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ON EPIMORPHISMS AND PROJECTIVITIES OF PROJECTIVE PLANES[1]

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We present geometric characterizations of those epimorphisms of (arbitrary) projective planes which stem from uniform valuations admitting an abelian value group. In particular, we study relations between such epimorphisms and the groups of projectivities of the projective planes involved.

Given an epimorphism $\varphi: \Pi \to \Pi'$ between projective planes Π and Π', it is an open question how the groups of projectivities of Π and Π' (regarded as permutation groups on projective lines) are related. Within this note we will not answer this sophisticated and hard problem in full, but we will address the question to which extend the projectivities of Π induce permutations on the lines of Π' which are distinct from the projectivities of Π'. Questions of this kind are especially of interest when functions on projective planes which are invariant under perspectivities, such as orderings or half orderings, are subject to be lifted via an epimorphism.
In particular, we show that for any pappian projective plane Π' there exists a projective plane Π and an epimorphism $\varphi: \Pi \to \Pi'$ such that the projectivities of Π induce the full symmetric group on the lines of Π' via φ (in this case no non-trivial function on Π' invariant under perspectivities lifts to Π). On the other hand, in terms of valuations and places of coordinatizing ternary fields we will characterize certain situations where only the projectivities of Π' and no further permutations are induced through φ.

We denote the points (resp. lines) of a *projective plane* Π by lower case (resp. upper case) Latin letters, and write pq for the line joining two distinct points p and q of Π. Given a frame (o,u,v,e) of Π, following Pickert [9, §1 p.31], we coordinatize the affine

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plane Π_{uv} with respect to uv as line at infinity and get a Hall *ternary field* K(o,u,v,e) = (K,T) of $\Pi = \Pi(K,T)$. We identify K with $ov \setminus \{v\}$ and simply write y for (0,y) and ∞ for v. Further let $a + b := T(1,a,b)$, $ab := T(a,b,0)$, $K^* := K \setminus \{0\}$, and take $a - b$, $-b$, a/c, and $c \setminus a$ to be the elements defined by $(a - b) + b = a$, $(-b) + b = 0$, $(a/c)c = a$, and $c(c \setminus a) = a$ respectively ($a,b,c \in K$, $c \neq 0$). Note that $(a + b) - b = a$, $(ac)/c = a$, $c \setminus (ca) = a$, and that in general (K, +) and (K^*, $\cdot$) are non-associative loops. If (K, +) is a group and if $T(m,x,c) = mx + c$ for all $m,x,c \in K$, then K is called a *cartesian field.*

An *epimorphism* φ of Π onto a projective plane Π' is a map from the point and line set of Π onto the point and line set of Π' that preserves incidence. If φ maps the frame (o,u,v,e) of Π onto a frame (o',u',v',e') of Π' then, by

$$\varphi(y) = \lambda(y) \qquad \text{for all points y on } ov = K \cup \{\infty\},$$

φ corresponds to a *place* λ between the associated ternary fields and vice versa, i.e. to a map $\lambda: K(o,u,v,e) \to K'(o',u',v',e') \cup \{\infty\}$ satisfying André's eight axioms [10, p.242]

(S1) $\lambda(0) = 0$ and $\lambda(1) = 1$,

(S2) $\lambda(m), \lambda(x), \lambda(c) \neq \infty \;\Rightarrow\; \lambda(T(m,x,c)) = T'(\lambda(m),\lambda(x),\lambda(c))$,

(S3) $\lambda(x) = \infty,\; \lambda(T(m,x,c)) \neq \infty,\; \lambda(c) \neq \infty \;\Rightarrow\; \lambda(m) = 0$,

(S4) $\lambda(m) = \infty,\; \lambda(T(m,x,c)) \neq \infty,\; \lambda(c) \neq \infty \;\Rightarrow\; \lambda(x) = 0$,

(S5) $\lambda(c) = \infty,\; \lambda(T(m,x,c)) \neq \infty \;\Rightarrow\; \infty \in \{\lambda(m), \lambda(x)\}$,

(S6) $y = T(m,x,c) = T(n,x,0),\; \lambda(x) = \lambda(y) = \infty,\; \lambda(c) \neq \infty \;\Rightarrow\; \lambda(m) = \lambda(n)$,

(S7) $0 = T(m,x,c),\; \lambda(m) = \lambda(c) = \infty,\; \lambda(T(m,u,c)) \neq \infty \;\Rightarrow\; \lambda(x) = \lambda(u)$,

(S8) $y = T(m,x,c) = T(n,x,0),\; 0 = T(m,u,c),\; \lambda(y),\lambda(m),\lambda(x),\lambda(c) = \infty \;\Rightarrow\; \infty \in \{\lambda(n), \lambda(u)\}$.

In case K is a commutative field, this notion reduces to the well-known valuation theoretic notion of a place. Paralleling this classical setting, let

$A_\lambda := \{ k \in K \mid \lambda(k) \neq \infty \}$ denote the *place ring* of λ,

$U_\lambda := \{ k \in K \mid \lambda(k) \neq 0, \infty \}$ denote the *set of units* of λ,

$I_\lambda := \{ k \in K \mid \lambda(k) = 0 \}$ denote the *place ideal* of λ,

and call two places $\lambda: K \to K' \cup \{\infty\}$ and $\mu: K \to K'' \cup \{\infty\}$ *equivalent*, if there exists an isomorphism $\alpha: K' \to K''$ such that $\mu(k) = \alpha\lambda(k)$ for all $k \in A_\lambda$. Accordingly, two epimorphisms φ, ψ of Π are *equivalent*, if there exists an isomorphism ρ with $\varphi = \rho\psi$.

We denote the *group of projectivities* from a line L of $\Pi = \Pi(K,T)$ onto L by $\mathscr{P}(L)$ and consider it as a permutation group of L. In particular, $\mathscr{P}(ov)$ is a permutation group of $K \cup \{\infty\}$. Since any two lines of Π can be mapped onto each other by a perspectivity, the permutation group $\mathscr{P}(L)$ does not depend on L, and we simply write $\mathscr{P}$ instead of $\mathscr{P}(ov)$ or $\mathscr{P}(L)$. $\mathscr{P}_p$ means the stabilizer of a point $p \in L$ in $\mathscr{P}$.

1. PROJECTIVITIES UNDER EPIMORPHISMS

Within this section let $\varphi: \Pi \to \Pi'$ be an epimorphism mapping the line L of Π onto L'. Given a projectivity $\pi \in \mathscr{P}$, via

$$\pi'(\varphi(p)) := \varphi(\pi(p)) \qquad \text{for all } p \in L$$

π induces a well-defined permutation π' of L', if and only if π lies in

$$\mathscr{G} := \{ \pi \in \mathscr{P} \mid \text{for all } p,q \in L\text{: } \varphi(p) = \varphi(q) \Leftrightarrow \varphi(\pi(p)) = \varphi(\pi(q)) \}.$$

One easily checks that $\mathscr{G}$ is a group, and that the mapping $\Phi: \mathscr{G} \to \mathrm{Sym}(L')$ sending π onto π' is a group homomorphism with kernel

$$\mathscr{T} := \{ \pi \in \mathscr{P} \mid \text{for all } p \in L\text{: } \varphi(p) = \varphi(\pi(p)) \},$$

i.e. its image, the *group* $\mathscr{G}^\varphi := \Phi(\mathscr{G})$ *of permutations of* L' *induced by* $\mathscr{G}$ fulfills

$$\mathscr{G}^\varphi = \mathscr{G} / \mathscr{T}.$$

Besides $\mathscr{G}^\varphi$ we have another permutation group acting on L', namely the group $\mathscr{P}'$ of projectivities of Π'. In what follows, we will study relations between these two groups.

