LOGARITHMIC-EXPONENTIAL SERIES

LOU VAN DEN DRIES*, ANGUS MACINTYRE** AND DAVID MARKER***

ABSTRACT. We extend the field of Laurent series over the reals in a canonical way to an ordered differential field of "LE-series" (logarithmic-exponential series), which is equipped with a well behaved exponentiation. We show that the LE-series with derivative 0 are exactly the real constants, and we invert operators to show that each LE-series has a formal integral. We give evidence for the conjecture that the field of LE-series is a universal domain for ordered differential algebra in Hardy fields. We define composition of LE-series and establish its basic properties, including the existence of compositional inverses. Various interesting subfields of the field of LE-series are also considered.

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Introduction

The ordered differential field $\mathbf{R}((x^{-1}))$ of formal Laurent series in *descending* powers of x over $\mathbf{R}$ consists of all series of the form

$$f(x) = \underbrace{a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x}_{\text{infinite part of } f} + \underbrace{a_0 + a_{-1}x^{-1} + a_{-2}x^{-2} + \cdots}_{\text{finite part of } f}$$

with real coefficients a_i. We have $x > \mathbf{R}$ for the ordering, and $x' = 1$ for the derivation. There is a formal composition operation that assigns to $f(x), g(x) \in \mathbf{R}((x^{-1}))$ with $g > \mathbf{R}$ an element $f(g(x)) \in \mathbf{R}((x^{-1}))$. However, x^{-1} has no antiderivative in $\mathbf{R}((x^{-1}))$. There is also no reasonable exponential operation defined on all of $\mathbf{R}((x^{-1}))$, since such an operation would have to satisfy $\exp x > x^n$ for all n. Exponentiation only makes sense for the finite elements of $\mathbf{R}((x^{-1}))$:

$$\begin{aligned}\exp(a_0 + a_{-1}x^{-1} + a_{-2}x^{-2} + \cdots) &= e^{a_0}\sum_{n=0}^{\infty}\frac{1}{n!}(a_{-1}x^{-1} + a_{-2}x^{-2} + \cdots)^n \\ &= e^{a_0}(1 + b_1x^{-1} + b_2x^{-2} + \cdots)\end{aligned}$$

for certain reals $b_1, b_2, \ldots$.

To remove these defects we extend here $\mathbf{R}((x^{-1}))$ to an ordered differential field $\mathbf{R}((x^{-1}))^{\mathrm{LE}}$, the field of *logarithmic-exponential series* (or *LE-series* for short) over $\mathbf{R}$. Its elements are infinite series of "LE-monomials" arranged **from left to right in decreasing order** and multiplied by real coefficients, for example

$$e^{e^x} - 3e^{x^2} + 5x^{1/2} - \log x + 1 + x^{-1} + x^{-2} + x^{-3} + \cdots + e^{-x} + x^{-1}e^{-x} \ .$$

The reversed order type of the set of LE-monomials that occur in a given LE-series can be any countable ordinal. (For the series displayed it is $\omega+2$.) Such series occur for example in solving implicit equations of the form $P(x, y, e^x, e^y) = 0$ for y as $x \to +\infty$, where P is a (nonconstant) polynomial in 4 variables over $\mathbf{R}$, see [5,6]. In general one can view LE-series as a somewhat novel kind of asymptotic (often divergent) expansions for real functions defined near $+\infty$. (The Stirling expansion for the Gamma function is an especially simple example.) They also arise as formal solutions to algebraic differential equations. Perhaps the most important natural source of LE-series is in the works of Écalle [8] and Il'yashenko [14] on the Dulac problem.

We give an explicit construction of $\mathbf{R}((x^{-1}))^{\mathrm{LE}}$ in section 2, but to give already now an idea of this remarkable mathematical structure we show here some typical computations in $\mathbf{R}((x^{-1}))^{\mathrm{LE}}$:

(1) **Taking a reciprocal**

$$\begin{aligned}\frac{1}{x - x^2e^{-x}} = \frac{1}{x(1 - xe^{-x})} &= x^{-1}(1 + xe^{-x} + x^2e^{-2x} + \cdots) \\ &= x^{-1} + e^{-x} + xe^{-2x} + \cdots\end{aligned}$$

(2) **Formal Integration**

$$\int \frac{e^x}{x}\,dx = constant + \sum_{n=0}^{\infty} n!x^{-1-n}e^x \qquad \text{(a divergent series).}$$

(3) **Formal Composition**

Let $f(x) = x + \log x$ and $g(x) = x \log x$. Then

$$\begin{aligned} f(g(x)) &= x\log x + \log(x\log x) \\ &= x\log x + \log x + \log(\log x) \end{aligned}$$

and

$$\begin{aligned} g(f(x)) = (x+\log x)\log(x+\log x) &= (x+\log x)\Big(\log x + \log\Big(1+\frac{\log x}{x}\Big)\Big) \\ &= x\log x + (\log x)^2 + (x+\log x)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\Big(\frac{\log x}{x}\Big)^n \\ &= x\log x + (\log x)^2 + \log x + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}\frac{(\log x)^{n+1}}{x^n}. \end{aligned}$$

(4) **Compositional Inversion**

The series $g(x) = x\log x$ has a compositional inverse of the form

$$\frac{x}{\log x}\Big(1 + F\Big(\frac{\log\log x}{\log x}, \frac{1}{\log x}\Big)\Big)$$

where $F(X,Y)$ is an ordinary convergent power series in the two variables X and Y over $\mathbf{R}$. (Cf. [6, §4], and see also [1, Chapter 2] for other interesting examples.)

In this paper we concentrate on developing the formalism of LE-series. Section 1 establishes notation and also contains some useful facts on generalized series fields $\boldsymbol{k}((G))$ for which we do not know a convenient reference. In section 2 we follow in essence the original method of Dahn and Göring [3, 4] in constructing the field $\mathbf{R}((x^{-1}))^{\mathrm{LE}}$, but present it, we hope, in a more intuitive way. (In addition it is notationally more natural than in our earlier paper [6].) We also show in section 2 that each LE-series over $\mathbf{R}$ contains only countably many monomials. In section 3 we define the (formal) derivative of an LE-series, and prove the non-trivial fact that the derivative of an LE-series vanishes if and only if the series is a real constant. In section 4 we show that the ordered differential field $\mathbf{R}((x^{-1}))^{\mathrm{LE}}$ satisfies the same kind of differential inequalities as Hardy fields do, cf. [22]. (It is an interesting question—which can be made precise in several ways— how far this analogy really goes.) In section 5 we prove that each LE-series has an antiderivative. In section 6 we define the (formal) composition of LE-series and establish the expected properties like associativity, Taylor expansion, the chain rule, and the existence of compositional inverses. In section 7 we characterize Écalle's field $\mathbf{R}[[[x]]]$ of transseries as a subfield of $\mathbf{R}((x^{-1}))^{\mathrm{LE}}$ and establish some of its properties.

All the above is strictly a one-variable theory, but instead of $\mathbf{R}$, any ordered field with a "good" exponential function can serve as a field of coefficients for a field of LE-series. As this extra generality could be useful in questions about dependence on parameters and doesn't cost any extra effort, we actually work in this somewhat more general setting.

It seems that the interest in logarithmic-exponential series has two rather different origins. Dahn and Göring [2, 3] were influenced by Tarski's problem about the real exponential field, while Écalle [8] and Il'yashenko [14] encountered LE-series in their work on the Dulac problem. There are also potential connections with the theory of surreal numbers of Conway and Kruskal, and "super exact asymptotics".

The geometric part of Tarski's problem has been solved in the mean time by Wilkie [27], but LE-series remain useful for deciding various questions about the functions definable in the real exponential field and expansions of it, as we showed in our previous paper [6] to which the present paper is a sequel. But we make a fresh start here and only assume familiarity with [6] in a few places.

Notational conventions on ordered sets and groups

Throughout we let m and n range over $\mathbf{N} = \{0, 1, 2, \ldots\}$. By "ordered set" we will always mean a linearly ordered set. Given an ordered set G and $a \in G$ we put
$G^{>a} := \{g \in G : g > a\}$, $G^{<a} := \{g \in G : g < a\}$,
$G^{\geq a} := \{g \in G : g \geq a\}$, $G^{\leq a} := \{g \in G : g \leq a\}$.
If in addition A and B are subsets of G we put:
$g > A$ if $g > a$ for all $a \in A$,
$g < A$ if $g < a$ for all $a \in A$,
$A < B$ if $a < b$ for all $a \in A$ and $b \in B$.

Let G be an ordered abelian (multiplicative) group. (As usual this includes the requirement that the ordering is invariant under multiplication: $a < b$ implies $ac < bc$, for all $a, b, c \in G$.) Given subsets A and B of G we put
$AB = A \cdot B := \{ab : a \in A, b \in B\}$,
$A^n := \{a_1 \cdots a_n \in G : a_1, \ldots, a_n \in A\}$ (with $A^0 = \{1\}$),
$[A] := \bigcup_n A^n$ (the submonoid of G generated by A),
$\langle A \rangle := [A \cup A^{-1}] =$ the subgroup of G generated by A .

We will deal often with sets $A \subseteq G$ that are *reverse well ordered*, that is, for which there is no infinite increasing sequence $g_0 < g_1 < g_2 < \ldots$ in A. The following, from [19], is a basic result about such subsets.

Neumann's Lemma. *If $A, B \subseteq G$ are reverse well ordered, so is AB, and for each $g \in AB$ there are only finitely many pairs $(a, b) \in A \times B$ with $ab = g$. If $A \subseteq G^{<1}$ is reverse well ordered, then $[A]$ is too, and for each $g \in [A]$ there are only finitely many tuples $(n, a_1, \ldots, a_n)$ with $a_1, \ldots, a_n \in A$ such that $g = a_1 \cdots a_n$.*

§1 Fields of Generalized Series

(1.1) Let G be a *multiplicative* ordered abelian group with unit 1_G (or just 1 if no confusion results) and let $\boldsymbol{k}$ be a field. We define $\boldsymbol{k}((G))$ to be the *field of series over $\boldsymbol{k}$ with monomials in G*; its elements are the formal sums

$$s = \sum_{g \in G} c_g g$$

with coefficients $c_g \in \boldsymbol{k}$, such that $\operatorname{Supp} s := \{g \in G : c_g \neq 0\}$ is **reverse well ordered** in G.

By Neumann's Lemma these series can be added and multiplied in the usual way, making $\boldsymbol{k}((G))$ into a field with subfield $\boldsymbol{k}$ (identifying $c \in \boldsymbol{k}$ with the series $s = c \cdot 1_G$) and G into a multiplicative subgroup of $\boldsymbol{k}((G))^\times$ (identifying $g \in G$ with the series $s = 1 \cdot g$). The *leading coefficient* $\operatorname{Lc}(s)$, the *leading monomial* $\operatorname{Lm}(s)$ and the *leading term* $\operatorname{Lt}(s)$ of a nonzero element $s = \sum c_g g$ are given by: $\operatorname{Lc}(s) = c_{g_0}$, $\operatorname{Lm}(s) = g_0$ and $\operatorname{Lt}(s) = \operatorname{Lc}(s)\operatorname{Lm}(s) = c_{g_0} g_0$, where $g_0 = \max \operatorname{Supp} s$. We extend the functions Lc, Lm and Lt to (multiplicative) functions on all of $\boldsymbol{k}((G))$ by setting $\operatorname{Lc}(0) = \operatorname{Lm}(0) = \operatorname{Lt}(0) = 0 \in \boldsymbol{k}((G))$. Then Lm is just a nonarchimedean absolute value, except that its values lie in the ordered set $G \cup \{0\}$, with $g > 0$ for all $g \in G$, instead of in $\mathbf{R}^{\geq 0}$; in particular $\operatorname{Lm}(s + s') \leq \max(\operatorname{Lm}(s), \operatorname{Lm}(s'))$ for $s, s' \in \boldsymbol{k}((G))$. For each subset S of G we put

$$\boldsymbol{k}((S)) := \{s \in \boldsymbol{k}((G)) : \operatorname{Supp} s \subseteq S\},$$

which is a $\boldsymbol{k}$-linear subspace of $\boldsymbol{k}((G))$. Note that

$$\boldsymbol{k}((G)) = \boldsymbol{k}((G^{>1})) \oplus \boldsymbol{k} \oplus \boldsymbol{k}((G^{<1})) \text{ (direct sum of } \boldsymbol{k}\text{-linear subspaces)}$$

and that $\boldsymbol{k}((G^{\leq 1})) = \boldsymbol{k} \oplus \boldsymbol{k}((G^{<1})) = \{s \in \boldsymbol{k}((G)) : \operatorname{Lm}(s) \leq 1\}$ is a valuation ring of $\boldsymbol{k}((G))$ with maximal ideal $\mathfrak{m} := \boldsymbol{k}((G^{<1}))$. Each $s \in \boldsymbol{k}((G))^\times$ can be written as $s = \operatorname{Lt}(s)(1 + \epsilon)$ with $\epsilon \in \mathfrak{m}$. We consider $\boldsymbol{k}((G))$ as a topological field by taking the valuation topology: the discs $D(g) := \{s : \operatorname{Lm}(s) < g\}$, $g \in G$, form a basis of open neighborhoods of 0. If G_1 is a subgroup of G, we shall consider $\boldsymbol{k}((G_1))$ as a subfield of $\boldsymbol{k}((G))$ in the obvious way.

If $\boldsymbol{k}$ is an ordered field we make $\boldsymbol{k}((G))$ into an ordered field by declaring a nonzero element s to be positive if $\operatorname{Lc}(s) > 0$; this ordering extends both the ordering of $\boldsymbol{k}$ and the ordering of the group G, and the interval topology given by this ordering coincides with the valuation topology if $G \neq \{1\}$.

