FIXED POINT THEOREMS FOR SPACES WITH A TRANSITIVE RELATION

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ABSTRACT. We present general fixed point theorems for spaces that are equipped with a transitive relation. We apply them to prove corresponding theorems for ultrametric spaces, topological spaces, complete lattices, and ordered abelian groups and fields.

1. INTRODUCTION

In our papers [1] and [3] we have developed a general framework for fixed point theorems that work with functions which are in some way contracting, or have other properties that allow an application of Zorn's Lemma. We consider **ball spaces** $(X, \mathcal{B})$, that is, nonempty sets X with a nonempty set $\mathcal{B}$ of distinguished nonempty subsets B. The completeness property we need for our fixed point theorems is inspired by the spherical completeness of ultrametric spaces. A **nest of balls** in $(X, \mathcal{B})$ is a nonempty totally ordered subset of $(\mathcal{B}, \subseteq)$. A ball space $(X, \mathcal{B})$ is called **spherically complete** if every nest of balls has a nonempty intersection.

Note that $\mathcal{B}$, a subset of the power set $\mathcal{P}(X)$, is a partially ordered set under reverse inclusion. However, spherically completeness in its simplest form, which we have defined here, does not mean that $\mathcal{B}$ is **inductively ordered**, i.e., that every increasing chain has an upper bound. But this would be true if for instance every singleton, or every intersection over a nonempty descending chain of balls, is a ball.

In this note we add a transitive relation R to the set X. We do not require that the relation has any other properties, such as reflexivity, symmetry or antisymmetry. We write xRy if $x, y \in X$ and x is in relation with y.

If X is the set of vertices of a graph (directed or not), and if we define xRy to mean that there is a path of finite length from x to y, then R is a transitive relation. The same is true when we also allow paths of infinite length. Fixed point theorems for spaces with additional graph structure have been abundant in the more recent literature, so it seems to be worthwile to adapt our above mentioned unified approach to this additional structure.

Given a function $f : X \to X$, we call a ball $B \in \mathcal{B}$ **f-contracting** if it is either a singleton containing a fixed point, or $f(B) \subsetneqq B$ holds.

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For the remainder of this introduction, we assume that $(X,\mathcal{B})$ is a ball space, R is a transitive relation on X, and $f:X\to X$ is a function. We will write fx in place of $f(x)$.

Theorem 1. *Assume that the following conditions hold:*

($\mathbf{A}_1$) *For every non-singleton f-contracting ball $B\in\mathcal{B}$ and every $x\in B$, the image $f(B)$ contains an f-contracting ball B' and an element $x'\in B'$ such that xRx'.*

($\mathbf{B}_1$) *If κ is a regular cardinal, $(B_\nu)_{\nu<\kappa}$ is a nest of f-contracting balls, and $x_\nu\in B_\nu$ are elements such that $x_\mu R x_\nu$ whenever $\mu<\nu<\kappa$, then $\bigcap_{\nu<\kappa}B_\nu$ contains an f-contracting ball B and there is some $z\in B$ such that $x_\nu Rz$ whenever $\nu<\kappa$.*

Then for every x in any f-contracting ball there is a fixed point z of f such that xRz.

Note that if $B=\{x\}$ is f-contracting, then by definition, x is a fixed point and we have that $f(B)=B$ contains the f-contracting ball B. However, as we do not require R to be symmetric, we may not have that xRx, and this is the reason why we exclude singleton f-contracting balls from condition (A_1).

The following theorem gives conditions for the existence of a unique fixed point.

Theorem 2. *Assume that the following conditions hold:*

($\mathbf{A}_2$) *For every f-contracting ball $B\in\mathcal{B}$ and every $x\in B$, the image $f(B)$ is again an f-contracting ball and contains an element $x'\in B'$ such that xRx'.*

($\mathbf{B}_2$) *If κ is a regular cardinal, $(B_\nu)_{\nu<\kappa}$ is a nest of f-contracting balls, and $x_\nu\in B_\nu$ are elements such that $x_\mu R x_\nu$ whenever $\mu<\nu<\kappa$, then $\bigcap_{\nu<\kappa}B_\nu$ is an f-contracting ball and contains an element z such that $x_\nu Rz$ whenever $\nu<\kappa$.*

Then every f-contracting ball B_0 contains a unique fixed point z of f, and we have that xRz for all $x\in B_0$.