(1.1) LEMMA. *Let* $\pi' \in \mathscr{P}'$ *and let* $x'_1, x'_2 \in L'$ *be distinct. Choose arbitrary inverse images* $x_i, y_i \in L$ *under* φ *of* x'_i *and* $y'_i := \pi'(x'_i)$, $i = 1, 2$. *Then there exists a projectivity* $\pi \in \mathscr{P}$ *of* Π *which induces* π' *and maps* x_i *onto* y_i, $i = 1, 2$.

Proof. Since each element of $\mathscr{P}'$ is a product of perspectivities, it suffices to show that each perspectivity $\alpha': G' \to H'$ with center z' of Π' (i.e. G', H' are distinct lines of Π' not incident with the point z', and $\alpha'(x') = z'x' \cap H'$ for all $x' \in G'$) mapping two given points $x'_1, x'_2 \in G'$ onto $y'_1, y'_2 \in H'$ lifts to a projectivity $\alpha: G \to H$ mapping x_i onto y_i where $G \in \varphi^{-1}(G')$, $H \in \varphi^{-1}(H')$, $x_i \in \varphi^{-1}(x'_i) \cap G$, and $y_i \in \varphi^{-1}(y'_i) \cap H$ can be taken arbitrarily ($i = 1, 2$).

In the case that x'_1, x'_2, y'_1, y'_2 form a frame, after choosing the elements G, H, x_i, y_i simply put $z := x_1y_1 \cap x_2y_2$ and take α to be the perspectivity from G onto H with center z. Then $\varphi(z) = z'$, $\alpha(x_i) = y_i$ ($i = 1, 2$), and $\alpha'(\varphi(x)) = \alpha(\varphi(x))$ for all $x \in G$.

In the case that x'_1, x'_2, y'_1, y'_2 do not form a frame, say $x'_1 = y'_1$, take a line M' of Π' with $x'_i, y'_i, z' \notin M'$ ($i \neq 1, 2$) and consider the perspectivities $\alpha'_1: G' \to M'$ with center z' and $\alpha'_2: M' \to H'$ with center z'. Clearly, we have $\alpha' = \alpha'_2\alpha'_1$. Since $x'_1, x'_2, \alpha'_1(x'_1), \alpha'_2(x'_2)$ as well as $\alpha'_1(x'_1), \alpha'_2(x'_2), y'_1, y'_2$ form frames, by the case settled above we obtain two perspectivities $\alpha_1$and α_2 of Π, such that the projectivity $\alpha := \alpha_2\alpha_1$ has the desired properties. □

(1.2) COROLLARY. *$\mathscr{P}'$ is a subgroup of $\mathscr{S}^{\varphi}$.*

(1.3) LEMMA. *Let $\pi' \in \mathscr{P}'$ and let $x'_1, x'_2, x'_3 \in L'$ be distinct. Choose arbitrary inverse images $x_i, y_i \in L$ under φ of x'_i and $y'_i := \pi'(x'_i)$, $i = 1, 2, 3$. Then there exists a projectivity $\pi \in \mathscr{P}$ of Π which induces π' and maps x_i onto y_i for $i = 1, 2, 3$.*

Proof. In view of Lemma (1.1) π' lifts to a projectivity $\sigma \in \mathscr{P}$ of Π with $\sigma(x_i) = y_i$ for $i = 1, 2$. We choose a frame (o, u, v, e) of Π which is mapped onto a frame by φ such that $o = y_1$, $v = y_2$, and $ov \cap ue = y_3$. Then, coordinatizing Π by the ternary field $K = K(o, u, v, e)$, φ is given by a place λ of K via

$$\varphi(x) = \lambda(x) \qquad \text{for all } x \in ov = K \cup \{\infty\},$$

and we have $0 = y_1$, $\infty = y_2$, $1 = y_3$. Put $c := \sigma(x_3) \in K$. Since σ induces π', we find

$$\lambda(c) = \varphi(\sigma(x_3)) = \pi'(\varphi(x_3)) = \pi'(x'_3) = y'_3 = \varphi(y_3) = \lambda(1) = 1.$$

Now let $\gamma \in \mathscr{P}$ be the projectivity mapping ∞ onto ∞ and $x \in K$ onto xc, and consider $\pi := \gamma^{-1}\sigma \in \mathscr{P}$. As desired, the latter fulfills

$$\pi(x_1) = \gamma^{-1}(y_1) = \gamma^{-1}(0) = 0 = y_1,$$
$$\pi(x_2) = \gamma^{-1}(y_2) = \gamma^{-1}(\infty) = \infty = y_2,$$
$$\pi(x_3) = \gamma^{-1}(c) = 1 = y_3,$$

and for all (other) $x \in K$ (by (S2), (S4), and $\lambda(c) = 1$)

$$\varphi(\pi(x)) = \lambda(\gamma^{-1}(\sigma(x))) = \lambda(\sigma(x)/c) = \lambda(\sigma(x)) = \varphi(\sigma(x)) = \pi'(\varphi(x)). \square$$

Given three points $a, b, c \in L$ with mutually distinct $\varphi(a)$, $\varphi(b)$, $\varphi(c)$, let $(\mathscr{S}_{a,b,c})^{\varphi} := \Phi(\mathscr{S}_{a,b,c})$ be the group of permutations of $L' = \varphi(L)$ which are induced via φ by those projectivities of Π from L onto L lying in $\mathscr{S}$ and stabilizing {a,b,c} pointwise. Instantly, from (1.3) we obtain the following.

(1.4) COROLLARY. $\mathscr{P}' \cap (\mathscr{S}_{a,b,c})^{\varphi} = \mathscr{P}'_{\varphi(a),\varphi(b),\varphi(c)}$.