(1.2) Comment. In the previous papers [5, 6] we started with an additive ordered abelian group Γ and formed a multiplicative copy t^Γ so that $\gamma \mapsto t^\gamma$ is an order reversing isomorphism from the additive group Γ onto the multiplicative group t^Γ. The elements of $\boldsymbol{k}((t^\Gamma))$ appear then as generalized *power* series $s = \sum a_\gamma t^\gamma$. In terms of [5, 6], the *support* supp s of this series is a subset of Γ rather than of t^Γ, and is related to Supp s as follows:

$$\operatorname{Supp} s = \{t^\gamma : \gamma \in \operatorname{supp} s\}.$$

(The notational difference is the capital S of "Support" versus the lower case s of "support".) In dealing with logarithmic-exponential series it is awkward to consider each logarithmic-exponential monomial artificially as a "power" t^γ. This was done in [6] with the unfortunate consequence that in general $t^\gamma \neq \exp(\gamma \log t)$, see remark 2.11 of [6]. When defining composition of logarithmic-exponential series this would become notationally confusing. That is why we have chosen here a better motivated and more flexible notation.

(1.3) Suppose $a_i \in \boldsymbol{k}((G))$ for all $i \in I$ for some, possibly infinite, set I. We say that the sum $\sum_{i\in I} a_i$ exists if the following two conditions are satisfied:

(a) For each $g \in G$ there are only finitely many $i \in I$ with $g \in \operatorname{Supp} a_i$;
(b) The union $\bigcup_{i\in I} \operatorname{Supp} a_i$ is reverse well ordered in G.

Clearly, if these conditions are satisfied we can associate with the family (a_i) a well defined element $\sum a_i \in \boldsymbol{k}((G))$. (Note that each element $a = \sum c_g g$ of $\boldsymbol{k}((G))$ is indeed the sum of the family $(c_g g)$ in this sense.) We will frequently use the following rules for manipulating such sums. Let $(a_i)_{i\in I}$, $(b_j)_{j\in J}$ and $(c_{i,j})_{(i,j)\in I\times J}$ be families of elements of $\boldsymbol{k}((G))$ such that $\sum a_i$, $\sum b_j$ and $\sum_{i,j} c_{i,j}$ exist. Then

(1) for each $d \in \boldsymbol{k}((G))$, $\sum da_i$ exists, and $\sum da_i = d\sum a_i$,
(2) if $I = J$, then $\sum(a_i + b_i)$ exists, and $\sum(a_i + b_i) = \sum a_i + \sum b_i$;
(3) $\sum_{i,j} a_i b_j$ exists, and $\sum_{i,j} a_i b_j = (\sum a_i)\cdot(\sum b_j)$;
(4) if $\sigma : I \to J$ is a bijection and $a_i = b_{\sigma(i)}$ for all i, then $\sum a_i = \sum b_j$,
(5) $\sum_j c_{i,j}$ exists for each $i \in I$, and $\sum_i \left(\sum_j c_{i,j}\right)$ exists, and

$$\sum_{i,j} c_{i,j} = \sum_i \left(\sum_j c_{i,j}\right).$$

These infinite sums occur everywhere in the subject. For instance, given an ordinary power series $F = \sum_{\omega\in\mathbf{N}^n} a_\omega Y_1^{\omega_1}\cdots Y_n^{\omega_n} \in \boldsymbol{k}[[Y_1,\dots,Y_n]]$ and infinitesimals $\epsilon_1,\dots,\epsilon_n \in \mathfrak{m}$ we have a well defined element $F(\epsilon_1,\dots,\epsilon_n) := \sum a_\omega \epsilon_1^{\omega_1}\cdots\epsilon_n^{\omega_n}$ of $\boldsymbol{k}((G))$, since this infinite sum exists by Neumann's Lemma.

We can also use these infinite sums to introduce efficiently certain useful operators on $\boldsymbol{k}((G))$. Let D be a $\boldsymbol{k}$-linear subspace of $\boldsymbol{k}((G))$ such that for each family $(a_i)_{i\in I}$ of elements in D for which $\sum a_i$ exists we have $\sum a_i \in D$. (Example: $D = \{s \in \boldsymbol{k}((G)) : \operatorname{Lm} s < A\}$ where A is any subset of G; note: $D = \boldsymbol{k}((G))$ for $A = \emptyset$.) Let us say that an operator $\mathrm{P} : D \longrightarrow D$ is *small* if there is a reverse well ordered subset R of $G^{<1}$ such that $\operatorname{Supp}\,\mathrm{P}(s) \subseteq R\cdot\operatorname{Supp} s$ for all $s \in D$ (we do not assume P is $\boldsymbol{k}$-linear or even additive). An obvious induction on n then shows that the n^{th} iterate P^n of P (with $\mathrm{P}^0 = \mathrm{I} = \mathrm{id}_D$) satisfies

$$\operatorname{Supp}\ \mathrm{P}^n(s) \subseteq R^n \cdot \operatorname{Supp} s$$

for all $s \in D$. Hence for each sequence (c_n) of coefficients in $\boldsymbol{k}$ we have a well defined operator

$$\sum c_n\,\mathrm{P}^n :\ s \mapsto\ \sum c_n\,\mathrm{P}^n(s) : D \longrightarrow D\ ,$$

the existence of the right hand sum being a consequence of Neumann's Lemma. In particular, if P is a small operator, then the perturbation $\mathrm{I} - \mathrm{P}$ of the identity is a bijection from D onto itself with inverse $\sum \mathrm{P}^n$.

We shall use small operators in section 5 to integrate logarithmic-exponential series, and in section 6 to obtain compositional inverses to them.

(1.4) Suppose now that G_1 is a convex subgroup of G and G is the internal direct product of subgroups G_1 and G_2, that is $G = G_1G_2$ and $G_1 \cap G_2 = \{1\}$. Then we have an isomorphism

$$\boldsymbol{k}((G)) \cong \boldsymbol{k}((G_1))((G_2))$$

of $\boldsymbol{k}((G_1))$-algebras given by

$$\sum_{g\in G} a_g g \ \mapsto \sum_{g_2\in G_2} \Big(\sum_{g_1 \in G_1} a_{g_1g_2} g_1 \Big) g_2. \tag{$*$}$$

Note that in $\boldsymbol{k}((G))$ we actually have

$$\sum_{g\in G} a_g g = \sum_{g_2\in G_2} \Big(\sum_{g_1 \in G_1} a_{g_1g_2} g_1 \Big) g_2$$

where all sums are interpreted according to the notation in (1.3). Note also that if $\boldsymbol{k}$ is an ordered field, then the isomorphism $(*)$ is order preserving. Thus there is no harm in identifying $\boldsymbol{k}((G))$ with $\boldsymbol{k}((G_1))((G_2))$ and we shall do so when convenient. With this identification we have

$$\boldsymbol{k}((G)) = A \oplus B \text{ (direct sum of } \boldsymbol{k}((G_1))\text{-linear subspaces)}$$

where $A = \{s \in \boldsymbol{k}((G)) : \operatorname{Supp} s > G_1\}$
and $B = \{s \in \boldsymbol{k}((G)) : \operatorname{Supp} s \leq g_1 \text{ for some } g_1 \in G_1\} = \boldsymbol{k}((G_1))((G_2^{\leq 1}))$.

(1.5) Exponential ordered fields

An *exponential ordered field* is a pair (K, E) where K is an ordered field and $E : K \to K^{>0}$ (the exponential map) is a strictly increasing homomorphism from the additive group of K into its positive multiplicative group $K^{>0}$. If in addition $E(K) = K^{>0}$ we call (K, E) a *logarithmic-exponential ordered field*. In that case the inverse to E is written as $\log : K^{>0} \to K$, and one also writes $a^r := E(r \log a)$ for $a, r \in K$, $a > 0$, and the usual identities hold: $a^{r+s} = a^r a^s$, $(ab)^r = a^r b^r$, and $a^{rs} = (a^r)^s$ for $a, b, r, s \in K$ with positive a, b.

A *pre-exponential ordered field* is a quadruple (K, A, B, E) such that K is an ordered field, A an additive subgroup of K and B a a convex subgroup of K with $K = A \oplus B$, and $E : B \to K^{>0}$ is a strictly increasing homomorphism from the additive group B into the multiplicative group $K^{>0}$.

(1.6) A basic example

Let $\boldsymbol{k}$ be an ordered exponential field with exponential map exp, and consider the ordered multiplicative group $x^{\boldsymbol{k}}$ with order preserving isomorphism $r \mapsto x^r$ from the additive group of $\boldsymbol{k}$ onto $x^{\boldsymbol{k}}$. This gives rise to the pre-exponential ordered field

$$(\boldsymbol{k}((x^{\boldsymbol{k}})), A, B, E)$$

by setting $A := \{s \in \boldsymbol{k}((x^{\boldsymbol{k}})) : \text{Supp } s > 1\}$, $B := \{s \in \boldsymbol{k}((x^{\boldsymbol{k}})) : \text{Supp } s \leq 1\}$, and defining $E : B \to (\boldsymbol{k}((x^{\boldsymbol{k}})))^{>0}$ by

$$E(a+\epsilon) = \exp(a) \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}$$

for $a \in \boldsymbol{k}$ and $\epsilon \in \mathfrak{m}(B) := \{s \in B : \text{Supp } s < 1\}$. To simplify notation we also write $\boldsymbol{k}((x^{\boldsymbol{k}}))$ for the resulting pre-exponential ordered field.

Note that upon setting $x := x^1$ we have $x > \boldsymbol{k}$, and $\boldsymbol{k}((x^{\boldsymbol{k}}))$ contains the field $\boldsymbol{k}(x)$ of rational functions in x over $\boldsymbol{k}$ as a subfield. We think of an element of $\boldsymbol{k}((x^{\boldsymbol{k}}))$ as a series

$$\sum_{r \in \boldsymbol{k}} a_r x^r$$

(in "descending powers of x", as one goes from left to right). Put $t := x^{-1}$ so that $0 < t < a$ for all positive $a \in \boldsymbol{k}$ and note that $\boldsymbol{k}((x^{\boldsymbol{k}}))$ contains the field $\boldsymbol{k}((t)) = \boldsymbol{k}((x^{-1})) := \boldsymbol{k}((x^{\mathbf{Z}}))$ of formal Laurent series in t as a subfield since $\mathbf{Z} \subseteq \boldsymbol{k}$ (as an ordered subgroup). For us it is more convenient to consider the infinitely large x as the "independent variable" rather than the infinitesimal t.

(1.7) Let (K, A, B, E) be a pre-exponential ordered field. We define its *first extension* (K', A', B', E') as follows: take a multiplicative copy $E(A)$ of the ordered additive abelian group A with order preserving isomorphism $E_A : A \to E(A)$. Let $K' := K((E(A)))$, $A' := \{s \in K' : \text{Supp } s > 1\}$ and $B' := \{s \in K' : \text{Supp } s \leq 1\}$, so that B' is a convex subring of K'. Note that $K' = A' \oplus B'$ and $B' = K \oplus \mathfrak{m}(B')$. We now extend E to $E' : B' \to (K')^{>0}$ by

$$E'(a+b+\epsilon) = E_A(a)E(b) \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}$$

for $a \in A$, $b \in B$ and $\epsilon \in \mathfrak{m}(B')$. One verifies easily that E' extends E and that (K', A', B', E') is again a pre-exponential ordered field. Note that the map E' is defined on the whole field K since $K \subseteq B'$.

We define for each n a pre-exponential ordered field (K_n, A_n, B_n, E_n): for $n = 0$ this is just (K, A, B, E) itself, and $(K_{n+1}, A_{n+1}, B_{n+1}, E_{n+1})$ is the first extension of (K_n, A_n, B_n, E_n). This gives an increasing sequence

$$K = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$$

of ordered fields. Let $K_\infty = \bigcup K_n$, and define $E_\infty : K_\infty \to (K_\infty)^{>0}$ to be the common extension of all the E_n. Then (K_∞, E_∞) is clearly an ordered exponential field.

§2 The Field of Logarithmic-Exponential Series

From now on in this paper we fix a logarithmic-exponential ordered field $\boldsymbol{k}$.

(2.1) Starting with the pre-exponential ordered field $(K, A, B, E) = \boldsymbol{k}((x^{\boldsymbol{k}}))$ as in (1.6), we denote the exponential field (K_∞, E_∞) that results from the construction in (1.7) by $\boldsymbol{k}((x^{-1}))^{\mathrm{E}}$ or by $\boldsymbol{k}((t))^{\mathrm{E}}$, and its exponential map E_∞ by E, to simplify notation. We define the increasing sequence $G_0 \subset G_1 \subset \ldots$ of multiplicative subgroups of $\boldsymbol{k}((t))^{\mathrm{E}}$ by $G_0 = x^{\boldsymbol{k}}$ and $G_{n+1} = G_n \, \mathrm{E}(A_n)$, where (K_n, A_n, B_n, E_n) is as in (1.7). By induction on n one easily checks that G_n is convex in G_{n+1}, $G_n \cap \mathrm{E}(A_n) = \{1\}$ and $K_n = \boldsymbol{k}((G_n))$, since $\boldsymbol{k}((G_n))((\mathrm{E}(A_n)) = \boldsymbol{k}((G_{n+1}))$ by the identification in (1.4). Unless we say otherwise we will consider K_n as the series field $\boldsymbol{k}((G_n))$ over $\boldsymbol{k}$. Note that $A_n = \{f \in K_n : \mathrm{Supp}\ f > G_{n-1}\}$, which is even true for $n = 0$ by setting $G_{-1} := \{1\} \subseteq G_0$. (Similarly, $K_{-1} := \boldsymbol{k}$ in the following.)

We put $G^{\mathrm{E}} := \bigcup G_n$, an ordered subgroup of the positive multiplicative group of the ordered field $\boldsymbol{k}((t))^{\mathrm{E}}$. The elements of G^{E} are called E-*monomials*. Thus we have the field inclusion $\boldsymbol{k}((t))^{\mathrm{E}} = \bigcup \boldsymbol{k}((G_n)) \subset \boldsymbol{k}((G^{\mathrm{E}}))$. The inclusion is proper because $\sum_n 1/\mathrm{E}^n(x) \in \boldsymbol{k}((G^{\mathrm{E}})) \setminus \boldsymbol{k}((t))^{\mathrm{E}}$, where $\mathrm{E}^0(x) := x$ and $\mathrm{E}^{n+1}(x) = \mathrm{E}(\mathrm{E}^n(x))$; to see this, note that $\mathrm{E}^n(x) \in G_n$ and $\mathrm{E}^n(x) > G_{n-1}$ for each n. Note that $\boldsymbol{k}((t))^{\mathrm{E}}$ is dense in the topological field $\boldsymbol{k}((G^{\mathrm{E}}))$.