A function $f:X\mapsto X$ will be called **R-compatible** if $xRfx$ for all $x\in X$. Condition (A_2) can be replaced by the condition

($\mathbf{A}'_2$) The function f is R-compatible and the image $f(B)$ of every f-contracting ball $B\in\mathcal{B}$ is again an f-contracting ball.

We will now state a third theorem which treats functions that are not necessarily contractive. A set $S\subseteq X$ is called **f-closed** if $f(S)\subseteq S$. A ball space is called **intersection closed** if the intersection of every nest of balls is again a ball, provided it is nonempty. For example, in every topological space the nonempty closed subsets form an intersection closed ball space.

Theorem 3. *Assume that $(X,\mathcal{B})$ is intersection closed and that the following conditions hold:*

($\mathbf{A}_3$) *For every non-singleton f-closed ball $B\in\mathcal{B}$ and every $x\in B$, there is an f-closed ball $B'\subsetneq B$ and some $x'\in B'$ such that xRx'.*
($\mathbf{B}_3$) *If κ is a regular cardinal, $(B_\nu)_{\nu<\kappa}$ is a nest of f-closed balls, and $x_\nu\in B_\nu$ are elements such that $x_\mu Rx_\nu$ whenever $\mu<\nu<\kappa$, then $\bigcap_{\nu<\kappa}B_\nu$ contains an element z such that $x_\nu Rz$ whenever $\nu<\kappa$.*

Then for every x in any f-closed ball there is a fixed point z of f such that xRz.

Finally, we present a fourth theorem that generalizes Theorem 4 of [1], which is helpful for applications in ordered abelian groups and fields.

A function f on a ball space $(X,\mathcal{B})$ will be called **contracting on orbits** if there is a function

$$X\ni x\ \mapsto\ B_x\in\mathcal{B}$$

such that for all $x\in X$, the following conditions hold:

(SC1) $x\in B_x$,
(SC2) $B_{fx}\subseteq B_x$, and if $x\neq fx$, then $B_{f^ix}\subsetneq B_x$ for some $i\geq 1$.

Note that (SC1) and (SC2) imply that $f^ix\in B_x$ for all $i\geq 0$.

Theorem 4. *Assume that the following conditions hold:*

($\mathbf{A}_4$) *The function f is R-compatible and contracting on orbits.*
($\mathbf{B}_4$) *If λ is a limit ordinal and $(B_{x_\nu})_{\nu<\lambda}$ is a nest such that $x_{\nu+1}=fx_\nu$ and $x_\mu Rx_\nu$ whenever $\mu<\nu<\lambda$, then $\bigcap_{\nu<\lambda}B_{x_\nu}$ contains an element z such that $B_z\subseteq\bigcap_{\nu<\lambda}B_{x_\nu}$ and $x_\nu Rz$ whenever $\nu<\lambda$.*

Then for every $x\in X$ there is a fixed point z of f such that xRz.

The proofs of these theorems will be given in Section 2. In Section 3, we will then derive fixed point theorems for ultrametric spaces. In Section 4, we apply our theorems to prove corresponding fixed point theorems for compact topological spaces. In Section 5, we give a fixed point theorem for order preserving functions on complete lattices We will conclude the paper in Section 6 with a fixed point theorem for ordered abelian groups and fields.

2. Proof of the main theorems

Take a ball space $(X,\mathcal{B})$, a transitive relation R on X, and a function $f:X\to X$. For the proofs of our first two theorems, we take $\mathcal{B}^f$ to be the subset of $\mathcal{B}$ consisting of all f-contracting balls. Then we introduce a partial order on the set

$$\mathcal{S}\ =\ \{(B,x)\mid B\in\mathcal{B}^f\text{ and }x\in B\}$$

as follows:

$$(B_1,x_1)\ <\ (B_2,x_2)\ \Leftrightarrow\ B_2\subsetneq B_1\text{ and }x_1Rx_2\,.$$

Lemma 5. *Take an f-contracting ball $B \in \mathcal{B}$ and an element $x \in B$. Set*

$$\mathcal{S}_{(B,x)} \;=\; \{(B',x') \in \mathcal{S} \mid (B,x) \leq (B',x')\} \;\subseteq\; \mathcal{S}\,.$$

If condition (B$_1$) of Theorem 1 holds, then $(\mathcal{S}_{(B,x)}, <)$ is inductively ordered. If condition (B$_2$) of Theorem 2 holds, then $(\mathcal{S}_{(B,x)}, <)$ is chain complete.