(1.5) PROPOSITION. *Let $\varphi: \Pi \to \Pi'$ be an epimorphism of projective planes, and let a, b, c be collinear points of Π with mutually distinct $\varphi(a)$, $\varphi(b)$, $\varphi(c)$. Then we have*

$$\mathscr{S}^{\varphi} = \mathscr{P}' \cdot (\mathscr{S}_{a,b,c})^{\varphi}.$$

Proof. Let $\sigma' \in \mathscr{S}^{\varphi}$ be induced by some $\sigma \in \mathscr{S}$. Since $\mathscr{P}'$ acts 3-transitive on $\varphi(ab)$, there exists a projectivity $\pi' \in \mathscr{P}'$ mapping $\varphi(a)$, $\varphi(b)$, and $\varphi(c)$ onto $\sigma'(\varphi(a))$, $\sigma'(\varphi(b))$, and $\sigma'(\varphi(c))$ respectively. In view of the preceding lemma, π' lifts to a projectivity $\pi \in \mathscr{S}$ with $\pi(a) = \sigma(a)$, $\pi(b) = \sigma(b)$, and $\pi(c) = \sigma(c)$. Hence $\rho := \pi^{-1}\sigma \in \mathscr{S}_{a,b,c}$, and the permutation ρ' induced by ρ on $\varphi(ab)$ lies in $(\mathscr{G}_{a,b,c})^{\varphi}$ and fulfills $\sigma' = \pi'\rho'$. This shows $\mathscr{G}^{\varphi} \subset \mathscr{P}'(\mathscr{G}_{a,b,c})^{\varphi}$. The reverse inclusion is clear by (1.2). □

Immediately from (1.4) and (1.5) we get:

(1.6) COROLLARY. *Let* $\varphi: \Pi \to \Pi'$ *be an epimorphism of projective planes, and let* a, b, c *be collinear points of* Π *with mutually distinct images under* φ. *Then we have*

$$\mathscr{G}^{\varphi} = \mathscr{P}' \Leftrightarrow \mathscr{P}'_{\varphi(a),\varphi(b),\varphi(c)} = (\mathscr{G}_{a,b,c})^{\varphi}.$$

In the case of pappian projective planes, which are characterized by the fact that any projectivity fixing three points is the identity (see, say [9]), we obtain the following specializations.

(1.7) COROLLARY. *Let* $\varphi: \Pi \to \Pi'$ *be an epimorphism of projective planes, and let* a, b, c *be collinear points of* Π *with mutually distinct images under* φ.

(a) *If* Π *is pappian, then* $\mathscr{G}^{\varphi} = \mathscr{P}'$.

(b) Π' *is pappian, if and only if* $\mathscr{P}' \cap (\mathscr{G}_{a,b,c})^{\varphi} = \{id\}$.

(1.8) REMARK. If φ is a proper epimorphism (i.e. not injective), then $\mathscr{G}$ is an imprimitive group which has the set of inverse images $\varphi^{-1}(p')$ of the points p' of L' as a system of blocks. Identifying this system with L', $\mathscr{G}^{\varphi}$ may be regarded as the induced permutation group of this system.

2. PROJECTIVITIES UNDER FRIENDLY EPIMORPHISMS

A *uniform valuation* of a ternary field (K, T) is a mapping $v: K \to \Gamma \cup \{0\}$ from K into an ordered loop $(\Gamma, \cdot)$ united with a least element 0 fulfilling:

(V1) $v(a) = 0 \Leftrightarrow a = 0$,

(V2) $v(ab) = v(a) \cdot v(b)$,

(V3) $v(a - b) \leq \max\{v(a), v(b)\}$,

(V4) $v(r) = 1$ for all $r \in R(K)$.

Herein, R = R(K) denotes the *radical* of K, i.e. the normal subloop of $(K^*, \cdot)$ generated by those elements $r \in K^*$ for which there are a, b, c, d, m, n, x, y $\in$ K with $a \neq b$, $n \neq m$, $y \neq x$, and $T(m,y,c) = T(n,y,d)$, such that at least one of the following equations holds

(i) $T(m,x,a) - T(m,x,b) = r \cdot (a-b)$

(ii) $T(n,x,d) - T(m,x,c) = r \cdot ((n-m) \cdot (x-y))$.

Obviously, v may be regarded as a loop homomorphism of $(K^*, \cdot)$ containing R in its kernel $U_v = \{k \in K \mid v(k) = 1\}$. Therefore, if v is onto, its value loop Γ is an abelian group, if and only if even the *extended radical* R_a of K lies in U_v, i.e. the normal subloop R_a of K^* generated by R(K) and by those $r \in K^*$ for which there exist $x,y,z \in K^*$ such that

(iii) $x(yz) = r \cdot ((xy)z)$ or

(iv) $xy = r \cdot (yx)$.

Note that for a cartesian field K, (i) and (ii) reduce to

(i') $a + e - a = r \cdot e$

(ii') $ac - ad + bd - bc = r \cdot [(a-b)(c-d)]$,

(a, b, c, d, e ranging over K with $a \neq b$, $c \neq d$, $e \neq 0$) and that $R_a(K) = \{1\}$, if and only if K is a commutative field. As shown in [1], each uniform valuation v of (K, T) gives rise to a place $\lambda: K \to A_v/I_v \cup \{\infty\}$ with *valuation ring* $A_v := \{k \in K \mid v(k) \leq 1\} = A_\lambda$ and *valuation ideal* $I_v := \{k \in K \mid v(k) < 1\} = I_\lambda$, and therefore yields an epimorphism of $\Pi(K)$. But in contrast to the situation for pappian planes, in general, not every epimorphism of an arbitrary projective plane $\Pi(K)$ can be obtained in this way (see say [7] and note that the uniform valuations of a skew field are exactly its Schilling valuations [6, 3.1]).

DEFINITION. An epimorphism $\varphi: \Pi \to \Pi'$ of a projective plane Π is called a *friendly epimorphism,* if Π can be coordinatized by a ternary field (K, T) admitting a uniform valuation v which induces φ up to equivalence and has an abelian value group.

(2.1) LEMMA. *Let* $\lambda: K \to K' \cup \{\infty\}$ *be a place of ternary fields containing the extended radical* $R_a(K)$ *in its place ring* A_λ. *Then, up to equivalence,* λ *is induced by a uniform valuation admitting an abelian value group, and therefore yields a friendly epimorphism* $\varphi: \Pi(K) \to \Pi(K')$.

Proof. Plainly, in view of the axioms (S1) up to (S5), the place ring A_λ is a subloop of (K, +), it is multiplicatively closed, contains the radical of K, and it is total (i.e. for all $k \in K$ we have $k \in A_\lambda$ or $1/k \in A_\lambda$). Further, A_λ is normal with respect to multiplication, since it even contains $R_a(K)$ and $K^*/R_a(K)$ is an abelian group. Hence, by virtue of [3, 1.4], A_λ is a uniform valuation ring of K, that means K carries a uniform valuation v: $K \to \Gamma \cup \{0\}$ with valuation ring A_λ and with an abelian value group Γ. As in [3, 1.5] one shows, that λ and the place induced by v are equivalent. □

(2.2) THEOREM. *Let* $\varphi: \Pi \to \Pi'$ *be an epimorphism of projective planes, and let* a, b *be points of* Π *with* $\varphi(a) \neq \varphi(b)$. *Then the following statements are equivalent.*

(a) φ *is a friendly epimorphism,*

(b) *for all* $c,x \in ab\setminus\{a,b\}$ *and all* $\pi \in \mathscr{P}_{a,b,c}$: $\varphi(x) = \varphi(b) \Leftrightarrow \varphi(\pi(x)) = \varphi(b)$,

(c) *for all* $c,x \in ab\setminus\{a,b\}$ *and all* $\pi \in \mathscr{P}_{a,b,c}$: $\varphi(x) \neq \varphi(a),\varphi(b) \Rightarrow \varphi(\pi(x)) \neq \varphi(a),\varphi(b)$.