The lemma below follows easily by induction on n and 1.4. We record it for later use.

Lemma 2.2.
i) $\boldsymbol{k}((G_n)) = A_n \oplus A_{n-1} \oplus \cdots \oplus A_0 \oplus \boldsymbol{k} \oplus \mathfrak{m}_n$ *(a direct sum of $\boldsymbol{k}$-linear spaces) where $\mathfrak{m}_n$ is the maximal ideal of $\boldsymbol{k}((G_n^{\preceq 1}))$;*
ii) if $a_n \in A_n, a_{n-1} \in A_{n-1}, \ldots, a_0 \in A_0, r \in \boldsymbol{k}$ and $\epsilon \in \mathfrak{m}_n$ are all positive, then $a_n > a_{n-1} > \cdots > a_0 > r > \epsilon$;
iii) $A_n \oplus A_{n-1} \oplus \cdots \oplus A_0 = \{f \in \boldsymbol{k}((G_n)) : \mathit{Supp}\ f > 1\}$;
iv) $G_n = x^{\boldsymbol{k}} \cdot \mathrm{E}(A_{n-1} \oplus \cdots \oplus A_0)$, *and* $x^{\boldsymbol{k}} \cap \mathrm{E}(A_{n-1} \oplus \cdots \oplus A_0) = \{1\}$.

Lemma 2.3. *Let $f \in K_n$, $f > 0$. Then*
$$f \in \mathrm{E}(\boldsymbol{k}((t))^{\mathrm{E}}) \Longleftrightarrow \mathrm{Lm}\, f \in \mathrm{E}(A_{n-1} \oplus \cdots \oplus A_0)\ .$$

Proof. By 2.2 iv) we have $\mathrm{Lm}\, f = x^r \,\mathrm{E}(a)$ where $r \in \boldsymbol{k}$ and $a \in A_{n-1} \oplus \cdots \oplus A_0$. Let $c = \mathrm{Lc}\, f > 0$. Then $f = cx^r \,\mathrm{E}(a)(1-\epsilon)$ with $\epsilon \in \mathfrak{m}_n$. If $\mathrm{Lm}\, f \in \mathrm{E}(A_{n-1} \oplus \cdots \oplus A_0)$, then $r = 0$ (by 2.2 iv)), so
$$f = \mathrm{E}\left(a + \log(c) - \sum_{i=1}^{\infty} \frac{\epsilon^i}{i!}\right) \in \mathrm{E}(K_{n-1}) \subseteq \mathrm{E}(\boldsymbol{k}((t))^{\mathrm{E}}).$$
Conversely, suppose $f = \mathrm{E}(g)$, with $g \in K_l$, $l \geq n$. Write $g = \alpha + s + \delta$ with $\alpha \in A_l \oplus \cdots \oplus A_0$, $s \in \boldsymbol{k}$ and $\delta \in \mathfrak{m}_l$. Then
$$f = \mathrm{E}(g) = \exp(s)\,\mathrm{E}(\alpha)\Big(1 + \sum_{i=1}^{\infty} \frac{\delta^i}{i!}\Big).$$

Hence $\operatorname{Lm} f = \mathrm{E}(\alpha) = x^r\,\mathrm{E}(a)$, and thus $r = 0$ by 2.2 iv).

(2.4) Notational Conventions about Infinite Sums

In accordance with 1.3 we shall write

$$f = \sum c_r x^r \text{ in } K_0$$

to mean that $c_r \in \boldsymbol{k}$ for all $r \in \boldsymbol{k}$, and that $f = \sum c_r x^r$ in the sense of the series field $\boldsymbol{k}((G_0))$. Similarly, if we write

$$f = \sum f_a\,\mathrm{E}(a) \text{ in } K_{n+1}$$

we mean that the sum is over all $a \in A_n$, that $f_a \in K_n$ for all $a \in A_n$, and that $f = \sum f_a\,\mathrm{E}(a)$ in the sense of the series field $K_{n+1} = K_n((\mathrm{E}(A_n)))$ over K_n. Thus in this case $f \in K_{n+1}$, and also $f = \sum f_a\,\mathrm{E}(a)$ in the sense of the series field $\boldsymbol{k}((G_{n+1}))$, by 1.3 and 1.4. Finally, we write

$$f = \sum f_i \text{ in } \boldsymbol{k}((t))^{\mathrm{E}}$$

to mean that there exists an n such that all $f_i \in K_n$ and that $f = \sum f_i$ in the sense of the series field $K_n = \boldsymbol{k}((G_n))$. (In particular $f \in K_n$ for this n, and clearly $\sum f_i$ is independent of the particular value of n for which all $f_i \in K_n$.)

(2.5) Comparison with [6]

The field $\mathbf{R}((t))^{\mathrm{E}}$ only differs notationally from the $\mathbf{R}((t))^{\mathrm{E}}$ of [6]. To facilitate comparison we specify the isomorphism θ (of exponential ordered fields) from our present $\mathbf{R}((t))^{\mathrm{E}}$ onto the $\mathbf{R}((t))^{\mathrm{E}}$ of [6]. Under θ the present $K_n = \mathbf{R}((G_n))$ maps onto $K_n = \mathbf{R}((t^{\Gamma_n}))$ of [6], with present G_n, A_n and B_n mapping onto t^{Γ_n}, J_n and O_n of [6], respectively. For $n = 0$ we have $K_0 = \mathbf{R}((x^{\mathbf{R}}))$ (present), and $K_0 = \mathbf{R}((t^{\mathbf{R}}))$ (in [6]) with $\theta(\sum a_r x^r) = \sum a_r t^{-r}$. Inductively, for $f = \sum f_a\,\mathrm{E}(a)$ in present K_{n+1} we put $\theta(f) := \sum \theta(f_a) t^{-\theta(a)}$. For all intents and purposes we may identify our present $\mathbf{R}((t))^{\mathrm{E}}$ via θ with the $\mathbf{R}((t))^{\mathrm{E}}$ of [6].

(2.6) The substitution map Φ

To extend $\boldsymbol{k}((t))^{\mathrm{E}}$ to a logarithmic-exponential ordered field $\boldsymbol{k}((t))^{\mathrm{LE}}$, we will need to have available a certain map $\Phi : \boldsymbol{k}((t))^{\mathrm{E}} \longrightarrow \boldsymbol{k}((t))^{\mathrm{E}}$, which plays a key role throughout this paper. It embeds $\boldsymbol{k}((t))^{\mathrm{E}}$ into itself as an ordered exponential field and is the identity on $\boldsymbol{k}$, in particular, $\Phi(\mathrm{E}(f)) = \mathrm{E}(\Phi(f))$ for $f \in \boldsymbol{k}((t))^{\mathrm{E}}$. We think of it as "substituting $\mathrm{E}(x)$ for x", symbolically: $\Phi(f(x)) = f(\mathrm{E}(x))$. (In §6 we justify this formula when we define composition of LE-series.) The definition of $\Phi(f)$ is by induction:

if $f = \sum a_r x^r$ in K_0, then $\Phi(f) := \sum a_r\,\mathrm{E}(rx)$,

if $f = \sum f_a\,\mathrm{E}(a)$ in K_{n+1}, then $\Phi(f) := \sum \Phi(f_a)\,\mathrm{E}(\Phi(a))$.

To make sense of this one shows simultaneously by induction on n that $\Phi(G_n) \subseteq G_{n+1}$, $\Phi(A_n) \subseteq A_{n+1}$ and $\Phi(K_n) \subseteq K_{n+1}$. (See also [6, §2] for details.) The properties of Φ stated above are easily derived, and in fact, one obtains $\Phi(G_n) \subseteq \mathrm{E}(A_n \oplus \cdots \oplus A_0)$. Thus $\mathrm{Lm}(\Phi(f)) \in E(A_n \oplus \cdots \oplus A_0)$ for all $f \in K_n$ and so, by 2.3, $\Phi(f) \in \mathrm{E}(\boldsymbol{k}((t))^{\mathrm{E}})$ for every positive element $f \in \boldsymbol{k}((t))^{\mathrm{E}}$. This means that each positive element of $\Phi(\boldsymbol{k}((t))^{\mathrm{E}})$ has a logarithm in $\boldsymbol{k}((t))^{\mathrm{E}}$.

It is easy to see that Φ also preserves infinite sums: if $f = \sum f_i$ in $\boldsymbol{k}((t))^{\mathrm{E}}$, then $\Phi(f) = \sum \Phi(f_i)$ in $\boldsymbol{k}((t))^{\mathrm{E}}$.

(2.7) The construction of $\boldsymbol{k}((t))^{\mathrm{LE}}$

We construct an increasing chain of exponential ordered fields

$$\boldsymbol{k}((t))^{\mathrm{E}} = L_0 \subset L_1 \subset L_2 \subset \cdots$$

and isomorphisms η_n from L_n onto $\boldsymbol{k}((t))^{\mathrm{E}}$. Let $L_0 = \boldsymbol{k}((t))^{\mathrm{E}}$ and let η_0 be the identity. Given L_n and η_n, take $L_{n+1} \supset L_n$ and an isomorphism of exponential ordered fields $\eta_{n+1} : L_{n+1} \to \boldsymbol{k}((t))^{\mathrm{E}}$ such that $\eta_{n+1}(z) = \Phi(\eta_n(z))$ for all $z \in L_n$. So L_n sits inside L_{n+1} as $\Phi(\boldsymbol{k}((t))^{\mathrm{E}})$ sits in $\boldsymbol{k}((t))^{\mathrm{E}}$. Thus by a remark in 2.6 every positive element of L_n has a logarithm in L_{n+1}.

Let $\boldsymbol{k}((t))^{\mathrm{LE}} = \bigcup L_n$. Clearly $\boldsymbol{k}((t))^{\mathrm{LE}}$ is a logarithmic-exponential ordered field. We call $\boldsymbol{k}((t))^{\mathrm{LE}}$ the field of *logarithmic-exponential series* over $\boldsymbol{k}$. We also write it as $\boldsymbol{k}((x^{-1}))^{\mathrm{LE}}$ and call its elements *LE-series* over $\boldsymbol{k}$.

There will be no harm in denoting the exponential map of $\boldsymbol{k}((t))^{\mathrm{LE}}$ by the same symbol E as the exponential map of $\boldsymbol{k}((t))^{\mathrm{E}}$. Its inverse will be written both as log and as L. We often use the nth iterates E^n and L^n of these maps, with

$$\mathrm{E}^0 = \mathrm{L}^0 := \text{ identity map on } \boldsymbol{k}((t))^{\mathrm{LE}} \ ,$$

while for $n > 0$, $\mathrm{L}^n(z)$ is only defined for $z > \mathrm{E}^{n-1}(0)$. As $\boldsymbol{k}((t))^{\mathrm{LE}}$ is a logarithmic-exponential ordered field we have by (1.5) a well defined element $f^g := \mathrm{E}(g \log f)$ for $f, g \in \boldsymbol{k}((t))^{\mathrm{LE}}$ with $f > 0$; note that for $f = x$ and $g = r \in \boldsymbol{k}$ this agrees, fortunately, with x^r viewed as an element of K_0.

Let $l_n := \mathrm{L}^n(x) \in L_n$, in particular $l_0 = x$ and $l_1 = \log x$. The isomorphism η_n^{-1} from $\boldsymbol{k}((t))^{\mathrm{E}}$ onto L_n maps x to l_n, and we want to view it as "substitution of l_n for x", that is $\eta_n^{-1}(f(x)) = f(l_n)$. In §6 we will make sense of this last formula.

For each n we have an isomorphism $\eta_n^{-1} \circ \eta_{n+1} : L_{n+1} \longrightarrow L_n$, of exponential ordered fields, and these isomorphisms have a unique common extension to an automorphism of the exponential ordered field $\boldsymbol{k}((t))^{\mathrm{LE}}$. This automorphism extends Φ, and we shall denote it by Φ as well. Thus we have an automorphism Φ^k for each (positive or negative) integer k. Note that $\Phi^n(z) = \eta_n(z)$ for $z \in L_n$. In particular, $\Phi^{-n}(x) = \mathrm{L}^n x$.

(2.8) $\boldsymbol{k}((t))^{\mathrm{LE}}$ as a field of generalized series

It will be convenient to construe L_n as a field of generalized series. Put $G^{\mathrm{E},n} := \eta_n^{-1}(G^{\mathrm{E}})$, an ordered subgroup of the positive multiplicative group of L_n. We have an isomorphism

$$\sum_{\mu \in G^{\mathrm{E}}} a_\mu \mu \mapsto \sum_{\mu \in G^{\mathrm{E},n}} a_\mu \eta_n^{-1}(\mu) : \boldsymbol{k}((G^{\mathrm{E}})) \longrightarrow \boldsymbol{k}((G^{\mathrm{E},n}))$$

of ordered fields. There is no loss of generality in taking η_n^{-1} to be the restriction of this isomorphism to $\boldsymbol{k}((t))^{\mathrm{E}}$, with image $L_n \subseteq \boldsymbol{k}((G^{\mathrm{E},n}))$, for each n. Consider the increasing sequence

$$G^{\mathrm{E}} = G^{\mathrm{E},0} \subset G^{\mathrm{E},1} \subset G^{\mathrm{E},2} \subset \cdots$$

of subgroups of the positive multiplicative group of $\boldsymbol{k}((t))^{\mathrm{LE}}$. Let $G^{\mathrm{LE}} := \bigcup G^{\mathrm{E},n}$, the (ordered) group of *LE-monomials*. Then the identifications above give an ordered field inclusion $\boldsymbol{k}((t))^{\mathrm{LE}} \subseteq \boldsymbol{k}((G^{\mathrm{LE}}))$. Thus for $f \in \boldsymbol{k}((t))^{\mathrm{LE}}$ its support Supp f is a reverse well ordered subset of G^{LE}, and we can speak of the *leading monomial* $\mathrm{Lm}(f)$ of f (in $G^{\mathrm{LE}} \cup \{0\}$), its *leading coefficient* $\mathrm{Lc}(f)$ (in $\boldsymbol{k}$), and its *constant term* (in $\boldsymbol{k}$), the latter being the coefficient of the LE-monomial 1 when f is written as an element of $\boldsymbol{k}((G^{\mathrm{LE}}))$. We consider $\boldsymbol{k}((t))^{\mathrm{LE}}$ as a topological field by taking the interval topology given by the ordering of $\boldsymbol{k}((t))^{\mathrm{LE}}$; this topology equals the topology induced by the valuation topology of $\boldsymbol{k}((G^{\mathrm{LE}}))$. An element $\epsilon \in \boldsymbol{k}((t))^{\mathrm{LE}}$ is said to be infinitesimal if $|\epsilon| < \boldsymbol{k}^{>0}$.