Proof: We take any chain $\mathcal{C}$ in $(\mathcal{S}, <)$, and let κ be its cofinality. Then κ is a regular cardinal. We may assume that κ is infinite since otherwise, $\mathcal{C}$ has a last element which then is a supremum for $\mathcal{C}$.

We choose a subchain $((B_\nu, x_\nu))_{\nu<\kappa}$ that is cofinal in $\mathcal{C}$. Then condition (B$_1$) means that this subchain has an upper bound (B^*, z). It follows that B^* is a proper subset of all B_ν and hence also of all balls in $\mathcal{C}$. Further, we have that $x_\nu R z$ for all $\nu < \kappa$. For every $(B',x') \in \mathcal{C}$, there is some $\nu < \kappa$ such that $(B',x') < (B_\nu, x_\nu) < (B^*, z)$, which shows that (B^*, z) is also an upper bound for $\mathcal{C}$. This proves assertion 1).

If condition (B$_2$) holds and we set $B^* = \bigcap_{\nu<\kappa} B_\nu$ and choose z according to this condition, then (B^*, z) is a supremum of $((B_\nu, x_\nu))_{\nu<\kappa}$ and hence also of $\mathcal{C}$. This proves assertion 2). □

In order to prove Theorem 1, we take an f-contracting ball $B \in \mathcal{B}$ and an element $x \in B$. We assume that conditions (A$_1$) and (B$_1$) hold. Then by Lemma 5, $(\mathcal{S}_{(B,x)}, <)$ is inductively ordered. By Zorn's Lemma, it admits a maximal element (B^*, z). If B^* is not a singleton, then by condition (A$_1$), $f(B) \subsetneqq B$ contains an f-contracting ball B' and an element $x' \in B'$ such that xRx'. It follows that $(B^*, z) < (B', x')$, which contradicts the maximality of (B^*, z). Therefore, B^* is a singleton. As $(B^*, z) \in \mathcal{S}_{(B,x)}$, B^* is f-contracting and it follows that z is a fixed point of f. The fact that $(B^*, z) \in \mathcal{S}_{(B,x)}$ also implies that xRz. This completes the proof of Theorem 1.

In order to prove Theorem 2, we take an f-contracting ball $B \in \mathcal{B}$. We assume that conditions (A$_2$) and (B$_2$) hold. This time, we will replace the use of Zorn's Lemma by that of transfinite induction. We build a chain of elements of $\mathcal{S}_{(B,x)}$ as follows. We set $B_0 := B$ and $x_0 := x$. Having constructed (B_ν, x_ν) for an ordinal ν, we stop the construction if B_ν is a singleton; otherwise, making use of condition (A$_2$) we set $B_{\nu+1} := f(B_\nu)$, which is again an f-contracting ball and is properly contained in B_ν, and choose $x_{\nu+1}$ in $f(B_\nu)$ such that $x_\nu R X_{\nu+1}$. We obtain that $(B_\nu, x_\nu) < (B_{\nu+1}, x_{\nu+1})$.

If λ is a limit ordinal and we have constructed B_ν for all $\nu < \lambda$, we proceed as follows. We take κ to be the cofinality of λ and choose a cofinal subsequence $(B_{\nu_\alpha}, x_{\nu_\alpha})_{\alpha<\kappa}$. By condition (B$_2$), $B_\lambda := \bigcap_{\alpha<\kappa} B_{\nu_\alpha} = \bigcap_{\nu<\lambda} B_\nu$ is an f-contracting ball, and there is some $z \in \bigcap_{\alpha<\kappa} B_{\nu_\alpha} = B_\lambda$ such that $x_{\nu_\alpha} R z$ for all $\alpha < \kappa$. As in the proof of the above lemma it follows that $x_\nu R z$ for all $\nu < \lambda$. So we can set $x_\lambda := z$ to obtain that $(B_\nu, x_\nu) < (B_\lambda, x_\lambda)$ for all $\nu < \lambda$.