Proof. ($a \Rightarrow b$). By definition, Π can be coordinatized by a ternary field $K = K(o,u,v,e)$ such that (up to equivalence) φ is induced by a uniform valuation of K containing R_a := $R_a(K)$ in its valuation ring A. For all $c' \in K^*$, all $\pi \in \mathscr{P}_{0,v,c'} = \mathscr{P}_{0,\infty,c'}$, and all $x' \in K^*$ we have $\pi(x') \in R_a x'$ by [4, 1.3], and therefore we get

(*) $\varphi(x') = \varphi(v) \Leftrightarrow x' \notin A \Leftrightarrow \pi(x') \notin A \Leftrightarrow \varphi(\pi(x')) = \varphi(v)$.

To show, that this property also holds for the given a, b, c, x instead of o, v, c', x' we first consider the case that $\varphi(a)$, $\varphi(b)$, $\varphi(o)$, $\varphi(v)$ form a frame. In this instance we make use of the perspectivity α: $ab \to ov$ with center $z := ao \cap bv$. It namely induces a perspectivity α': $\varphi(ab) \to \varphi(ov)$ of Π' such that

$$\varphi(\alpha(x)) = \alpha'(\varphi(x)) \quad \text{for all } x \in ab,$$

and such that for each $\pi \in \mathscr{P}_{a,b,c}$ we have $\alpha\pi\alpha^{-1} \in \mathscr{P}_{0,v,c'}$ with $c' := \alpha(c)$. Hence, by (*), we obtain for all $x \in ab$

$$\begin{aligned} \varphi(x) = \varphi(b) &\Leftrightarrow \varphi(\alpha^{-1}\alpha(x)) = \varphi(\alpha^{-1}(v)) &&\Leftrightarrow \alpha'^{-1}(\varphi(\alpha(x))) = \alpha'^{-1}(\varphi(v)) \\ &\Leftrightarrow \varphi(\alpha(x)) = \varphi(v) &&\Leftrightarrow_{(*)} \varphi(\alpha\pi\alpha^{-1}(\alpha(x))) = \varphi(v) \\ &\Leftrightarrow \varphi(\alpha\pi(x)) = \varphi(\alpha(b)) &&\Leftrightarrow \alpha'(\varphi(\pi(x))) = \alpha'(\varphi(b)) \\ &\Leftrightarrow \varphi(\pi(x)) = \varphi(b). \end{aligned}$$

The case that three or all of $\varphi(a)$, $\varphi(b)$, $\varphi(o)$, $\varphi(v)$ are collinear can now be settled by considering additional points $\varphi(a_o)$, $\varphi(b_o)$ and applying the step above twice.

($b \Rightarrow c$) is trivial, for exchanging the points a and b in claim (b) yields a claim equivalent to (b).

($c \Rightarrow a$). Choose a frame (o, u, v, e) of Π which is mapped onto a frame by φ and where a = o and b = v. Then, coordinatizing Π by $K = K(o, u, v, e)$, φ corresponds to a place λ of K, and on $ab = K \cup \{\infty\}$ we have $a = 0$, $b = \infty$, and $\varphi(x) \neq \varphi(a),\varphi(b) \Leftrightarrow x \in U_\lambda$. Given $r \in R_a(K)$, in light of [4, 1.3], there exist $\pi_i \in \mathscr{P}_{0,\infty,c_i}$, $c_i \in K^*$, $i = 1, \ldots, n$ ($n \in \mathbf{N}$), such that $r = \pi_n\pi_{n-1}\ldots\pi_1(1)$. Successively applying (c), we get from $1 \in U_\lambda$ first $\pi_1(1) \in U_\lambda$, then $\pi_2\pi_1(1) \in U_\lambda$, and so on up to $r = \pi_n\pi_{n-1}\ldots\pi_1(1) \in U_\lambda$. Hence $R_a(K) \subset U_\lambda$, and by the lemma above, φ is a friendly epimorphism. □

From these geometric characterizations of friendly epimorphisms and from their proof one immediately infers, that -in contrast to the definition- the property of being friendly does not depend on the special coordinatization of Π. So we have the following.

(2.3) COROLLARY. *If φ is a friendly epimorphism of Π, then each underlying ternary field the coordinatizing frame of which is mapped onto a frame by φ carries a uniform valuation* v *inducing φ up to equivalence and admitting an abelian value group.*

(2.4) COROLLARY. *A place λ: $K \to K' \cup \{\infty\}$ of ternary fields induces a friendly epimorphism, if and only if its place ring A_λ contains the extended radical $R_a(K)$.*

(2.5) COROLLARY. *If a projective plane Π admits a friendly epimorphism which is proper (i.e. not injective), then the group of projectivities of Π is not 4-transitive.*

(2.6) LEMMA. *Let φ: $\Pi \to \Pi'$ be a friendly epimorphism of a projective plane Π, and let π be a projectivity from the line L of Π onto itself. If there exist* a, b, c $\in$ L *with mutually distinct* $\varphi(a)$, $\varphi(b)$, $\varphi(c)$ *and mutually distinct* $\varphi(\pi(a))$, $\varphi(\pi(b))$, $\varphi(\pi(c))$, *then we have for all* $x \in L$

$$\varphi(x) = \varphi(b) \iff \varphi(\pi(x)) = \varphi(\pi(b)).$$

Proof. Since $\mathscr{P}'$ acts 3-transitive on $L' = \varphi(L)$, there exists a projectivity $\sigma' \in \mathscr{P}'$ with $\sigma'(\varphi(\pi(a))) = \varphi(a)$, $\sigma'(\varphi(\pi(b))) = \varphi(b)$, and $\sigma'(\varphi(\pi(c))) = \varphi(c)$. In view of (1.3), σ' lifts to a projectivity $\sigma \in \mathscr{G}$ fulfilling

$$\sigma(\pi(a)) = a,\ \ \sigma(\pi(b)) = b,\ \ \sigma(\pi(c)) = c,\ \text{ and } \sigma'(\varphi(x)) = \varphi(\sigma(x)) \text{ for all } x \in L.$$

Hence $\sigma\pi \in \mathscr{P}_{a,b,c}$. By Theorem (2.2b), the hypothesis $\varphi(x) = \varphi(b)$ leads to $\varphi(\sigma\pi(x)) = \varphi(b) = \varphi(\sigma\pi(b))$, which in face of $\sigma \in \mathscr{G}$ means $\sigma'(\varphi(\pi(x)) = \sigma'(\varphi(\pi(b))$, and therefore finally $\varphi(\pi(x)) = \varphi(\pi(b))$. □

(2.7) THEOREM. *Let φ: $\Pi \to \Pi'$ be a friendly epimorphism of a projective plane Π, and let π be a projectivity from the line L of Π onto itself. If there exist at least three points on L with mutually distinct images under φ and mutually distinct images under $\varphi \circ \pi$, then π induces a well defined permutation on $\varphi(L)$, that is $\pi \in \mathscr{G}$.*

Proof. Let a, b, c $\in$ L with mutually distinct $\varphi(a)$, $\varphi(b)$, $\varphi(c)$ and mutually distinct $\varphi(\pi(a))$, $\varphi(\pi(b))$, $\varphi(\pi(c))$. Given p, q $\in$ L with $\varphi(p) = \varphi(q)$, we have to show that $\varphi(\pi(p))$