To make sense of infinite sums in $\boldsymbol{k}((t))^{\mathrm{LE}}$ we put

$$L_{m,n} := \eta_m^{-1}(K_n) \subseteq L_m \text{ and } G_{m,n} := \eta_m^{-1}(G_n) \subseteq G^{\mathrm{E},m}$$

so that $L_{m,n} = \boldsymbol{k}((G_{m,n}))$, and $L_m = \bigcup_n L_{m,n}$. We write

$$f = \sum f_i \text{ in } \boldsymbol{k}((t))^{\mathrm{LE}}$$

to mean that there exist m and n such that all $f_i \in L_{m,n}$, and that $f = \sum f_i$ in the sense of $L_{m,n} = \boldsymbol{k}((G_{m,n}))$. In that case also $f \in L_{m,n}$, and $\sum f_i$ is independent of the particular values of m and n for which all $f_i \in L_{m,n}$. Note that if $f = \sum f_i$ in $\boldsymbol{k}((t))^{\mathrm{LE}}$ then $\Phi^k(f) = \sum \Phi^k(f_i)$ in $\boldsymbol{k}((t))^{\mathrm{LE}}$ for all integers k.

(2.9) The valuation on $\boldsymbol{k}((t))^{\mathrm{LE}}$

The leading monomial function Lm behaves like a nonarchimedean absolute value, and the corresponding valuation (in the sense of Krull)

$$v : \boldsymbol{k}((t))^{\mathrm{LE}} \to \log(G^{\mathrm{LE}}) \cup \{\infty\}$$

is defined by $v(f) := -\log(\mathrm{Lm}(f))$ for $f \neq 0$. The value group $\log(G^{\mathrm{LE}})$ is here ordered as an additive subgroup of $\boldsymbol{k}((t))^{\mathrm{LE}}$. The valuation ring of v is of course the 'closed unit disc' $\{f : \mathrm{Lm}(f) \leq 1\}$. Note also that the restriction of v to $\boldsymbol{k}((t))^{\mathrm{E}}$ has value group $(\bigoplus_n A_n) \oplus \boldsymbol{k}\log(x)$, by 2.2.

(2.10) A countability property of logarithmic-exponential series

Here we show that each LE-series over $\mathbf{R}$ contains only countably many LE-monomials. This ultimately stems from the well known fact that each well ordered subset of the real line $\mathbf{R}$ is countable, via the following lemma due to Esterle [9].

Lemma. *Suppose each (reverse) well ordered subset of the ordered abelian multiplicative group G is countable. Then also each (reverse) well ordered subset of the ordered field $\mathbf{R}((G))$ is countable.*

Corollary. *Each (reverse) well ordered subset of the ordered field $\mathbf{R}((t))^{\mathrm{LE}}$ is countable, in particular, Supp f is countable for each $f \in \mathbf{R}((t))^{\mathrm{LE}}$. The cardinality of $\mathbf{R}((t))^{\mathrm{LE}}$ is $2^{\aleph_0}$.*

Proof. The lemma implies by an induction on n that each (reverse) well ordered subset of the ordered field $K_n = \mathbf{R}((G_n))$ is countable, and that K_n has cardinality $2^{\aleph_0}$. Thus $\mathbf{R}((t))^{\mathrm{E}}$ also has these properties. Since each ordered field L_n is isomorphic to $\mathbf{R}((t))^{\mathrm{E}}$ it follows that their union $\mathbf{R}((t))^{\mathrm{LE}}$ inherits these properties as well.

Remark. By a theorem of Harrington and Shelah [12] each well ordered subset of a Borel order is countable. However, this fact does not contain the corollary above as a special case: the main result in Harrington, Marker and Shelah [13] implies that the underlying ordered set of $\mathbf{R}((t))^{\mathrm{LE}}$ is not order embeddable in a Borel order. On the other hand, in §7 we will introduce the logarithmic-exponential ordered subfield $\mathbf{R}((t))^{\mathrm{LE,ft}}$ of $\mathbf{R}((t))^{\mathrm{LE}}$ whose elements are the LE-series with "support of hereditarily finite type". This substructure is probably large enough for most purposes, and is a Borel structure. See §7 for more details.

(2.11) Other approaches

We obtained $\boldsymbol{k}((t))^{\mathrm{LE}}$ as a proper subfield of the series field $\boldsymbol{k}((G^{\mathrm{LE}}))$. Couldn't we have constructed a similar logarithmic-exponential ordered field as a full series field $\boldsymbol{k}((G))$? The answer is negative: Kuhlmann, Kuhlmann and Shelah [16] showed that the ordered additive group of $\boldsymbol{k}((G))$ is not isomorphic to its positive multiplicative group when $G \neq \{1\}$.

On the other hand, Kuhlmann and Kuhlmann [15] construct logarithmic-exponential ordered fields as countable unions of fields of generalized series that are closed under a natural logarithmic operation at each stage. We have not considered in detail how their field of "exponential-logarithmic" series is related to our field of LE-series. While the construction in [15] may have its virtues, the fact that our series are of the form $\Phi^{-n}(f)$ $(= f(l_n))$, where $f = f(x)$ is an *exponential* series, will be of great use to us in later sections. This byproduct of the construction routinely reduces problems to the case of exponential series.

Ressayre [R] proved that every model of the theory of the real exponential field is isomorphic to a "truncation closed" subfield of a generalized series field over $\mathbf{R}$ with the usual exponentiation on infinitesimals. This fact can be used to give alternative proofs for some results in [6] that depend there on LE-series.

Recently, see (7.23), we became aware of Van der Hoeven's thesis, which contains also various constructions of logarithmic-exponential series fields.

§3 The derivation on $\boldsymbol{k}((t))^{\mathrm{LE}}$

(3.1) We start by defining inductively a derivation $f \mapsto f'$ on $\boldsymbol{k}((t))^{\mathrm{E}}$, to be thought of as "differentiation with respect to x".

For $f = \sum a_r x^r$ in K_0 we set $f' = \sum r a_r x^{r-1}$. Then the map $f \mapsto f' : K_0 \to K_0$ is easily seen to be a derivation. Assume we have defined already the derivation $f \mapsto f' : K_n \to K_n$. We then extend this map to K_{n+1} as follows: for $f = \sum_{a \in A_n} f_a \,\mathrm{E}(a)$ in K_{n+1} we set

$$f' = \sum_{a \in A_n} (f'_a + f_a a')\,\mathrm{E}(a).$$

It is easy to check that then the map $f \mapsto f' : K_{n+1} \to K_{n+1}$ is again a derivation.

This inductive definition provides us with a derivation $f \mapsto f'$ on $\boldsymbol{k}((t))^{\mathrm{E}}$, as promised. It has the following strong additivity property.

Lemma 3.2. *Suppose $\sum_{i \in I} f_i$ exists in K_n. Then $\sum f'_i$ also exists in K_n and $(\sum f_i)' = \sum f'_i$.*

The (inductive) proof is entirely routine and left to the reader.

For $F \in \boldsymbol{k}[[X_1, \ldots, X_m]]$ we let $\frac{\partial F}{\partial X_i}$ denote the formal partial derivative of F with respect to X_i.

Corollary 3.3. *Let $F \in \boldsymbol{k}[[X_1, \ldots, X_m]]$ and let $\epsilon_1, \ldots, \epsilon_m$ be infinitesimals of $\boldsymbol{k}((t))^{\mathrm{E}}$. Put $\epsilon := (\epsilon_1, \ldots, \epsilon_m)$. Then*

$$F(\epsilon)' = \sum_{i=1}^{m} \frac{\partial F}{\partial X_i}(\epsilon)\epsilon'_i.$$

Proof. Write $F = \sum_{\omega \in \mathbf{N}^m} a_\omega X^\omega$, and suppose all $\epsilon_i \in K_n$. Then $F(\epsilon) = \sum a_\omega \epsilon^\omega$ in K_n, and the desired result is almost immediate from the previous lemma.

Corollary 3.4. *If $f \in \boldsymbol{k}((t))^{\mathrm{E}}$, then $\mathrm{E}(f)' = f'\,\mathrm{E}(f)$.*

Proof. From the definition of the derivation we see that $\mathrm{E}(a)' = a'\,\mathrm{E}(a)$for $a \in A_n$. An easy induction shows that this is also true for $a \in A_n \oplus \cdots \oplus A_0$.

Suppose $f \in K_n$. By 2.2 i) we have $f = a + r + \epsilon$ where $a \in A_n \oplus \cdots \oplus A_0$, $r \in \boldsymbol{k}$ and $\epsilon \in K_n$ is infinitesimal. Then

$$\mathrm{E}(f) = \mathrm{E}(a)\exp(r)\sum_{i=0}^{\infty} \frac{\epsilon^i}{i!}.$$

By 3.3 we have $(\sum \frac{\epsilon^i}{i!})' = \epsilon' \sum \frac{\epsilon^i}{i!}$. Thus

$$\begin{aligned}
\mathrm{E}(f)' &= a'\,\mathrm{E}(a)\exp(r)\sum \frac{\epsilon^i}{i!} + \mathrm{E}(a)\exp(r)\epsilon' \sum \frac{\epsilon^i}{i!} \\
&= (a' + \epsilon')\,\mathrm{E}(f) \\
&= f'\,\mathrm{E}(f).
\end{aligned}$$

Next we show that the field of constants of our derivation is exactly $\boldsymbol{k}$. To obtain this fact we have to prove a little more.

Lemma 3.5. *Let $f \in K_n \setminus \{0\}$ have constant term 0, and let $g \in A_n \setminus \{0\}$. Then:*

(i) $f' \neq 0$,

(ii) $\mathrm{Lm}\,(\frac{f'}{f}) = x^{-1}$ *if* $n = 0$, *and* $\mathrm{Lm}\,(\frac{f'}{f}) \in G_{n-1}$ *if* $n > 0$;

(iii) $g' \in A_n \setminus \{0\}$ *if* $n > 0$.

Proof. By induction on n. For $n = 0$ we have $\mathrm{Lm}\, f = x^r$ for some non-zero $r \in \boldsymbol{k}$, hence $\mathrm{Lm}\, f' = x^{r-1}$, so that $f' \neq 0$. This proves the case $n = 0$. Assume the lemma holds for a certain value of n, and let $f \in K_{n+1} \setminus \{0\}$ have constant term 0. Write $f = \sum_{a \in A_n} f_a \,\mathrm{E}(a)$ in K_{n+1}. Put $\alpha := \max\{a \in A_n : f_a \neq 0\}$. To show that (i) holds for our f we distinguish several cases.

Case $\alpha = 0$. Then

$$f = f_0 + \sum_{a \in A_n, a<0} f_a \,\mathrm{E}(a)$$

where $f_0 \in K_n \setminus \{0\}$ has constant term 0 and

$$f' = f_0' + \sum_{a \in A_n, a<0} (f_a' + f_a a') \,\mathrm{E}(a).$$

By the inductive assumption $f_0' \neq 0$, hence $f' \neq 0$.

Case $\alpha \neq 0$. Then

$$f' = (f_\alpha' + f_\alpha \alpha') \,\mathrm{E}(\alpha) + \sum_{a \in A_n, a<\alpha} (f_a' + f_a a') \,\mathrm{E}(a).$$

We claim that $f_\alpha' + f_\alpha \alpha' \neq 0$, from which it follows in particular that $f' \neq 0$.

Subcase $f_\alpha' = 0$. Then by the inductive assumption $f_\alpha \in \boldsymbol{k}$ and $\alpha' \neq 0$, hence $f_\alpha' + f_\alpha \alpha' = f_\alpha \alpha' \neq 0$.

Subcase $f_\alpha' \neq 0$. Then we write $f_\alpha = \phi + c$ where $c \in \boldsymbol{k}$ and $\phi \in K_n \setminus \{0\}$ has constant term 0. Then

$$f_\alpha' + f_\alpha \alpha' = \phi' + (\phi + c)\alpha' = \phi\Big(\frac{\phi'}{\phi} + \alpha'\Big(\frac{\phi + c}{\phi}\Big)\Big).$$

Clearly

$$\begin{aligned} \mathrm{Lm}\,\Big(\frac{\phi + c}{\phi}\Big) &= 1 \text{ if } \mathrm{Lm}\,\phi > 1 \text{ or c} = 0 \\ &> 1 \text{ if } \mathrm{Lm}\,\phi < 1 \text{ and } c \neq 0. \end{aligned}$$

Hence by the inductive assumption applied to ϕ and α:

$$\mathrm{Lm}\,\Big(\frac{\phi'}{\phi}\Big) < \mathrm{Lm}(\alpha') \leq \mathrm{Lm}\,\Big(\alpha'\Big(\frac{\phi + c}{\phi}\Big)\Big),$$

and thus $f'_\alpha + f_\alpha \alpha' \neq 0$.