The chain of balls thus constructed is strictly descending. Hence there must be an ordinal ν^*, bounded by the cardinality of X, where the construction stops. Then B_{ν^*} must be a singleton, that is, $B_{\nu^*} = \{x_{\nu^*}\}$. As it is also an f-contracting ball, x_{ν^*} is a fixed point of f. If $x_{\nu^*} \neq y \in B$, then $y \notin B_{\nu^*}$, which means that there is some $\mu < \nu^*$ such that $y \in B_\mu$, but $y \notin B_{\mu+1} = f(B_\mu)$. This shows that y cannot be a fixed point of f. Therefore, x_{ν^*} is the unique fixed point of f. Since $(B_{\nu^*}, x_{\nu^*}) \in \mathcal{S}_{(B,x)}$, we also know that xRx_{ν^*}. Since $x \in B$ was arbitrary, this holds for all $x \in B$.

We turn to the proof of Theorem 3. Now we take $\mathcal{B}^f$ to be the set of all f-closed balls in $\mathcal{B}$. Using this new meaning of $\mathcal{B}^f$, we define $\mathcal{S}$ and $\mathcal{S}_{(B,x)}$ as before.

Lemma 6. *Assume that $(X, \mathcal{B})$ is an intersection closed ball space and take an f-closed ball $B \in \mathcal{B}$ and an element $x \in B$. If condition (B$_3$) of Theorem 3 holds, then $(\mathcal{S}_{(B,x)}, <)$ is chain complete.*

Proof: We take any chain $\mathcal{C}$ in $(\mathcal{S}, <)$. As in the proof of Lemma 5, we let κ be its cofinality, assume that it is infinite, and choose a subchain $((B_\nu, x_\nu))_{\nu<\kappa}$ that is cofinal in $\mathcal{C}$. By condition (B$_3$) of Theorem 3, $\bigcap_{\nu<\kappa} B_\nu$ contains an element z such that $x_\nu Rz$ whenever $\nu < \kappa$. In particular, $((B_\nu, x_\nu))_{\nu<\kappa}$ is nonempty, and as $(X, \mathcal{B})$ is intersection closed, $\bigcap_{\nu<\kappa} B_\nu$ is a ball. Finally, being an intersection of f-closed sets, also $\bigcap_{\nu<\kappa} B_\nu$ is f-closed. Therefore, $\bigcap_{\nu<\kappa} B_\nu \in \mathcal{S}_{(B,x)}$, and it is the supremum of $\mathcal{C}$. This proves our assertion. □

In order to prove Theorem 3, we take an f-closed ball $B \in \mathcal{B}$ and an element $x \in B$. We assume that conditions (A$_3$) and (B$_3$) hold. Then by Lemma 6, $(\mathcal{S}_{(B,x)}, <)$ is chain complete. By Zorn's Lemma, it admits a maximal element (B^*, z). If B^* is not a singleton, then by condition (A$_3$), there is an f-closed ball $B' \subsetneqq B$ and an element $x' \in B'$ such that xRx'. It follows that $(B^*, z) < (B', x')$, which contradicts the maximality of (B^*, z). Therefore, B^* is a singleton. As B^* is f-closed, it follows that z is a fixed point of f. The fact that $(B^*, z) \in \mathcal{S}_{(B,x)}$ also implies that xRz. This completes the proof of Theorem 3.

Finally, we turn to the proof of Theorem 4. Again, we take an f-closed ball $B \in \mathcal{B}$ and an element $x \in B$. We assume that conditions (A$_4$) and (B$_4$) hold.

We consider the set $\mathcal{S}_x$ that consists of all nests of the form $(B_{x_\nu})_{\nu<\lambda}$, where λ is any ordinal, such that $x_{\nu+1} = fx_\nu$ and $x_\mu Rx_\nu$ whenever $\mu < \nu < \lambda$. We introduce a partial order on $\mathcal{S}_x$ by defining that $\mathcal{C} \leq \mathcal{C}'$ if and only if the sequence $\mathcal{C}$ is an initial segment of $\mathcal{C}'$. It follows that the union over an ascending chain of nests in $\mathcal{S}_x$ is again a nest in $\mathcal{S}_x$. Hence by Zorn's Lemma, $\mathcal{S}_x$ admits a maximal nest $\mathcal{N}$.