$= \varphi(\pi(q))$. This, together with the same argument applied to π^{-1}, yields $\pi \in \mathcal{G}$.
We first settle the case that $\varphi(q) \in \{\varphi(a), \varphi(b), \varphi(c)\}$, say $\varphi(q) = \varphi(b)$. Using the lemma above, we find $\varphi(\pi(q)) = \varphi(\pi(b))$, and again by the lemma above, from $\varphi(p) = \varphi(b)$ we get $\varphi(\pi(p)) = \varphi(\pi(b))$, and therefore $\varphi(\pi(p)) = \varphi(\pi(q))$.
Now we turn to the case that $\varphi(q) \notin \{\varphi(a), \varphi(b), \varphi(c)\}$. Then, in view of the lemma above, we also have that $\varphi(\pi(q)) \notin \{\varphi(\pi(a)), \varphi(\pi(b)), \varphi(\pi(c))\}$. Now applying the lemma with q instead of b and p instead of x we get the desired equality $\varphi(\pi(p)) = \varphi(\pi(q))$. □

Of course, also the reverse claim of (2.7) holds, i.e. the elements π of $\mathcal{G}$ are characterized by the existence of three points with mutually distinct images under the friendly epimorphism φ and under $\varphi \circ \pi$. In particular this leads to the following.

(2.8) COROLLARY. *Let* $\varphi: \Pi \to \Pi'$ *be a friendly epimorphism of a projective plane* Π, *and let* a, b, c *be collinear points of* Π *with mutually distinct* $\varphi(a), \varphi(b), \varphi(c)$. *Then*

$$\mathcal{P}_{a,b,c} = \mathcal{G}_{a,b,c}.$$

(2.9) REMARKS.
(a) Note that (2.7), and also (2.8), are wrong, if one requires only two points to have mutually distinct images under φ and under $\varphi \circ \pi$. For instance, let $K = K'((t))$ be a commutative field of Laurent-series, let $\lambda: K \to K' \cup \{\infty\}$ be the place associated to the degree valuation v of K, and let $\varphi: \Pi(K) \to \Pi(K')$ be the friendly epimorphism induced by v. Then the projectivity mapping ∞ onto ∞ and $x \in K$ onto tx fixes the two points 0 and ∞, but is obviously not in $\mathcal{G}$.
(b) Apart from the theorems above, friendly epimorphisms can be described geometrically by valuations of Π in the matroid theoretical sense of Dress and Wenzel (see [5]), or by certain multiple valued halforderings of Π in the sense of Junkers (see [8]). In particular, corollaries (2.3) and (2.4) are also consequences of each one of these two approaches.

3. PROJECTIVITIES UNDER VERY FRIENDLY EPIMORPHISMS

DEFINITION. An epimorphism $\varphi: \Pi \to \Pi'$ of a projective plane Π is called a *very friendly epimorphism,* if Π can be coordinatized by a ternary field (K, T) admitting a uniform valuation v which induces φ (up to equivalence), has an abelian value group, and contains $R_a(K)$ in its loop $1 - I_v$ of one-units.

(3.1) THEOREM. *Let* $\varphi: \Pi \to \Pi'$ *be an epimorphism of projective planes, and let* a, b *be two points of* Π *with* $\varphi(a) \neq \varphi(b)$. *Then the following statements are equivalent.*

(a) φ *is a very friendly epimorphism,*

(b) for all $c, x \in ab \setminus \{a,b\}$ *and all* $\pi \in \mathscr{P}_{a,b,c}$ *we have:* $\varphi(\pi(x)) = \varphi(x)$.

Proof. (a ⇒ b). By hypothesis, we may coordinatize Π by a ternary field $K = K(o,u,v,e)$ such that (up to equivalence) φ is induced by a uniform valuation of K with $R_a(K) \subset 1 - I_v$. Hence we have $\lambda(r) = 1$ for all $r \in R_a(K)$ and for the place λ associated to φ. Given $c' \in K^*$, $\pi \in \mathscr{P}_{o,v,c'} = \mathscr{P}_{0,\infty,c'}$, and $x' \in K^*$, from [4, 1.3] we obtain that $\pi(x') = rx'$ for some $r \in R_a(K)$, and thus we find

(*) $\varphi(\pi(x')) = \lambda(rx') = \lambda(x') = \varphi(x')$.

Now, in the very same vein as in the proof of (2.2), one shows that this property also holds for the given a, b, c, x instead of o, v, c', x'.

(b ⇒ a). Coordinatize Π by a ternary field $K = K(a, u, b, e)$, such that $a = 0$, $b = \infty$, and φ corresponds to a place λ of K. Given $r \in R_a(K)$, in view of [4, 1.3], there exist $\pi_i \in \mathscr{P}_{0,\infty,c_i}$, $c_i \in K^*$, $i = 1, \ldots, n$ ($n \in \mathbf{N}$), such that $r = \pi_n\pi_{n-1}\ldots\pi_1(1)$. Applying (b), we obtain $1 = \lambda(1) = \varphi(1) = \varphi(\pi_1(1)) = \ldots = \varphi(\pi_n\pi_{n-1}\ldots\pi_1(1)) = \lambda(r)$, proving that $R_a \subset 1 - I_v \subset A_v$. In view of (2.1), φ is a very friendly epimorphism. □

Again, the theorem above and its proof show that -in contrast to the definition- the property of being very friendly does not depend on the coordinatization of Π. So, in view of (2.1) and (3.1), we have the following.

(3.2) COROLLARY. *If* φ *is a very friendly epimorphism of* Π, *then each underlying ternary field* K *the coordinatizing frame of which is mapped onto a frame by* φ *carries a uniform valuation* v *inducing* φ *up to equivalence, admitting an abelian value group, and satisfying* $R_a(K) \subset 1 - I_v$.

(3.3) COROLLARY. *A place* $\lambda: K \to K' \cup \{\infty\}$ *of ternary fields induces a very friendly epimorphism, if and only if it maps each element of the extended radical* $R_a(K)$ *of* K *onto* 1.

(3.4) COROLLARY. *Let* $\varphi: \Pi \to \Pi'$ *be a very friendly epimorphism of projective planes. Then* Π' *is a pappian projective plane, and the permutations induced by the projectivities of* Π *on the lines of* Π' *are exactly the projectivities of* Π', *i.e. we have* $\mathscr{P}^{\varphi} = \mathscr{P}'$.

Proof. Given collinear points a, b, c of Π with mutually distinct images, immediately from (3.1) we obtain $(\mathscr{S}_{a,b,c})^{\varphi} = \{id\}$. The claim now follows from (1.6) and (1.7b). □

(3.5) REMARK. Of course, if Π is pappian, then each epimorphism of Π is a very friendly epimorphism, since the extended radical of a commutative field is always trivial. However, in general there exist friendly epimorphisms which are not very friendly. In particular, an isomorphism between projective planes is always friendly (it is induced by a trivial valuation $v|_{K^*} \equiv 1$ with $A_v = K$, $1-I_v = \{1\}$), but it needs not to be very friendly (namely if $R_a \neq \{1\}$, which holds exactly for non-pappian planes).

4. EPIMORPHISMS ONTO PAPPIAN PLANES

In this section we will present examples for the settings described in the sections above. We shall show that $\mathscr{S}^{\varphi}$, which obviously varies between $\mathscr{B}'$ and the full symmetric group on the lines of Π', actually takes both values, even if φ is a friendly epimorphism onto a pappian projective plane.