Next we prove (ii) above for our f (with n replaced by $n+1$). If $\alpha = 0$, then

$$\mathrm{Lm}\Big(\frac{f'}{f}\Big) = \mathrm{Lm}\Big(\frac{f'_0}{f_0}\Big) \in G_{n-1} \subseteq G_n$$

by the inductive assumption, while if $\alpha \neq 0$, then

$$\mathrm{Lm}\Big(\frac{f'}{f}\Big) = \mathrm{Lm}\Big(\frac{f'_\alpha + f_\alpha \alpha'}{f_\alpha}\Big) \in G_n.$$

Next we prove (iii) (with n replaced by $n+1$) for $g \in A_{n+1} \setminus \{0\}$. Note that g has constant term 0, so that $g' \neq 0$. Write $g = \sum_{0<a\in A_n} g_a \, \mathrm{E}(a)$ in K_{n+1}. Then $g' = \sum_{0<a\in A_n} (g'_a + g_a a') \, \mathrm{E}(a)$, and thus $g' \in A_{n+1}$.

Remark. Suppose that $f \in K_{n+1} \setminus \{0\}$ has constant term 0 and that $\alpha \neq 0$, where α is defined as in the proof of 3.5. For later use we note that then $\mathrm{Lt}(f'_\alpha + f_\alpha \alpha') = \mathrm{Lt}(f_\alpha \alpha')$ (with the notations of the proof of 3.5) and thus

$$\mathrm{Lt}\, f = \mathrm{Lt}(f_\alpha \alpha')\, \mathrm{E}(\alpha)\ .$$

This is clear if $f'_\alpha = 0$. If $f'_\alpha \neq 0$, then $f_\alpha = \phi + c$ as in the corresponding subcase in the proof of 3.5, where it is shown that then $\mathrm{Lm}(\frac{\phi'}{\phi}) < \mathrm{Lm}(\frac{f_\alpha}{\phi}\alpha')$. Hence $\mathrm{Lt}(\frac{\phi'}{\phi} + \frac{f_\alpha}{\phi}\alpha') = \mathrm{Lt}(\frac{f_\alpha}{\phi}\alpha')$ and thus $\mathrm{Lt}(f'_\alpha + f_\alpha\alpha') = \mathrm{Lt}\big(\phi(\frac{\phi'}{\phi} + \frac{f_\alpha}{\phi}\alpha')\big) = \mathrm{Lt}(\phi)\,\mathrm{Lt}(\frac{f_\alpha}{\phi}\alpha') = \mathrm{Lt}(f_\alpha\alpha')$.

As an immediate consequence of 3.5 we obtain:

Corollary 3.6. *If $f \in \boldsymbol{k}((t))^{\mathrm{E}}$ and $f' = 0$, then $f \in \boldsymbol{k}$.*

Our next goal is to extend the derivation to $\boldsymbol{k}((t))^{\mathrm{LE}}$. For this we need the following lemma, where $\Phi : \boldsymbol{k}((t))^{\mathrm{E}} \to \boldsymbol{k}((t))^{\mathrm{E}}$ is as in (2.6).

Lemma 3.7. *Let $f \in \boldsymbol{k}((t))^{\mathrm{E}}$. Then*

(1) $\Phi(f)' = \Phi(f')\,\mathrm{E}(x)$,
(2) $\Phi^m(f)' = \Phi^m(f') \prod_{i=1}^m \mathrm{E}^i(x)$.

Proof. Identity (1) is an absolutely straigtforward induction on n, where $f \in K_n$, and (2) follows by induction on m from (1).

Let D denote the derivation $f \mapsto f'$ on $\boldsymbol{k}((t))^{\mathrm{E}}$. To motivate how we extend D to $\boldsymbol{k}((t))^{\mathrm{LE}}$, consider an element $f \in L_n$, and put $\widetilde{f} := \eta_n(f)$. Thinking of f as $\widetilde{f}(l_n)$, the chain rule suggests that we should have

$$f' := (D\widetilde{f})(l_n) \cdot l'_n = \frac{(D\widetilde{f})(l_n)}{\prod_{i=0}^{n-1} l_i} = \frac{\eta_n^{-1}(D\eta_n(f))}{\prod_{i=0}^{n-1} l_i}.$$

Thus we introduce the derivation

$$D_n := \frac{1}{\prod_{i=0}^{n-1} l_i}\left(\eta_n^{-1} \circ D \circ \eta_n\right)$$

on L_n: this is indeed a derivation on L_n, since the conjugate $\eta_n^{-1} \circ D \circ \eta_n$ of D is one. Note that $D_0 = D$. We now verify that D_{n+1} extends D_n. Let $f \in L_n$. Then

$$\begin{aligned} D_{n+1}f &= \frac{1}{\prod_{i=0}^{n} l_i}\eta_{n+1}^{-1} D\eta_{n+1}(f) = \frac{1}{\prod_{i=0}^{n} l_i}\eta_{n+1}^{-1} D(\Phi\eta_n(f)) \\ &= \frac{1}{\prod_{i=0}^{n} l_i}\eta_{n+1}^{-1}\left(\Phi D\eta_n(f) \cdot \mathrm{E}(x)\right) \text{ (by 3.7(1))} \\ &= \frac{1}{\prod_{i=0}^{n} l_i}\eta_n^{-1}(D\eta_n(f)) \cdot l_n = \frac{1}{\prod_{i=0}^{n-1} l_i}\eta_n^{-1}(D\eta_n(f)) \\ &= D_n f. \end{aligned}$$

The desired extension of D to $\boldsymbol{k}((t))^{\mathrm{LE}}$ is the derivation $\bigcup_n D_n$ on $\boldsymbol{k}((t))^{\mathrm{LE}}$, which we also denote by $\frac{d}{dx}$. For $f \in \boldsymbol{k}((t))^{\mathrm{LE}}$ we often write f' instead of $\frac{df}{dx}$. We summarize some basic facts about this derivation in the following theorem.

Theorem 3.9. *Let $f \in \boldsymbol{k}((t))^{\mathrm{LE}}$ and $f_i \in \boldsymbol{k}((t))^{\mathrm{LE}}$ for all $i \in I$. Then:*

1. *If $f = \sum_{i \in I} f_i$ in $\boldsymbol{k}((t))^{\mathrm{LE}}$, then also $f' = \sum f_i'$ in $\boldsymbol{k}((t))^{\mathrm{LE}}$.*
2. *Let $F \in \boldsymbol{k}[[X_1, \dots, X_m]]$ and let $\epsilon_1, \dots, \epsilon_m$ be infinitesimals of $\boldsymbol{k}((t))^{\mathrm{LE}}$. Put $\epsilon := (\epsilon_1, \dots, \epsilon_m)$. Then*

$$F(\epsilon)' = \sum_{i=1}^{m} \frac{\partial F}{\partial X_i}(\epsilon)\epsilon_i'.$$

3. *$\mathrm{E}(f)' = f'\,\mathrm{E}(f)$, and $(\log|f|)' = f'/f$ for nonzero f.*
4. *If $f' = 0$, then $f \in \boldsymbol{k}$.*
5. *$\Phi^m(f)' = \Phi^m(f') \prod_{i=1}^{m} \mathrm{E}^i(x)$.*

Proof. These results follow easily from 3.2, 3.3, 3.4, 3.6, 3.7 and 3.8. To give an example of how this goes, let us derive the first part of (3). Take n such that $f \in L_n$ and put $a := \frac{1}{\prod_{i=0}^{n-1} l_i}$. Then also $\mathrm{E}(f) \in L_n$, hence

$$\begin{aligned} \mathrm{E}(f)' &= a\eta_n^{-1}((\eta_n \mathrm{E}(f))') = a\eta_n^{-1}(\mathrm{E}(\eta_n f)') \\ &= a\eta_n^{-1}(\mathrm{E}(\eta_n f)(\eta_n f)') \text{ (by 3.4)} \\ &= a\,\mathrm{E}(f)\eta_n^{-1}(\eta_n f)' \text{ (since } \eta_n^{-1} \text{ commutes with E)} \\ &= f'\,\mathrm{E}(f). \end{aligned}$$

Let us also derive (5) for $m = 1$, using the same notations :

$$\begin{aligned}
\Phi(f)' &= a\eta_n^{-1}((\eta_n \Phi f)') = a\eta_n^{-1}(\Phi(\eta_n f)') \\
&= a\eta_n^{-1}(\Phi((\eta_n f)')\,\mathrm{E}(x)) \text{ (by 3.7)} \\
&= a\Phi(\eta_n^{-1}(\eta_n f)')\frac{1}{l_{n-1}} \text{ (since } \eta_n^{-1} \text{ commutes with } \Phi) \\
&= a\Phi(a^{-1}f')\frac{1}{l_{n-1}} = \Phi(f')a\Phi(a^{-1})\frac{1}{l_{n-1}} \\
&= \Phi(f')\,\mathrm{E}(x).
\end{aligned}$$

Now an induction on m gives (5).

(3.10) Remark. "The derivation of $\boldsymbol{k}((t))^{\mathrm{LE}}$ " always refers to $\frac{d}{dx}$. There are other useful derivations on $\boldsymbol{k}((t))^{\mathrm{LE}}$. For example, $\frac{d}{dt} := -x^2\frac{d}{dx}$ is a derivation that would be natural to consider if we had taken t as the independent variable instead of x. The properties of $\frac{d}{dt}$ are of course immediate consequences of those of $\frac{d}{dx}$.

(3.11) We know from [6, Corollary 2.8] that $\mathbf{R}((t))^{\mathrm{LE}}$—more precisely, a certain natural expansion of it, also denoted by $\mathbf{R}((t))^{\mathrm{LE}}$—is an elementary extension of $\mathbf{R}_{\mathrm{an,exp}}$. Let $H(\mathbf{R}_{\mathrm{an,exp}})$ be the Hardy field of germs at $+\infty$ of the functions $f : \mathbf{R} \to \mathbf{R}$ that are definable in $\mathbf{R}_{\mathrm{an,exp}}$. Then $H(\mathbf{R}_{\mathrm{an,exp}})$ is also naturally an elementary extension of $\mathbf{R}_{\mathrm{an,exp}}$, in fact, generated as such over $\mathbf{R}_{\mathrm{an,exp}}$ by the germ of the identity function, see [5, §5]. Thus there is a unique elementary $\mathcal{L}_{\mathrm{an,exp}}$-embedding of $H(\mathbf{R}_{\mathrm{an,exp}})$ into $\mathbf{R}((t))^{\mathrm{LE}}$ that sends the germ of the identity function to x. We shall call this the natural embedding. Note that, as a Hardy field, $H(\mathbf{R}_{\mathrm{an,exp}})$ is in particular a differential field.

Corollary 3.12. *The natural embedding of $H(\mathbf{R}_{\mathrm{an,exp}})$ into $\mathbf{R}((t))^{\mathrm{LE}}$ is a differential field embedding.*

Proof. From [5] we know that the functions definable in $\mathbf{R}_{\mathrm{an,exp}}$ are given piecewise by terms in the language $\mathcal{L}_{\mathrm{an,exp,log}}$. The result then follows easily from 3.9 by induction on terms.

§4 Valuation Theoretic Properties of the Derivation

The canonical derivation on a Hardy field has some useful valuation theoretic properties related to L'Hospital's rule, see [22]. In this section we show that these properties also hold for $\boldsymbol{k}((t))^{\mathrm{LE}}$. We state the results in terms of the multiplicative "leading monomial" function instead of the corresponding (additive) valuation.

Proposition 4.1. *Let $f, g \in \boldsymbol{k}((t))^{\mathrm{LE}}$. Then the following hold:*

(1) *If $\operatorname{Lm} f \neq 1$ and $\operatorname{Lm} g \neq 1$, then $\operatorname{Lm} f \leq \operatorname{Lm} g$ if and only if $\operatorname{Lm} f' \leq \operatorname{Lm} g'$.*
(2) *If $\operatorname{Lm} f < \operatorname{Lm} g \neq 1$, then $\operatorname{Lm} f' < \operatorname{Lm} g'$.*
(3) *If $\operatorname{Lm} f \leq 1$, then $\operatorname{Lm} f' < 1$.*
(4) *If $\operatorname{Lm} f \geq 1$, then $|f'| < |f|^{1+\epsilon}$ for each $\epsilon \in \boldsymbol{k}^{>0}$.*
(5) *If $0 < \operatorname{Lm} f \leq 1$, then $|f'| < |f|^{1-\epsilon}$ for each $\epsilon \in \boldsymbol{k}^{>0}$.*

Remark. Property (1) implies that the canonical valuation on $\boldsymbol{k}((t))^{\mathrm{LE}}$ is a *differential valuation* in the sense of Rosenlicht [21 , Cor. 1], where it is also shown that (2) and (3) follow from that fact. Properties (4) and (5) are also easily derived, as in [22] for Hardy fields.

Property (5), with $\epsilon = 1/2$, say, implies that the derivation is a continuous operation on the topological field $\boldsymbol{k}((t))^{\mathrm{LE}}$.

Proof. As already mentioned it suffices to prove (1) since the other properties then follow as in [21, 22]. Let $\operatorname{Lm} f \neq 1$ and $\operatorname{Lm} g \neq 1$. We may also assume $f \neq 0$ and $g \neq 0$, since otherwise (1) is trivial. If $f, g \in K_0$ the desired result follows from $\operatorname{Lm} f' = x^{-1} \operatorname{Lm} f$, and $\operatorname{Lm} g' = x^{-1} \operatorname{Lm} g$.

Suppose (1) holds for elements of K_n and let $f, g \in K_{n+1}$. By subtracting constant terms we may as well assume that the constant terms of f and g are zero. Let $f = \sum_{a \in A_n} f_a \operatorname{E}(a)$ in K_{n+1} and put $\alpha := \max\{a \in A_n : f_a \neq 0\}$. Then $\operatorname{Lm} f = \operatorname{Lm}(f_\alpha) \operatorname{E}(\alpha)$, while, by the remark following 3.5,

$$\operatorname{Lm} f' = \begin{cases} \operatorname{Lm} f_0' & \text{if } \alpha = 0 \\ \operatorname{Lm}(f_\alpha \alpha') \operatorname{E}(\alpha) & \text{if } \alpha \neq 0. \end{cases}$$

Let $g = \sum_{a \in A_n} g_a \operatorname{E}(a)$ in K_{n+1} and put $\beta := \max\{a \in A_n : g_a \neq 0\}$. If $\alpha \neq \beta$, then $\operatorname{Lm} f \leq \operatorname{Lm} g$ if and only if $\alpha < \beta$ if and only if $\operatorname{Lm} f' \leq \operatorname{Lm} g'$.