Suppose that $\mathcal{N}$ is of the form $(B_{x_\nu})_{\nu<\lambda}$ with λ a limit ordinal. But then condition (B_4) of Theorem 4 shows the existence of a ball B_z such that $\mathcal{N} \cup \{B_{f^k z} \mid k \in \mathbb{N}\}$ properly contains $\mathcal{N}$. Since f is R-compatible, we have that $zRf^k z$ for all $k \in \mathbb{N}$, and by the transitivity of R, we obtain that $x_\nu R f^k z$ for all $\nu < \lambda$ and all $k \in \mathbb{N}$. This shows that $\mathcal{N} \cup \{B_{f^k z} \mid k \in \mathbb{N}\} \in \mathcal{S}_x$, contradicting the maximality of $\mathcal{N}$. Therefore, $\mathcal{N}$ contains a smallest ball B_z.

We wish to show that z is a fixed point of f. If we would have that $z \neq fz$, then by (SC2), $B_{f^i z} \subsetneq B_z \subseteq \bigcap \mathcal{N}$ for some $i \geq 1$, and the nest $\mathcal{N} \cup \{B_{f^k z} \mid k \in \mathbb{N}\}$ would again properly contain $\mathcal{N}$. As before, we would obtain a cotradiction to the maximality of $\mathcal{N}$.

Hence, z is a fixed point of f. Since $\mathcal{N} \in \mathcal{S}_x$, we also have that xRz. This completes the proof of Theorem 4.

3. Ultrametric spaces

In this section we consider ultrametric spaces (X, d) which are defined as follows. An ultrametric d on a set X is a function from $X \times X$ to a partially ordered set Γ with smallest element 0, such that for all $x, y, z \in X$ and all $\gamma \in \Gamma$,

(U1) $d(x, y) = 0$ if and only if $x = y$,
(U2) if $d(x, y) \leq \gamma$ and $d(y, z) \leq \gamma$, then $d(x, z) \leq \gamma$,
(U3) $d(x, y) = d(y, x)$ (symmetry).

(U2) is the ultrametric triangle law; if Γ is totally ordered, it can be replaced by

(UT) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.

A **closed ultrametric ball** is a set $B_\alpha(x) := \{y \in X \mid d(x, y) \leq \alpha\}$, where $x \in X$ and $\alpha \in \Gamma$. The problem with general ultrametric spaces is that closed balls $B_\alpha(x)$ are not necessarily **precise**, that is, there may not be any $y \in X$ such that $d(x, y) = \alpha$. Therefore, we prefer to work only with precise ultrametric balls, which we can write in the form

$$B(x, y) := \{z \in X \mid d(x, z) \leq d(x, y)\},$$

where $x, y \in X$. We obtain the **ultrametric ball space** $(X, \mathcal{B})$ from (X, d) by taking $\mathcal{B}$ to be the set of all such balls $B(x, y)$.

It follows from symmetry and the ultrametric triangle law that $B(x, y) = B(y, x)$ and that

$$B(t, z) \subseteq B(x, y) \quad \text{if and only if} \quad t \in B(x, y) \text{ and } d(t, z) \leq d(x, y). \tag{1}$$

In particular,

$$B(t, z) \subseteq B(x, y) \quad \text{if } t, z \in B(x, y).$$

Two elements γ and δ of Γ are **comparable** if $\gamma \leq \delta$ or $\gamma \geq \delta$. Hence if $d(x, y)$ and $d(y, z)$ are comparable, then $B(x, y) \subseteq B(y, z)$ or $B(y, z) \subseteq$

$B(x,y)$. If $d(y,z) < d(x,y)$, then in addition, $x \notin B(y,z)$ and thus, $B(y,z) \subsetneqq B(x,y)$. We note:

$$d(y,z) < d(x,y) \Longrightarrow B(y,z) \subsetneqq B(x,y)\,. \tag{2}$$

If Γ is totally ordered and B_1 and B_2 are any two balls with nonempty intersection, then $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.

As for ball spaces, we consider nests of (precise) ultrametric balls, and we will represent them in the form $(B(x_i,y_i))_{i\in I}$ where I is a totally ordered set and $B(x_j,y_j) \subsetneqq B(x_i,y_i)$ whenever $i,j \in I$ with $i<j$. The ultrametric space (X,d) with a transitive relation R on X will be called **R-spherically complete** if for every nest of balls $(B(x_i,y_i))_{i\in I}$ satisfying x_iRx_j whenever $i,j\in I$ with $i<j$ there is $z \in \bigcap_{i\in I} B(x_i,y_i)$ such that x_iRz for all $i\in I$. Recall that a function $f: X\to X$ is said to be R-compatible if $xRfx$ for all $x\in X$. Further, f is **non-expanding** if $d(fx,fy) \le d(x,y)$ for all $x,y\in X$, **contracting** if $d(fx,fy) < d(x,y)$ for all distinct $x,y\in X$, and **contracting on orbits** if $d(fx,f^2x) < d(x,fx)$ for all $x\in X$ with $x\ne fx$.