Recall that a uniformly valued ternary field (K, T, v) yields an ultrametric space (K, d) with respect to the metric defined by $d(x,y) := v(x - y)$. In particular, K is said to be *spherically complete,* if every chain of balls of K has a non-empty intersection; a mapping φ: K → K is called an *isometry of* K, if it is a bijection fulfilling $d(\varphi(x), \varphi(y)) = d(x, y)$; and a mapping φ: K → K is called a *contraction,* if $d(\varphi(x), \varphi(y)) < d(x, y)$ for all $x, y \in K$ with $x \neq y$. We will make use of the notion of a *cartesian field of formal power series* $C((\Gamma))$ over the cartesian field C on the ordered loop $(\Gamma, \cdot, \leq)$ as introduced in [2]. Its elements are of the form $x = \sum_{\gamma\in\Gamma} x_\gamma t^\gamma$ with well ordered support s(x) and *degree* $\partial(x) := \min(s(x))$ for $x \neq 0$ ($\partial(0) := 0$, $x_\gamma \in C$). Recall that the degree gives rise to a uniform valuation on $C((\Gamma))$ with residue class cartesian field C and value loop Γ (carrying the dual ordering $\leq^d$), and that, with respect to this valuation, $C((\Gamma))$ is spherically complete (see, say [12, 5.1 and 5.2]).

(4.1) LEMMA. *Let* (K, +, ·) *be a (cartesian) field admitting a uniform valuation* v: K → $\Gamma\cup\{0\}$ *such that* (K, v) *is spherically complete. Further let* $\Phi = (\Phi_\gamma)_{\gamma\in\Gamma\cup\{0\}}$ *be any family of isometries of K fixing* 1 *and* 0 *and with* $\Phi_1 = id_K$. *With the new product*

$$a \lozenge b \ := \ a \cdot \Phi_{v(a)}(b)$$

$K^\Phi := (K, +, \lozenge, v)$ *is a uniformly valued cartesian field.*

Proof. Clearly, (K, +) is an abelian group with neutral element 0, and for all $a \in K$ we have $a \lozenge 0 = 0 \lozenge a = 0$ and $a \lozenge 1 = 1 \lozenge a = a$. We have to check that for a, b, c ∈ K, $a \neq b$, the equation

$$-a \lozenge x + b \lozenge x = c$$

has a unique solution x in K. In the case $v(a) = v(b) =: \gamma$, the equation reads $c = -a\Phi_\gamma(x) + b\Phi_\gamma(x)$, which has a unique solution $\Phi_\gamma(x)$ in the cartesian field (K, +, ·). In the case $v(a) \neq v(b)$, say $v(a) < v(b)$, the mapping $f(x) := (\Phi_{v(b)})^{-1}(b \backslash (a \cdot \Phi_{v(a)}(x) + c))$ is a contraction on K, because for all $x, y \in K$ with $x \neq y$ we have

$$\begin{aligned} v(f(x) - f(y)) &= v(\,(\Phi_{v(b)})^{-1}(b\backslash(a \cdot \Phi_{v(a)}(x) + c)) - (\Phi_{v(b)})^{-1}(b\backslash(a \cdot \Phi_{v(a)}(y) + c))\,) \\ &= v(\,b\backslash(a \cdot \Phi_{v(a)}(x) + c) - b\backslash(a \cdot \Phi_{v(a)}(y) + c)) && \Phi_{v(b)} \text{ isometry} \\ &= v(b) \backslash v(\,a \cdot \Phi_{v(a)}(x) - a \cdot \Phi_{v(a)}(y)\,) && \text{by [1, 1.2]} \\ &= v(b) \backslash (v(a) \cdot v(\,\Phi_{v(a)}(x) - \Phi_{v(a)}(y)\,)\,) && \text{by [1, 1.2]} \\ &= v(b) \backslash (v(a) \cdot v(\,x - y\,)\,) && \Phi_{v(a)} \text{ isometry} \\ &< v(\,x - y\,). \end{aligned}$$

Hence, by Priess-Crampe's fixed point theorem [11], f admits a unique fixed point x_0, which is the desired unique solution. Analogously, one shows that for a, b, c ∈ K, $a \neq b$, also the equation $x \lozenge a - x \lozenge b = c$ has a unique solution. Hence $(K, +, \lozenge)$ is a cartesian field.

Obviously, the mapping v fulfills (V1) and (V3) also for this new cartesian field. (V2) is immediate by

$$v(a \lozenge b) = v(a\,\Phi_{v(a)}(b)) = v(a) \cdot v(\Phi_{v(a)}(b)) = v(a) \cdot v(b),$$

since $\Phi_{v(a)}$ is an isometry. For (V4) we only have to check that (with $a \neq b$, $c \neq d$)

$$v(\,a\lozenge c - a\lozenge d + b\lozenge d - b\lozenge c\,) = v(a - b) \cdot v(c - d).$$

The left hand side translates into

$$\gamma := v(\,a\Phi_{v(a)}(c) - a\Phi_{v(a)}(d) + b\Phi_{v(b)}(d) - b\Phi_{v(b)}(c)\,).$$

In the case $v(a) = v(b)$, by [1, 1.2] γ equals $v(a - b) \cdot v(\Phi_{v(a)}(c) - \Phi_{v(a)}(d))$, which yields the desired expression, because $\Phi_{v(a)}$ is an isometry. In the case $v(a) \neq v(b)$, say $v(a) < v(b)$, we have $v(a\Phi_{v(a)}(c) - a\Phi_{v(a)}(d)) = v(a) \cdot v(c - d) < v(b) \cdot v(c - d) = v(b\Phi_{v(b)}(d) - b\Phi_{v(b)}(c))$, and the principle of domination implies

$$\gamma = v(b\Phi_{v(b)}(d) - b\Phi_{v(b)}(c)) = v(b) \cdot v(c - d) = v(a - b) \cdot v(c - d). \quad \square$$

(4.2) PROPOSITION. *Let Π' be an arbitrary pappian projective plane. Then there exist a projective plane Π and a friendly epimorphism φ: Π → Π' such that the projectivities of Π induce the full symmetric group on each line of Π'.*

Proof. Let K be a commutative field coordinatizing Π' with respect to the frame (o', u', v', e'), and let Ω be the full permutation group of the line L' := o'v' = K ∪ {∞}. Choose an ordered abelian group (Γ, ·, ≤) which is large enough to allow a (set theoretic) injection ι of the (pointwise) stabilizer $\Omega_{0,1,\infty}$ into Γ\{1} (large ordered abelian groups exist, see, say [10]). For each α ∈ Γ ∪ {0} we define a permutation π_α of L' fixing 0, 1 and ∞ as follows

$$\pi_\alpha(k) := \begin{cases} \omega(k) & \text{if } \alpha \in \iota(\Omega_{0,1,\infty}) \text{ and } \alpha = \iota(\omega) \\ k & \text{else} \end{cases} \qquad \text{for all } k \in K \cup \{\infty\}.$$