If $\alpha = \beta = 0$, then $\operatorname{Lm} f_0 \neq 1$ and $\operatorname{Lm} g_0 \neq 1$. By induction $\operatorname{Lm} f_0 \leq \operatorname{Lm} g_0$ if and only if $\operatorname{Lm} f_0') \leq \operatorname{Lm} g_0'$. Thus $\operatorname{Lm} f \leq \operatorname{Lm} g$ if and only if $\operatorname{Lm} f' \leq \operatorname{Lm} g'$. If $\alpha = \beta \neq 0$, then $\operatorname{Lm} f \leq \operatorname{Lm} g$ if and only if $\operatorname{Lm} f_\alpha \leq \operatorname{Lm} g_\alpha$ if and only if $\operatorname{Lm} f' \leq \operatorname{Lm} g'$.

This proves (1) for $f, g \in \boldsymbol{k}((t))^{\mathrm{E}}$. The general case easily reduces to this special case.

(4.2) As an application of this result we show that the differential equation $y'' + y = 0$ has no nontrivial solutions in $\boldsymbol{k}((t))^{\mathrm{LE}}$.

Suppose $f \in \boldsymbol{k}((t))^{\mathrm{LE}} \setminus \{0\}$ and $f'' + f = 0$. Then

$$((f')^2 + f^2)' = 2f'f'' + 2f'f = 0.$$

Thus, by 3.9 , $(f')^2 + f^2 = c$ for some $c \in \boldsymbol{k}^{>0}$. If $\mathrm{Lm}\, f > 1$, then f is infinite so $(f')^2 + f^2 \neq c$, a contradiction. If $\mathrm{Lm}\, f < 1$, then, by 4.1, $\mathrm{Lm}(f') < 1$, so $\mathrm{Lm}((f')^2 + f^2) < 1$, contradicting $c \neq 0$. Thus $\mathrm{Lm}(f) = 1$. Then 4.1 gives $\mathrm{Lm}(f'') < 1$, so that $f'' + f \neq 0$, again a contradiction.

Adjoining $i = \sqrt{-1}$ to $\boldsymbol{k}((t))^{\mathrm{LE}}$ (and extending the derivation) doesn't change matters since if $f + gi$ were a solution of $y'' + y = 0$ (with $f, g \in \boldsymbol{k}((t))^{\mathrm{LE}}$), then f and g would also be solutions.

The next result relates the derivation and the ordering.

Proposition 4.3. *Suppose $f \in \boldsymbol{k}((t))^{\mathrm{LE}}$ and $f > \boldsymbol{k}$. Then $f' > 0$.*

Proof. We easily reduce to the case that $f \in \boldsymbol{k}((t))^{\mathrm{E}}$. The case $f \in K_0$ is immediate. Assuming the result holds for all $f \in K_n$, let $f \in K_{n+1}$. After subtracting from f its constant term we may assume f has constant term 0. Write f as in the proof of 4.1 and define α as in that proof. Then by the remark following 3.5

$$\mathrm{Lt}\, f' = \begin{cases} \mathrm{Lt}\, f_0' & \text{if } \alpha = 0 \\ \mathrm{Lt}(f_\alpha \alpha')\, \mathrm{E}(\alpha) & \text{if } \alpha \neq 0. \end{cases}$$

Since $f_\alpha \geq 0$ and $\alpha \geq 0$ we can apply the inductive hypothesis to give the desired result.

Corollary 4.4. *Suppose $f \in \boldsymbol{k}((t))^{\mathrm{LE}}$ and $f > x^{\boldsymbol{k}}$. Then $f' > f^{1-\epsilon}$ for each $\epsilon \in \boldsymbol{k}^{>0}$.*

Proof. The hypothesis gives $\log f > \boldsymbol{k} \log x$, which implies $\log f - \log x > \boldsymbol{k}$. Hence $f'/f > 1/x$ by 4.3. Also $1/x > f^{-\epsilon}$ for $\epsilon \in \boldsymbol{k}^{>0}$. Thus $f'/f > f^{-\epsilon}$, that is, $f' > f^{1-\epsilon}$.

Note that 4.1 (4) and 4.4 together say that the derivative of any $f > x^{\boldsymbol{k}}$ is close to f in a certain "relative to f" sense.

Many analogues of results about Hardy fields can be deduced without any difficulty, by copying (with minor modifications) Rosenlicht's proofs in [21–24]. Here is one such result. For $f \in \boldsymbol{k}((t))^{\mathrm{LE}}$ with $f > \boldsymbol{k}$, let the comparability class $\mathrm{Cl}(f)$ of f be the set of all $\phi \in \boldsymbol{k}((t))^{\mathrm{LE}}$ for which there exist positive $r, s \in \boldsymbol{k}$ such that $f^r \leq \phi \leq f^s$. For $f, g \in \boldsymbol{k}((t))^{\mathrm{LE}}$ with $f, g > \boldsymbol{k}$ we set $\mathrm{Cl}(f) < \mathrm{Cl}(g)$ if $\phi < \chi$ for all $\phi \in \mathrm{Cl}(f)$ and for all $\chi \in \mathrm{Cl}(g)$. This defines a linear order on the set of comparability classes.

Proposition 4.5. *Suppose $f, g \in \boldsymbol{k}((t))^{\mathrm{LE}}$ and $f, g > \boldsymbol{k}$. Then $\mathrm{Cl}(f) \leq \mathrm{Cl}(g)$ if and only if $\mathrm{Lm}(f'/f) \leq \mathrm{Lm}(g'/g)$ if and only if $\mathrm{Lm}(\log f) \leq \mathrm{Lm}(\log g)$.*

§5 Integration

In this section we will prove that we can integrate in $\boldsymbol{k}((t))^{\mathrm{LE}}$; that is, given $g \in \boldsymbol{k}((t))^{\mathrm{LE}}$ we can find $f \in \boldsymbol{k}((t))^{\mathrm{LE}}$ with $f' = g$. We call such an f an *integral* of g. Indeed we will show that each series in L_n has an integral in L_{n+1}. We begin by considering $g \in \boldsymbol{k}((t))^{\mathrm{E}}$ and showing that there is only one obstruction to finding an integral in $\boldsymbol{k}((t))^{\mathrm{E}}$.

(5.1) The *residue* of a series $g \in \boldsymbol{k}((t))^{\mathrm{E}}$ is by definition the coefficient of the monomial x^{-1} in g, and is denoted by $\operatorname{res} g$.

Proposition 5.2. *Let $g \in \boldsymbol{k}((t))^{\mathrm{E}}$. Then g has an integral in $\boldsymbol{k}((t))^{\mathrm{E}}$ if and only if* $\operatorname{res} g = 0$.

Remark. The residue obstruction is natural since if $f \in \boldsymbol{k}((t))^{\mathrm{LE}}$ and $f' = x^{-1}$, then $f = \log x + r$ for some $r \in \boldsymbol{k}$, and $\log x$ does not belong to $\boldsymbol{k}((t))^{\mathrm{E}}$.

(5.3) Towards the proof of 5.2, let $g = \sum a_r x^r \in K_0$ and $\operatorname{res} g = a_{-1} = 0$. Then clearly

$$f := \sum_{r \neq -1} \frac{a_r}{r+1} x^{r+1}$$

satisfies $f' = g$. Next, assume inductively that each element of K_n with residue 0 has an integral in K_n. Let

$$g := \sum_{a \in A_n} g_a \operatorname{E}(a)$$

in K_{n+1}. Then $f = \sum f_a \operatorname{E}(a)$ in K_{n+1} is an integral of g if and only if $f_a' + f_a a' = g_a$ for all $a \in A_n$. Thus if g has an integral in K_{n+1}, then the equation $y' + ya' = g_a$ is solvable in K_n, for each $a \in A_n$ such that $g_a \neq 0$. Conversely, if each such equation has a solution $f_a \in K_n$, then $f := \sum_{g_a \neq 0} f_a \operatorname{E}(a) \in K_{n+1}$ is an integral of g. For $a = 0$ the equation $y' + ya' = g_a$ reads $y' = g_a$, and has a solution in K_n by the inductive assumption. Suppose $a \in A_n$ and $a \neq 0$. Then the equation $y' + ya' = g_a$ has the same solutions in K_n as the equation $y - \mu y' = h$, where $\mu := (-a')^{-1}$ and $h := (a')^{-1} g_a$. Note that $\mu, h \in K_n$, and that if $n = 0$, then $\operatorname{Lm} \mu < x$ while if $n > 0$, then μ is infinitesimal with respect to K_{n-1} by 3.5(iii). Therefore the "if" part of 5.2 is a consequence of the following lemma.

Lemma 5.4. *Let $h, \mu \in K_n$, with $\operatorname{Lm} \mu < x$ if $n = 0$, and μ infinitesimal with respect to K_{n-1} if $n > 0$. Then the equation $y - \mu y' = h$ has a solution in K_n.*

Proof. Let P be the operator on K_n defined by $\mathrm{P}\, y = \mu y'$, and let I be the identity operator on K_n. Then the equation $y - \mu y' = h$ (to be solved for $y \in K_n$) becomes $(\mathrm{I} - \mathrm{P})(y) = h$. For $n = 0$ we view K_0 as the series field $\boldsymbol{k}((G_0))$, and we note that then P is a small operator in the sense of 1.3, since

$$\operatorname{Supp}\ \mathrm{P}\, y \subseteq (\operatorname{Supp}\ \mu) \cdot (\operatorname{Supp}\ y') \subseteq (x^{-1} \operatorname{Supp}\ \mu) \cdot (\operatorname{Supp}\ y)$$

for all $y \in K_0$. Thus, by 1.3, the equation $(\mathrm{I} - \mathrm{P})(y) = h$ has the solution $y = (\mathrm{I} - \mathrm{P})^{-1}(h) = \sum \mathrm{P}^i(h)$ in K_0. Next, suppose $n > 0$. Let $\operatorname{Supp}^* y$ denote the support of an element $y \in K_n$, where we consider K_n as the series field

$K_{n-1}((\mathrm{E}(A_{n-1})))$ over K_{n-1}. Thus $\mathrm{Supp}^*\ y \subseteq \mathrm{E}(A_{n-1})$ and $\mathrm{Supp}^*\ y' \subseteq \mathrm{Supp}^*\ y$ for $y \in K_n$. When K_n is viewed in this way as a series field over K_{n-1}, then P is again a small operator, because

$$\mathrm{Supp}^*\ \mathrm{P}\, y \subseteq (\mathrm{Supp}^*\ \mu)\cdot(\mathrm{Supp}^*\ y)$$

for all $y \in K_n$. The desired result follows again from 1.3.

(5.5) We have now established the “if” direction of 5.2. For the “only if” direction, suppose $g \in \boldsymbol{k}((t))^{\mathrm{E}}$ has an integral in $\boldsymbol{k}((t))^{\mathrm{E}}$. Let $c := \operatorname{res} g$. Then $g - cx^{-1}$ has residue 0, hence has an integral in $\boldsymbol{k}((t))^{\mathrm{E}}$ by the "if" part of 5.2. Thus $g - (g - cx^{-1}) = cx^{-1}$ has an integral in $\boldsymbol{k}((t))^{\mathrm{E}}$, which implies $c = 0$ by the remark following 5.2. Therefore $\operatorname{res} g = 0$, as desired.

Theorem 5.6. *Each $g \in \boldsymbol{k}((t))^{\mathrm{LE}}$ has an integral.*

Proof. By 5.2 each $g \in \boldsymbol{k}((t))^{\mathrm{E}} = L_0$ has an integral in L_1. Suppose now that $g \in L_m$. Then we have for $f \in \boldsymbol{k}((t))^{\mathrm{LE}}$:

$$\begin{aligned} f' = g &\Longleftrightarrow \Phi^m(f') = \Phi^m(g) \\ &\Longleftrightarrow \Phi^m(f)' = \Phi^m(g)\prod_{i=1}^{m}\mathrm{E}^i(x) \ \text{(by 3.9)}\ . \end{aligned}$$

Since $\Phi^m(g)\prod_{i=1}^m \mathrm{E}^i(x) \in \boldsymbol{k}((t))^{\mathrm{E}}$ there is $y \in L_1$ such that $y' = \Phi^m(g)\prod_{i=1}^m \mathrm{E}^i(x)$. Then $f := \Phi^{-m}(y)$ is an integral of g and belongs to K_{m+1}.

Since we can take exponentials and integrals we can solve first order linear differential equations.

Corollary 5.7. *If $f, g \in \boldsymbol{k}((t))^{\mathrm{LE}}$, then there is $y \in \boldsymbol{k}((t))^{\mathrm{LE}}$ such that $y' + fy = g$, and in the homogeneous case ($g = 0$) we can take $y \neq 0$.*

Proof. Let $h_0, h_1 \in \boldsymbol{k}((t))^{\mathrm{LE}}$ such that $h_0' = f$ and $h_1' = \mathrm{E}(h_0)g$. Then $y = h_1/E(h_0)$ is a solution. If $g = 0$ we can take $h_1 = 1$ which gives a non-trivial solution of the homogeneous equation.

Second order linear differential equations

(5.8) Corollary 5.7 contains all there is to say about first order linear ODE's in one unknown over $\boldsymbol{k}((t))^{\mathrm{LE}}$. It would be nice to extend this to linear ODE's in one unknown of any order. By remark (4.2) the second order linear equation $y'' + y = 0$ has no non-trivial solutions in $\boldsymbol{k}((t))^{\mathrm{LE}}$. The following result completely settles the question which homogeneous second order linear ODE's in one unknown have non-trivial solutions in $\boldsymbol{k}((t))^{\mathrm{LE}}$. (This is because such an equation $u'' + \alpha u' + \beta u = 0$ ($\alpha, \beta \in \boldsymbol{k}((t))^{\mathrm{LE}}$) transforms into an equation of the form $y'' + fy = 0$ by the change of variables $u := \gamma y$ where $0 \neq \gamma \in \boldsymbol{k}((t))^{\mathrm{LE}}$ satisfies $2\gamma' + \alpha\gamma = 0$.)