Theorem 7. *Take an ultrametric space (X,d), an element $x\in X$, a transitive relation R on X, and an R-compatible function $f: X\to X$ which is non-expanding and contracting on orbits.*

If (X,d) is R-spherically complete, then f admits a fixed point $z\in B(x,fx)$ that satisfies xRz. If in addition f is contracting, then f admits a unique fixed point z; it is independent of the choice of x and satisfies xRz for all $x\in X$.

Proof: To begin with, we note that for every $x\in X$ the ball $B(x,fx)$ is f-contracting. Indeed, if $x\ne fx$ then $d(fx,f^2x) < d(x,fx)$ which by (2) shows that $B(fx,f^2x) \subsetneqq B(x,fx)$.

If the ball $B(x,y)$ is f-contracting, then $fx\in B(x,y)$, whence $B(x,fx)\subseteq B(x,y)$. By what we have just shown, $B(x,y)$ properly contains $B(fx,f^2x)$. This proves that condition (A_1) of Theorem 1 holds. Our assumption that (X,d) is R-spherically complete implies that also condition (B_1) holds. So by Theorem 1, for every $x\in X$ there is a fixed point $z\in B(x,fx)$ of f.

If f is contracting, then it is non-expanding and contracting on orbits. Hence by what we have shown, it admits a fixed point z. If $y\ne z$ were another fixed point, then $d(x,y) = d(fx,fy) < d(x,y)$, a contradiction. Therefore, z is the only fixed point of f, and it follows that xRz holds for all $x\in X$. □

4. Topological spaces

In this section we will consider topological spaces X, equipped with a transitive relation R. A ball space is associated with X by taking $\mathcal{B}$ to be the collection of all nonempty closed sets.

We will say that X is **R-compact** if for every regular cardinal κ, every descending chain $(B_\nu)_{\nu<\kappa}$ of nonempty closed sets, and any choice of elements $x_\nu \in B_\nu$ such that $x_\mu R x_\nu$ whenever $\mu < \nu < \kappa$, the intersection $\bigcap_{\nu<\kappa} B_\nu$ contains an element z such that $x_\nu R z$ for all $\nu < \kappa$. If xRy for all $x, y \in X$, then every R-compact topological space is compact.

Theorem 8. *Take a topological space X with a transitive relation R on X and a function $f : X \to X$ such that for every non-singleton closed f-closed set B and every $x \in B$ there is a closed f-closed set $B' \subsetneq B$ and some $x' \in B'$ such that xRx'. If X is R-compact, then for every $x \in X$ there is a fixed point z of f that satisfies xRz.*

Proof: We take $\mathcal{B}$ to be the set of all nonempty closed sets in X. Then $(X, \mathcal{B})$ is intersection closed. By our assumptions, condition (A_3) of Theorem 3 is satisfied. If X is R-compact, then also condition (B_3) holds. Hence, our assertion follows from Theorem 3 together with the fact that X itself is a closed f-closed set. □

5. Complete lattices

We consider a complete latice $(L, <)$, together with a transitive relation R on L. We denote the top element of L by $\top$ and the bottom element by $\bot$. For any $a, b \in L$ with $a \le b$, we define the **interval**

$$[a, b] \;:=\; \{c \in L \mid a \le c \le b\} \,.$$

If we talk of an interval $[a, b]$, we will always implicitly assume that it is nonempty. The ball space associated with the lattice is obtained by setting

$$\mathcal{B} \;:=\; \{[a, b] \mid a, b \in L \text{ with } a \le b\} \,.$$

In [3] we prove:

Proposition 9. *The ball space associated with a complete lattice is spherically complete and intersection closed.*

A function $f : L \to L$ is **order preserving** if $a < b$ implies $fa < fb$. For such a function, an interval $[a, b]$ is f-closed if and only if $fa \ge a$ and $fb \le b$.