Let F = K((Γ)) be the field of formal power series over K on Γ with degree valuation v, which is a uniform valuation of F with respect to the dual ordering $\leq^d$ on Γ. For each α ∈ Γ ∪ {0} we define a bijection Φ_α of F by

$$\Phi_\alpha(\textstyle\sum_{\gamma\in\Gamma} k_\gamma t^\gamma) := \pi_\alpha(k_1)\, t^1 + \sum_{\gamma\in\Gamma\setminus\{1\}} k_\gamma t^\gamma.$$

Then clearly, each Φ_α is an isometry of F fixing 0 and 1, and we have $\Phi_1 = id_L$, because $1 \notin \iota(\Omega_{0,1,\infty})$. Now consider the uniformly valued cartesian field F^Φ as defined in the preceding lemma and the projective plane Π over F^Φ.
For the place λ: F → K ∪ {∞} associated to v we have

$$\lambda(\textstyle\sum_{\gamma\in\Gamma} k_\gamma t^\gamma) = \begin{cases} 0 & \text{if } v(\sum_{\gamma\in\Gamma} k_\gamma t^\gamma) <^d 1 \\ k_1 & \text{if } v(\sum_{\gamma\in\Gamma} k_\gamma t^\gamma) = 1 \\ \infty & \text{if } v(\sum_{\gamma\in\Gamma} k_\gamma t^\gamma) >^d 1 \end{cases} .$$

And for the projectivities $\rho_\alpha \in \mathscr{P}$, α ∈ Γ\{1}, defined by $\rho_\alpha(x) := (t^\alpha \lozenge x) \lozenge t^{\alpha^{-1}}$ for all x ∈ F and by $\rho_\alpha(\infty) := \infty$, we observe

$$\begin{aligned} \rho_\alpha(\textstyle\sum_{\gamma\in\Gamma} k_\gamma t^\gamma) &= (t^\alpha\, \Phi_\alpha(\textstyle\sum_{\gamma\in\Gamma} k_\gamma t^\gamma)) \lozenge t^{\alpha^{-1}} \\ &= (t^\alpha (\pi_\alpha(k_1)\, t^1 + \textstyle\sum_{\gamma\in\Gamma\setminus\{1\}} k_\gamma t^\gamma)) \lozenge t^{\alpha^{-1}} \\ &= (\pi_\alpha(k_1)\, t^\alpha + \textstyle\sum_{\gamma\in\Gamma\setminus\{1\}} k_\gamma t^{\gamma\alpha}) \cdot \Phi_{v(\ldots)}(t^{\alpha^{-1}}) \\ &= (\pi_\alpha(k_1)\, t^\alpha + \textstyle\sum_{\gamma\in\Gamma\setminus\{1\}} k_\gamma t^{\gamma\alpha}) \cdot t^{\alpha^{-1}} \\ &= \pi_\alpha(k_1)\, t^1 + \textstyle\sum_{\gamma\in\Gamma\setminus\{1\}} k_\gamma t^\gamma. \end{aligned}$$

So we have $\rho_\alpha \in \mathscr{P}_{0,1,\infty}$ and

$$\begin{aligned} \lambda(\rho_\alpha(\textstyle\sum_{\gamma\in\Gamma} k_\gamma t^\gamma)) &= \lambda(\pi_\alpha(k_1)\, t^1 + \textstyle\sum_{\gamma\in\Gamma\setminus\{1\}} k_\gamma t^\gamma) \\ &= \pi_\alpha(\lambda(\textstyle\sum_{\gamma\in\Gamma} k_\gamma t^\gamma)), \end{aligned}$$

which means that ρ_α lies in $\mathscr{S}$ with respect to the epimorphism φ: Π → Π' associated to λ, and that ρ_α induces the permutation π_α on L'. Hence we find

$$(\mathscr{S}_{0,1,\infty})^\varphi = \Omega_{0,1,\infty}.$$

Since $\mathscr{P}'$ is three-fold transitive on L', (1.5) finally yields $\mathscr{S}^\varphi = \Omega$. □

(4.3) REMARK. The proposition above is supposed to hold for arbitrary non-pappian projective planes Π', too. Indeed, if Π' can be coordinatized by a (not necessarily proper) cartesian field C, then with C instead of K the proof of (4.2) goes through nearly word by word. For weaker projective planes Π', the ideas above should extend to Schörner's notion of formal power series over arbitrary ternary fields [13].

(4.4) PROPOSITION. *Let* $(K, +, \cdot, v)$ *be a valued commutative field, and let* $(r_\gamma)_{\gamma\in\Gamma\cup\{0\}}$, $(s_\gamma)_{\gamma\in\Gamma\cup\{0\}}$ *be two families of elements of* K *with* $r_0 = r_1 = s_0 = s_1 = 0$ *and* $v(r_\gamma), v(s_\gamma) < \gamma$ *for all* $\gamma \in \Gamma$. *Then the new product*

$$a \lozenge b \quad := \quad a \cdot b \ + \ r_{v(a)} \cdot s_{v(b)}$$

makes $(K, +, \lozenge, v)$ *a uniformly valued cartesian field the extended radical of which lies in* $1 - I_v$.

Proof. By [6, 3.8], $C := (K, +, \lozenge, v)$ is a uniformly valued cartesian field. Hence $R = R(C)$ lies in U_v. To check that R even lies in the loop $1 - I_v$ of one-units, we simply write r_a and s_b instead of $r_{v(a)}$ and $s_{v(b)}$. In view of (i') and (ii') from the beginning of section two, R is generated by those $r \in K^*$ which satisfy (note $r_r = 0$, since $v(r) = 1$)

$$\begin{aligned} ac + r_a s_c - r_a s_d - ad + bd + r_b s_d - r_b s_c - bc &= r \lozenge ((a-b)(c-d) + r_{a-b} s_{c-d}) \\ &= r \cdot ((a-b)(c-d) + r_{a-b} s_{c-d}), \end{aligned}$$

that is by elements of the shape

$$\begin{aligned} r &= (ac - ad + bd - bc + r_a s_c - r_a s_d + r_b s_d - r_b s_c) / ((a-b)(c-d) + r_{a-b} s_{c-d}) \\ &= ((a-b)(c-d) + (r_a - r_b)(s_c - s_d)) / ((a-b)(c-d) + r_{a-b} s_{c-d}) \\ &= (1 + A \backslash ((r_a - r_b)(s_c - s_d))) \ / \ (1 + A \backslash (r_{a-b} s_{c-d})) \end{aligned}$$

with $A := (a-b)(c-d)$. Now, if $v(a) = v(b)$ or $v(c) = v(d)$, then $(r_a - r_b)(s_c - s_d) = 0$, and if $v(a) \neq v(b)$ and $v(c) \neq v(d)$, say $v(a) < v(b)$ and $v(c) < v(d)$, then we have $v((r_a - r_b)(s_c - s_d)) \leq \max\{v(r_a s_c), v(r_a s_d), v(r_b s_c), v(r_b s_d)\} < \max\{v(ac), v(ad), v(bc), v(bd)\} = v(bd) = v(A)$. In both cases we find $v(A\backslash((r_a - r_b)(s_c - s_d))) < 1$, and therefore $1 + A\backslash((r_a - r_b)(s_c - s_d)) \in 1 - I_v$. Further we immediately get $v(A\backslash(r_{a-b}s_{c-d})) < 1$, so also $1 + A\backslash(r_{a-b}s_{c-d}) \in 1 - I_v$, which finally shows $r \in 1 - I_v$. Hence we have that R lies in $1 - I_v$.