Theorem 5.9. *Let $f \in \boldsymbol{k}((t))^{\mathrm{LE}}$. The equation $y'' + fy = 0$ has a non-trivial solution in $\boldsymbol{k}((t))^{\mathrm{LE}}$ if and only if*

$$f < \frac{1}{4x^2} + \frac{1}{4x^2(\log x)^2} + \frac{1}{4x^2(\log x)^2(\log\log x)^2} + \cdots + \frac{1}{4x^2(\log x)^2\cdots(\underbrace{\log\ldots\log}_{n\text{-times}}(x))^2}$$

for some n.

(5.10) Our present proof of 5.9 seems too long to us. That is why we do not present it here, but only give a few indications below. (Details are available from the authors upon request.) The topic of solvability of ODE's in $\mathbf{R}((t))^{\mathrm{LE}}$ is a large one, and we expect to return to it in a later article with a more definitive treatment and more general results.

In our present proof of 5.9 we construct solutions explicitly using an extension of the "method of undetermined coefficients". Proofs that equations have no nontrivial solutions use the valuation theoretic methods as in 4.2.

The precise cut in the ordered differential field $\boldsymbol{k}((t))^{\mathrm{LE}}$ that occurs in 5.9 has to do with the fact that each positive infinite element of $\boldsymbol{k}((t))^{\mathrm{LE}}$ is bounded below by $\underbrace{\log\ldots\log}_{n\text{-times}}(x)$ for some n. There is evidence that the solvability of differential equations in $\mathbf{R}((t))^{\mathrm{LE}}$ reflects solvability in Hardy fields (of germs at $+\infty$) of finite rank in the sense of [23]. (In these Hardy fields each infinitely increasing element is also bounded below by some iterate $\underbrace{\log\ldots\log}_{n\text{-times}}(x)$ of the logarithm. (In the Hardy field setting we let "x" denote the germ at $+\infty$ of the identity function.)

More generally we conjecture that $\mathbf{R}((t))^{\mathrm{LE}}$ is a kind of universal domain for ordered differential algebra in Hardy fields of finite rank. We hope to return to this important issue in a later paper. Here we only mention without proof the following exact analogue of the theorem above for such Hardy fields.

Theorem 5.11. *Let f belong to a Hardy field H of finite rank. Then the equation $y'' + fy = 0$ has a non-trivial solution in a Hardy field extension of H if and only if*

$$f < \frac{1}{4x^2} + \frac{1}{4x^2(\log x)^2} + \frac{1}{4x^2(\log x)^2(\log\log x)^2} + \cdots + \frac{1}{4x^2(\log x)^2\cdots(\underbrace{\log\ldots\log}_{n\text{-times}}(x))^2}$$

for some n.

This theorem is closely related to the discussion in [7, pp. 78-80] and also to [25, Theorem 3, part (3)]. (The main point is that a "nonoscillating" solution y of $y'' + fy = 0$ in the sense of [7] gives rise to a solution $z := -y'/y$ defined on some interval $(a, +\infty)$ of the corresponding 1st order Riccati equation $z' - z^2 = f$. If the germ of f at $+\infty$ lies in a Hardy field, then the germ at $+\infty$ of such a solution of the Riccati equation lies in a bigger Hardy field.)

§6 Composition and Inversion

(6.1) The goal of this section is to introduce the (formal) composition $f \circ g$ of two series $f, g \in \boldsymbol{k}((t))^{\mathrm{LE}}$ with $g > \boldsymbol{k}$, and to show that this operation has the expected properties. This is an elaborate affair. At the end of this section we comment on how this section relates to similar material in Écalle [8]. In order to state the main results we now introduce some conventions and notations. We put

$$\boldsymbol{k}((t))^{\mathrm{LE}}_{\infty} := \{g \in \boldsymbol{k}((t))^{\mathrm{LE}} : g > \boldsymbol{k}\}.$$

Throughout this section f, g and h range over $\boldsymbol{k}((t))^{\mathrm{LE}}$, with $g > \boldsymbol{k}$ and $h > \boldsymbol{k}$. Further restrictions on f, g and h may be imposed in some particular context.

Theorem 6.2. *There is a unique operation*

$$(f, g) \mapsto f \circ g : \boldsymbol{k}((t))^{\mathrm{LE}} \times \boldsymbol{k}((t))^{\mathrm{LE}}_{\infty} \longrightarrow \boldsymbol{k}((t))^{\mathrm{LE}}$$

such that:

(1) $r \circ g = r$ *for all* $r \in \boldsymbol{k}$,
(2) $f \circ x = f$ *and* $x \circ g = g$,
(3) *the map* $\phi \mapsto \phi \circ g : \boldsymbol{k}((t))^{\mathrm{LE}} \to \boldsymbol{k}((t))^{\mathrm{LE}}$ *is an ordered field embedding,*
(4) *if* $f = \sum f_i$ *in* $\boldsymbol{k}((t))^{\mathrm{LE}}$, *then* $f \circ g = \sum f_i \circ g$ *in* $\boldsymbol{k}((t))^{\mathrm{LE}}$.
(5) $\mathrm{E}(f) \circ g = \mathrm{E}(f \circ g)$ *and* $\Phi(f) = f \circ \mathrm{E}(x)$,
(6) $(f \circ g) \circ h = f \circ (g \circ h)$.

We also establish the chain rule and the existence of compositional inverses:

Proposition 6.3. *Let $\circ$ be the composition operation of the theorem. Then:*

$$(f \circ g)' = (f' \circ g) \cdot g' \ .$$

Moreover, $\boldsymbol{k}((t))^{\mathrm{LE}}_{\infty}$ is a group under the operation $\circ$, with identity element x.

Before we define composition we make a few simple observations:

Lemma 6.4. *Let $\circ$ be a composition operation as in the theorem (without assuming uniqueness). Then:*

(1) $(\log f) \circ g = \log(f \circ g)$ *for* $f > 0$,
(2) $x^r \circ g = g^r$,
(3) $\Phi^m(f) \circ g = f \circ \mathrm{E}^m\, g$ *and* $\Phi^{-m}(f) \circ g = f \circ L^m g$.

Proof. If $f > 0$ we have by (5) of the Theorem:

$$\mathrm{E}((\log f) \circ g) = \mathrm{E}(\log f) \circ g = f \circ g = \mathrm{E}(\log(f \circ g)),$$

hence $(\log f) \circ g = \log(f \circ g)$, which is (1). Thus by the Theorem:

$$x^r \circ g = \mathrm{E}(r \log x) \circ g = \mathrm{E}((r \log x) \circ g)) = \mathrm{E}(r \log g) = g^r,$$

which is (2). Now the first identity of (3) follows easily by associativity from the second identity of (5) in the Theorem. To obtain the second identity of (3) note that

$$\Phi^m(\Phi^{-m}(f)) = f = (f \circ \mathrm{L}^m x) \circ \mathrm{E}^m x = \Phi^m(f \circ \mathrm{L}^m x),$$

by (1), associativity, and the first identity. Hence $\Phi^{-m}(f) = f \circ \mathrm{L}^m x$, and thus

$$\Phi^{-m}(f) \circ g = (f \circ \mathrm{L}^m x) \circ g = f \circ \mathrm{L}^m g,$$

again by (1) and associativity.

(6.5) The theorem and the lemma suggest an inductive scheme for defining composition. The three key inductive formulas are as follows.

$$\text{If } f = \sum a_r x^r \text{ in } K_0, \text{ then } f \circ g = \sum a_r g^r \ .$$

$$\text{If } f = \sum f_a \,\mathrm{E}(a) \text{ in } K_{m+1}, \text{ then } f \circ g = \sum (f_a \circ g)\,\mathrm{E}(a \circ g) \ .$$

$$\text{If } f = \Phi^{-m}(\widetilde{f}) \text{ with } \widetilde{f} \in \boldsymbol{k}((t))^E, \text{ then } f \circ g = \widetilde{f} \circ \mathrm{L}^m g \ .$$

It is easy to show that the first infinite sum $\sum a_r g^r$ exists. The difficulty in showing existence of the second infinite sum on the right is to control Supp $(f \circ g)$ in terms of Supp f and Supp g, during the induction. The next two (easy) technical lemmas will be useful in this regard. Notation (also used later in this section): for any subset K of $\boldsymbol{k}((t))^{\mathrm{LE}} \subseteq \boldsymbol{k}((G^{\mathrm{LE}}))$ we put

$$\text{Supp } K := \bigcup_{s \in K} \text{Supp } s \ .$$

Lemma 6.6. *Let K be a subfield of $L_{M,N} = \boldsymbol{k}((G_{M,N}))$ with $\boldsymbol{k} \subseteq K$, and let $\mathcal{A}$ be a $\boldsymbol{k}$-linear subspace of K such that $\mathrm{E}(\alpha) > K$ for all $\alpha \in \mathcal{A}^{>0}$. (Hence $\alpha > \boldsymbol{k}$ for all $\alpha \in \mathcal{A}^{>0}$.)*

Then there is an injective homomorphism $\alpha \mapsto u_\alpha$ from the additive group $\mathcal{A}$ into the multiplicative group $G_{M,N+1}$ such that for all $\alpha \in \mathcal{A}$

(i) $\mathrm{E}(\alpha) = u_\alpha \cdot y_\alpha$ *with* $y_\alpha \in L_{M,N}$, *and Supp* $y_\alpha \subseteq [(\mathit{Supp}\ \alpha)^{\leq 1}] \subseteq \langle \mathit{Supp}\ K\rangle$.

(ii) $\alpha > 0 \Longrightarrow u_\alpha > K$.

Proof. Put $A_{M,i} := \Phi^{-M}(A_i)$, and $\mathfrak{m}_{M,N} := \Phi^{-M}(\mathfrak{m}_N)$ so that

$$\boldsymbol{k}((G_{M,N})) = A_{M,N} \oplus A_{M,N-1} \oplus \ldots \oplus A_{M,0} \oplus \boldsymbol{k} \oplus \mathfrak{m}_{M,N}.$$

Write each $\alpha \in \mathcal{A}$ as $\alpha = a + r + \epsilon$ with $a \in A_{M,N} \oplus A_{M,N-1} \oplus \ldots \oplus A_{M,0}$, $r \in \boldsymbol{k}$ and $\epsilon \in \mathfrak{m}_N$, and put $u_\alpha = \mathrm{E}(a)$ and $y_\alpha = \exp(r) \sum \frac{\epsilon^p}{p!}$. One easily checks that the lemma holds with these choices of u_α's and y_α's.

Lemma 6.7. *Let A be an ordered abelian (multiplicative) group, C a convex subgroup of A, and U and V reverse well ordered subsets of A such that*
(i) $(u_1 < u_2 \text{ in } U) \Longrightarrow u_2/u_1 > C$,
(ii) $v \in V \Longrightarrow v < C$.

Suppose for each $u \in U$ and $p \in \mathbf{N}$ we are given a reverse well ordered set $S_{u,p} \subseteq C$. Then the subset

$$W := \bigcup_{u \in U,\ p \in \mathbf{N}} S_{u,p} \cdot u \cdot V^p$$

of A is reverse well ordered. Moreover, for each $w \in W$ there are only finitely many tuples $(u, p, s, v_1, \dots, v_p)$ with $u \in U$, $p \in \mathbf{N}$, $s \in S_{u,p}$ and $v_1, \dots, v_p \in V$ such that

$$w = s \cdot u \cdot v_1 \cdot \ldots \cdot v_p.$$

Proof. Let $\overline{U}$, $\overline{V}$ and $\overline{W}$ be the images of U, V and W under the canonical map from A onto $\overline{A} := A/C$. Then clearly

$$\overline{W} = \overline{U} \cdot \bigcup_{p \in \mathbf{N}} \overline{V}^p = \overline{U} \cdot [\overline{V}] \,.$$

Using that $\overline{U}$ and $\overline{V}$ are reverse well ordered subsets of the ordered abelian group $\overline{A}$ and that $\overline{V} < 1$, we easily obtain the desired result from Neumann's Lemma.

(6.8) We first define the formal composition $f \circ g$ when f belongs to $\boldsymbol{k}((t))^{\mathrm{E}}$. This is done in stages, the m^{th} stage defining $f \circ g$ for all $f \in K_m$.

Instead of $f \circ g$ we will also write $f(g)$, because we think of $f \circ g$ as the result of substituting g for the indeterminate x in the series $f = f(x)$.

Stage 0. Write $f \in K_0$ as $f = \sum_{r \in \boldsymbol{k}} a_r x^r$ (all $a_r \in \boldsymbol{k}$), and put

$$f(g) := \sum a_r g^r .$$

For this definition to make sense we have to check that the right hand sum exist in $\boldsymbol{k}((t))^{\mathrm{LE}}$. Suppose $g \in L_{M,n} = \boldsymbol{k}((G_{M,n}))$, and write $g = c\mu(1+\epsilon)$ with $c = \mathrm{Lc}(g)$, $\mu = \mathrm{Lm}(g)$, and $\epsilon \in \mathfrak{m}_{M,n}$. Put $S := \mathrm{Supp}\ \epsilon$, so $S \subseteq G^{<1}_{M,n}$. Then, with $r \in \boldsymbol{k}$:

$$g^r = c^r \mu^r \sum_p \binom{r}{p} \epsilon^p$$

which gives $\mathrm{Supp}\ g^r \subseteq \mu^r[S]$. Since $\mu > 1$ and $\{r \in \boldsymbol{k} : a_r \neq 0\}$ is reverse well ordered, the set $\{\mu^r : r \in \boldsymbol{k}, a_r \neq 0\}$ is also reverse well ordered (in $G_{M,n}$). Thus by Neumann's lemma $\sum a_r g^r$ exists in $L_{M,n}$. The same argument shows that if $f = \sum_{i \in I} f_i$ in K_0, then $f(g) = \sum f_i(g)$ in $L_{M,n}$. This fact is easily seen to imply that the map $f \mapsto f(g) : K_0 \to L_{M,n}$ is an ordered field embedding over $\boldsymbol{k}$.