From Theorem 3, we derive the following result:

Theorem 10. *Take an order preserving R-compatible function $f : L \to L$ and assume that for every regular cardinal κ, every descending chain $([a_\nu, b_\nu])_{\nu<\kappa}$ of nonempty intervals, and any choice of elements $x_\nu \in [a_\nu, b_\nu]$ such that $x_\mu R x_\nu$ whenever $\mu < \nu < \kappa$, the intersection $\bigcap_{\nu<\kappa} [a_\nu, b_\nu]$ contains an element z such that $x_\nu R z$ for all $\nu < \kappa$. Then for every $x \in L$ there is a fixed point z of f that satisfies xRz.*

Proof: Take a non-singleton f-closed interval $[a,b]$ and $x \in [a,b]$. Since we assume f to be order preserving, we find that $fx \in [fa,fb] \subseteq [a,b]$ and that also $[fa,fb]$ is f-closed. Since f is R-compatible, we have that $xRfx$. If $[fa,fb] = [a,b]$, then we replace fa by fx if $f^2x \geq fx$, and fb by fx if $f^2x \leq fx$. The so obtained interval is again f-closed, and it is properly contained in $[a,b]$ (the proof by case distinction is straightforward). We have proved that condition (A_3) of Theorem 3 is satisfied.

Since $(L,\mathcal{B})$ is intersection closed, the assumptions of the theorem yield that also condition (B_3) holds. Hence, the assertion of our theorem follows from Theorem 3 together with the fact that $L = [\bot,\top] \in \mathcal{B}$ is f-closed. □

6. Ordered abelian groups and fields

For the conclusion of this paper, we consider an ordered abelian group $(G,<)$ together with a transitive relation R on G. Since the underlying additive group of an ordered field is an ordered abelian group $(G,<)$, we are implicitly covering also the case of ordered fields.

The associated ball space is given by the collection of all (nonempty) closed bounded intervals in $(G,<)$:

$$\mathcal{B} \;:=\; \{[a,b] \mid a,b \in G \text{ with } a \leq b\}\,.$$

We call $(G,<)$ **symmetrically complete** if this ball space is spherically complete. See [4] for more information on this notion and for a characterization of symmetrically complete ordered abelian groups and fields. In particular, we know from this characterization that every symmetrically complete ordered abelian group is divisible.

Take a function $f : G \to G$. We call it **non-expanding** if

$$|fx - fy| \;\leq\; |x-y|$$

for all $x,y \in G$, and **contracting on orbits** if there is a positive rational number $\frac{m}{n} < 1$ with $m,n \in \mathbb{N}$ such that

$$|fx - f^2x| \;\leq\; \frac{m}{n}|x - fx|$$

for all $x \in G$. Note that the right hand side makes sense since G is divisible.

From Theorem 4, we derive the following result:

Theorem 11. *Take a symmetrically complete ordered abelian group $(G,<)$, a transitive relation R on G, and an R-compatible non-expanding function $f : G \to G$ which is contracting on orbits. Assume that for every regular cardinal κ, every descending chain $([a_\nu,b_\nu])_{\nu<\kappa}$ of nonempty intervals, and any choice of elements $x_\nu \in [a_\nu,b_\nu]$ such that $x_\mu R x_\nu$ whenever $\mu < \nu < \kappa$, the intersection $\bigcap_{\nu<\kappa}[a_\nu,b_\nu]$ contains an element z such that $x_\nu R z$ for all $\nu < \kappa$. Then for every $x \in L$ there is a fixed point z of f that satisfies xRz.*

Proof: We set $C = \frac{m}{n}$. As in Section 8 of [1], for every $x \in G$ we take B_x to be the closed interval

$$B_x := \left\{ y \in X \;\middle|\; |x-y| \le \frac{|x-fx|}{1-C} \right\}$$

to obtain that $f^i x \in B_x$ for all $i \ge 0$. In particular, $x \in B_x$, hence (SC1) holds. It is further shown in Section 8 of [1] that our condition that f is contracting on orbits implies that also (SC2) holds.

In the proof of Theorem 21 of [1] it is shown that for every element z in the intersection of a nest as given in condition (B_4) of Theorem 4, the whole interval B_z is contained in the intersection. Together with the assumptions of our theorem, this shows that condition (B_4) holds. Hence, the assertion of our theorem follows from Theorem 4. □

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