The extended radical $R_a = R_a(C)$ is generated by R and additionally by those $r \in K^*$ fulfilling at least one of the equations

$$a \lozenge b = r \lozenge (b \lozenge a),$$

$$(a \lozenge b) \lozenge c = r \lozenge (a \lozenge (b \lozenge c))$$

where a,b,c vary over K^*. Since $v(a \lozenge b) = v(ab)$, we find $v(r) = 1$ also for these r. Hence these elements are of the shape

$$\begin{aligned} r &= (ab + r_a s_b) \cdot (ba + r_b s_a)^{-1} \\ &= (1 + r_a s_b/(ab)) \cdot (1 + r_b s_a/(ab))^{-1} \end{aligned}$$

or of the shape

$$\begin{aligned} r &= ((ab + r_a s_b)c + r_{ab} s_c) \cdot (a(bc + r_b s_c) + r_a s_{bc})^{-1} \\ &= (abc + r_a s_b c + r_{ab} s_c) \cdot (abc + a r_b s_c + r_a s_{bc})^{-1} \\ &= (1 + r_a s_b/(ab) + r_{ab} s_c/(abc)) \cdot (1 + r_b s_c/(bc) + r_a s_{bc}/(abc))^{-1}, \end{aligned}$$

and both are easily seen to be in $1 - I_v$. So we finally have $R_a \subset 1 - I_v$. □

Given any pappian projective plane Π', say over the field K', we take the field $K := K'((\Gamma))$ where Γ is the multiplicative group of the positive real numbers and put $r_0 := s_0 := r_1 := s_1 := 0$ and $r_\gamma := s_\gamma := t^{2\gamma}$ for all $\gamma \in \Gamma \setminus \{1\}$ (recall that the degree is a valuation of K with respect to the dual ordering of Γ). Then (4.4) provides us with the projective plane Π over the cartesian field $(K, +, \lozenge)$ admitting a very friendly epimorphism onto Π'. So, in view of (3.4), here we have $\mathcal{G}^\varphi = \mathcal{B}'$.

The following example yields proper translation planes Π with this property, i.e. planes over cartesian fields fulfilling one distributive law (so called quasifields).

(4.5) PROPOSITION. *Let* $(K, +, \cdot)$ *be a commutative field carrying a valuation* $v: K \to \Gamma \cup \{0\}$ *such that* K *is spherically complete. Further let* $(\Phi_\gamma)_{\gamma \in \Gamma \cup \{0\}}$ *be any family of isometries* $\Phi_\gamma \in \mathrm{Aut}(K, +)$ *fixing* 1. *With the new product*

$$a \lozenge b := (\Phi_{v(a)})^{-1} (\Phi_{v(a)}(a) \cdot \Phi_{v(a)}(b))$$

$K_\Phi := (K, +, \lozenge, v)$ *becomes a uniformly valued quasifield, the extended radical* R_a *of which lies in* U_v. *If additionally* $\Phi_\gamma(k) \in (1 - I_v)k$ *for all* $k \in K^*$ *and all* $\gamma \in \Gamma \cup \{0\}$, *then* R_a *lies in* $1 - I_v$.

Proof. By [6, 3.3], K_Φ is a uniformly valued (left) quasifield. In particular, $R = R(K_\Phi)$ lies in U_v. Since the image of K^* under v is an abelian group, we even find $R_a \subset U_v$.

Now suppose that $\Phi_\gamma(k) \in (1 - I_v)k$ for all $k \in K^*$ and all $\gamma \in \Gamma \cup \{0\}$. We simply write Φ_k instead of $\Phi_{v(k)}$. First note, that for all $a,b \in K$ we have

(*) $a \lozenge b \in (1 - I_v)ab$,

since $1 - I_v$ is a (normal) subgroup of $(K^*, \cdot)$, and since $\Phi_a(a \lozenge b) \in (1 - I_v)(a \lozenge b)$ but also $\Phi_a(a \lozenge b) = (\Phi_a(a) \cdot \Phi_a(b)) \in (1 - I_v)a \cdot (1 - I_v)b = (1 - I_v)ab$.

For a left quasifield, R is generated by elements r of the shape

$$a \lozenge c - b \lozenge c = r \lozenge ((a-b) \lozenge c),$$

with $a,b,c \in K$, $a \neq b$, $c \neq 0$. We first settle the case $v(a) = v(b)$. Then $\Phi_a = \Phi_b$, and we observe

$$\begin{aligned} \Phi_a^{-1}(\Phi_a(a-b)\,\Phi_a(c)) &= \Phi_a^{-1}((\Phi_a(a)-\Phi_a(b))\,\Phi_a(c)) \\ &= \Phi_a^{-1}(\Phi_a(a)\cdot\Phi_a(c)) - \Phi_a^{-1}(\Phi_a(b)\cdot\Phi_a(c)) \\ &= a \lozenge c - b \lozenge c \\ &= r \lozenge ((a-b) \lozenge c) \\ &\in (1-I_v)\cdot r(a-b)c, \end{aligned}$$

by (*). Therefore, using $\Phi_a(k) \in (1-I_v)k$, we get

$$\Phi_a(a-b)\,\Phi_a(c) \in (1-I_v)\cdot r(a-b)c \cap (1-I_v)(a-b)c,$$

which finally leads to $r \in 1-I_v$, as desired.

Now we consider the case $v(a) \neq v(b)$, say $v(a) > v(b)$. In view of (*) there exist e_1, e_2 $\in 1-I_v$ with $e_1ac - e_2bc = a\lozenge c - b\lozenge c = r\lozenge((a-b)\lozenge c) \in (1-I_v)\cdot r(a-b)c$, which leads to

$$e_1a - e_2b \in (1-I_v)\cdot r(a-b),$$

$$e_1 - e_2ba^{-1} \in (1-I_v)\cdot r(1-ba^{-1}),$$

and thus to

$$r \in (1-I_v)\cdot(e_1 - e_2ba^{-1})\cdot(1-ba^{-1})^{-1} \subset 1-I_v,$$

because $ba^{-1} \in I_v$. Hence we have shown $R \subset 1-I_v$. By virtue of (*), it is easily to be seen that also the extended radical R_a lies in $1-I_v$. □

For a concrete example illustrating (4.5) take a commutative field of Laurent series $K = L((t)) = L((\Gamma))$ and put

$$\Phi_\gamma\Big(\sum_{i \in \mathbf{Z}} x_i \cdot t^i\Big) := x_0 + (1+t)^\gamma \cdot \sum_{i \neq 0} x_i \cdot t^i,$$

where $\Gamma = (\mathbf{Z}, +)$ is written additively. Then Φ_γ is an additive automorphism and an isometry fixing 1. It also fulfills $\Phi_\gamma(k) \in (1-I_v)k$, since k and $\Phi_\gamma(k)$ have the same degree and the same leading coefficient. Hence $\Pi := \Pi(K_\Phi)$ is a translation plane admitting a very friendly epimorphism onto $\Pi' := \Pi(L)$.

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