiom (A1) may be replaced by the weaker axiom (A′1).

(b) Generally, examples of ternary fields with a small radical can be obtained by the classical constructions of projective planes out of Pappian or desarguesian planes via bending lines or via disturbing the sum and/or the product of the underlying field. If these distortions are 'small' (say, with respect to a uniform structure, a topology, a norm, a valuation, etc.), chances are good that also the radical of the arising ternary field is 'small'. Examples along this line can be found in section three of [7].

(c) Friendly epimorphisms also have some significance within other geometrical settings. For instance, they are related to the matroid theoretical notion of valuations due to Dress and Wenzel, they occur in Maldeghem's classification of triangle buildings, they play a role in the description of Hjelmslev planes, they can be characterized by certain Junker's order functions, and so on. For literature on these topics see [4], [7], [8] and [10].

(d) [7] and the results of this note suggest, that each topological ternary field in which the extended radical (or maybe only the radical) is bounded in some suitable way should coordinatize a topological projective plane.

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