Stage $m+1$. Assume inductively that we have defined $f(g)$ as an element of $\boldsymbol{k}((t))^{\mathrm{LE}}$ for all $f \in K_m$ and all g. We will also assume inductively that for $g \in L_{M,n}$ the following four properties hold.

$(1)_m$ For all $f \in K_m$ we have $f(g) \in L_{M,m+n}$.
If $f = \sum_{i \in I} f_i$ in K_m, then $f(g) = \sum_{i \in I} f_i(g)$ in $L_{M,m+n}$;

$(2)_m$ The map $f \mapsto f(g) : K_m \to L_{M,m+n}$ is an ordered field embedding over $\boldsymbol{k}$.
We shall denote the image of this map by $\boldsymbol{k}((G_m(g)))$.

$(3)_m$ If $\lambda \in \operatorname{Supp} \boldsymbol{k}((G_m(g)))$, then $\lambda > \sigma$ for some $\sigma \in \boldsymbol{k}((G_m(g)))^{>0} \cdot \langle \operatorname{Supp} g \rangle$.

$(4)_m$ If $m = 0$ and $f \in K_0$, $\epsilon \in L_{M,n}$ with $\operatorname{Lm}(\epsilon) < \operatorname{Lm}(g)$, then

$$f(g+\epsilon) = \sum_{p=0}^{\infty} \frac{f^{(p)}(g)}{p!} \epsilon^p \text{ in } L_{M,n}\ .$$

If $m > 0$ and $f \in K_m$, $\epsilon \in L_{M,n}$, $|\epsilon| < \boldsymbol{k}((G_{m-1}(g)))^{>0}$, then

$$f(g+\epsilon) = \sum_{p=0}^{\infty} \frac{f^{(p)}(g)}{p!} \epsilon^p \text{ in } L_{M,m+n}\ .$$

(6.9) To justify the inductive assumption for $m = 0$ we now show that $(1)_0$ through $(4)_0$ hold for $g \in L_{M,n}$. The arguments of Stage 0 have already established $(1)_0$ and $(2)_0$. For $(3)_0$, write $g = c \cdot (\mu + h)$ with $c = \operatorname{Lc}(g)$ and $\mu = \operatorname{Lm}(g)$. Then $g^r = c^r \mu^r \sum_p \binom{r}{p} (h/\mu)^p$. Thus for each $\lambda \in \operatorname{Supp} g^r$ there are $p \in \mathbf{N}$ and $\lambda_1, \dots, \lambda_p \in \operatorname{Supp} h \subseteq \operatorname{Supp} g$ such that

$$\lambda = \mu^r \cdot (\lambda_1/\mu) \cdots (\lambda_p/\mu) > \frac{1}{2} c^{-r} g^r \cdot (\lambda_1/\mu) \cdots (\lambda_p/\mu) \in \boldsymbol{k}((G_0(g)))^{>0} \cdot \langle \operatorname{Supp} g \rangle$$

which proves $(3)_0$. For $(4)_0$, let $f = \sum_r a_r x^r$ in K_0. Then $\frac{f^{(p)}}{p!} = \sum_r a_r \binom{r}{p} x^{r-p}$, hence $\frac{f^{(p)}(g)}{p!} = \sum_r a_r \binom{r}{p} g^{r-p}$, and thus

$$\begin{aligned} f(g+\epsilon) &= \sum_r a_r (g+\epsilon)^r = \sum_r a_r (g(1+g^{-1}\epsilon))^r \\ &= \sum_r \left(a_r g^r \sum_p \binom{r}{p} (g^{-1}\epsilon)^p \right) = \sum_{(p,r) \in \mathbf{N} \times \boldsymbol{k}} a_r \binom{r}{p} g^{r-p} \epsilon^p \\ &= \sum_p \frac{f^{(p)}(g)}{p!} \epsilon^p \end{aligned}$$

as desired.

(6.10) We are now ready to define $f(g)$ for $f \in K_{m+1}$. Write $f = \sum_{a \in A_m} f_a \operatorname{E}(a)$ in K_{m+1}, and put

$$(*) \qquad f(g) := \sum f_a(g) \operatorname{E}(a(g)).$$

For this definition to make sense we have to check that the right hand sum in (*) exists in $\boldsymbol{k}((t))^{\mathrm{LE}}$. That is where the four part inductive assumption is essential.

Remark. At the beginning of "stage $m+1$" we assumed that $f(g)$ has already been assigned a value for $f \in K_m$. This agrees with (*): if $f \in K_m$ is considered as an element of K_{m+1}, then only one term in the infinite sum in (*) can be nonzero, and this term equals $f(g)$ as given by the original assignment. So there is no conflict of notation.

(6.11) To complete the induction we have to establish the following:

(I) The infinite sum in (*) defining $f(g)$ for $f \in K_{m+1}$ exists in $\boldsymbol{k}((t))^{\mathrm{LE}}$.

(II) Properties $(1)_{m+1}$ through $(4)_{m+1}$ hold for $g \in L_{M,n}$.

(6.12) *Proof of* (I). Let $f = \sum_{a \in A_m} f_a \,\mathrm{E}(a)$ in K_{m+1} and let $g \in L_{M,n}$. We will show that then $\sum_a f_a(g)\,\mathrm{E}(a(g))$ exists in $L_{M,m+n+1}$. This amounts to checking:

(a) For each $\mu \in G_{M,m+n+1}$ there are only finitely many $a \in A_m$ such that $\mu \in \mathrm{Supp}\ f_a(g)\,\mathrm{E}(a(g))$.

(b) The subset
$$\bigcup_{a \in A_m} \mathrm{Supp}\ f_a(g)\,\mathrm{E}(a(g))$$
of $G_{M,m+n+1}$ is reverse well ordered.

Suppose for a moment that every monomial of Supp $\boldsymbol{k}((G_m(g)))$ is greater than some element of $\boldsymbol{k}((G_m(g)))^{>0}$. It is then not hard to verify (a) and (b), by first deriving from this assumption that

$$a_1 < a_2 \text{ in } A_m \Longrightarrow \mathrm{Supp}\ f_{a_1}(g)\,\mathrm{E}(a_1(g)) < \mathrm{Supp}\ f_{a_2}(g)\,\mathrm{E}(a_2(g)).$$

The problem, however, is that Supp $\boldsymbol{k}((G_m(g)))$ may contain monomials that are smaller than all positive elements of $\boldsymbol{k}((G_m(g)))$. (For example, this happens for $m = 0$ and $g = x + \mathrm{E}(-x)$ with the monomial $\mathrm{E}(-x)$.) In that case there may be interference among the sets Supp $f_a(g)\,\mathrm{E}(a(g))$ as a ranges over A_m, and this causes trouble in verifying (a) and (b). It is because of this possibility that the following more complicated argument (using $(3)_m$ and $(4)_m$) seems to be unavoidable.

Write $g = h + \delta$ where $h \in L_{M,n}$ contains those monomials of g that are greater than some element of $\boldsymbol{k}((G_m(g)))^{>0}$ and where $\delta \in L_{M,n}$ contains those monomials of g that are less than $\boldsymbol{k}((G_m(g)))^{>0}$. We now claim that $\mathrm{Lt}(f(g)) = \mathrm{Lt}(f(h))$ for $f \in K_m^{>0}$. To see this, note that by $(4)_m$ we have

$$f(h) = f(g - \delta) = f(g) + \sum_{p=1}^{\infty} (-1)^p \frac{f^{(p)}(g)}{p!} \delta^p.$$

All monomials of the infinite sum on the right hand side are less than $\mathrm{Lm}(f(g))$, since $f(g) \in \boldsymbol{k}((G_m(g)))^{>0}$ and $|\delta| < \boldsymbol{k}((G_m(g)))^{>0}$. Thus $\mathrm{Lt}(f(g)) = \mathrm{Lt}(f(h))$ as claimed. It follows that each element of $\boldsymbol{k}((G_m(g)))^{>0}$ (no matter how small) is greater than some element of $\boldsymbol{k}(\!(G_m(h))\!)^{>0}$, and conversely, that each element of $\boldsymbol{k}(\!(G_m(h))\!)^{>0}$ is greater than some element of $\boldsymbol{k}(\!(G_m(g))\!)^{>0}$. The last statement implies that $|\delta| < \boldsymbol{k}(\!(G_m(h))\!)^{>0}$.

Hence by $(4)_m$ we have for $a \in A_m$ (and h in the role of g):

$$a(g) = a(h) + \sum_{p=1}^{\infty} \frac{a^{(p)}(h)}{p!}\delta^p.$$

The infinite sum in this formula is $\boldsymbol{k}$-infinitesimal, and therefore

$$\mathrm{E}(a(g)) = \mathrm{E}(a(h)) \left(1 + \sum_{p=1}^{\infty} F_p(a'(h), \dots, a^{(p)}(h))\delta^p\right)$$

where $F_p(X_1, \dots, X_p) \in \mathbf{Q}[X_1, \dots, X_p]$ is a polynomial independent of a and g. We also let $F_0 = 1 \in \mathbf{Q}$. Applying $(4)_m$ once more we get

$$f_a(g) = \sum_{p=0}^{\infty} \frac{f_a^{(p)}(h)}{p!}\delta^p$$

and multiplying the last two identities gives

$$(\dagger) \qquad f_a(g)\,\mathrm{E}(a(g)) = \mathrm{E}(a(h)) \sum_{p=0}^{\infty} H_{a,p}\delta^p$$

where $H_{a,p} := \sum_{i=0}^{p} \frac{f_a^{(i)}(h)}{i!} F_{p-i}(a'(h), \dots, a^{(p-i)}(h)) \in \boldsymbol{k}(\!(G_m(h))\!)$.

We now apply lemma 6.6 to the field $K := \boldsymbol{k}(\!(G_m(h))\!)$, with $N := m+n$ and with $\mathcal{A} := \{a(h) : a \in A_m\}$. The lemma tells us that for each $a \in A_m$ we can write $\mathrm{E}(a(h)) = u_a y_a$, where $u_a \in G_{M,N+1}$ and $y_a \in L_{M,N}$, such that $\mathrm{Supp}\ y_a \subseteq \langle \mathrm{Supp}\ K\rangle$ and such that $u_{a_1}/u_{a_2} > K$ whenever $a_1 > a_2$ in A_m. Next we apply lemma 6.7 with $A := G_{M,N+1}$, $C :=$ convex hull of $\langle \mathrm{Supp}\ K\rangle$ in $G_{M,N}$, $U := \{u_a : f_a \neq 0\}$, $V := \mathrm{Supp}\ \delta$, and where for each $u = u_a \in U$ and $p \in \mathbf{N}$ we take

$$S_{u,p} := \mathrm{Supp}\ H_{a,p} \cdot \mathrm{Supp}\ y_a$$

which is a reverse well ordered subset of C. Our application of Lemma 6.6 shows that hypothesis (i) of Lemma 6.7 is satisfied in this situation. In order to verify that hypothesis (ii) of lemma 6.7 is also satisfied we first derive the following:

($\triangle$) Each element of $\langle \mathrm{Supp}\ K\rangle$ is greater than some element of $K^{>0}$.

This reduces to showing that each element of $\mathrm{Supp}\ K$ is greater than some element of $K^{>0}$. By $(3)_m$ applied to h instead of g this in turn will be the case if each element of $\langle \mathrm{Supp}\ h\rangle$ is greater than some element of $K^{>0}$. But each element of $\langle \mathrm{Supp}\ h\rangle$ is a product of finitely many monomials each of which is greater than some element of $\boldsymbol{k}(\!(G_m(g))\!)^{>0}$, hence greater than some element of $K^{>0}$ (by an earlier remark), and thus their product is greater than some element of $K^{>0}$. This finishes the proof of ($\triangle$).

Now each element of Supp δ is less than $\boldsymbol{k}((G_m(g)))^{>0}$, and thus less than $K^{>0}$, which by ($\triangle$) implies that each element of Supp δ is less than $\langle \text{Supp } K \rangle$, hence less than C. Thus hypothesis (ii) of lemma 6.7 is satisfied in our situation. Note that by (†) above we have for each $u = u_a \in U$:

$$\text{Supp } f_a(g) \, \mathrm{E}(a(g)) \subseteq \bigcup_p S_{u,p} \cdot u \cdot V^p.$$

Hence by lemma 6.7 we obtain the properties (a) and (b) above, which express the existence of $\sum f_a \, \mathrm{E}(a(g))$ in $L_{M,m+n+1}$. This finishes the proof of (I).

(6.13) *Proof of* (II). The way we established (I) also leads to the first part of $(1)_{m+1}$, and the second part follows in a similar way. Property $(2)_{m+1}$ follows easily from the definitions and $(1)_{m+1}$. To prove $(3)_{m+1}$, let $f = \sum f_a \, \mathrm{E}(a)$ in K_{m+1}, and let $\lambda \in \text{Supp } f(g)$. Following the proof of (I) and using the notations introduced there it follows from (†) that $\lambda = s u_a v$ with $s \in \langle \text{Supp } K \rangle$, $u_a \in U$, and $v \in \langle \text{Supp } \delta \rangle \subseteq \langle \text{Supp } g \rangle$. Since $u_a > c \, \mathrm{E}(a(h)) > (c/2) \, \mathrm{E}(a(g))$ for some $c \in \boldsymbol{k}^{>0}$ it follows easily from other facts obtained in the proof of (I) that λ is greater than some element of $\boldsymbol{k}((G_{m+1}(g)))^{>0} \cdot \langle \text{Supp } g \rangle$. This concludes the proof of $(3)_{m+1}$.

Now $(4)_{m+1}$. Let $f \in K_{m+1}$ and $\epsilon \in L_{M,n}$ with $|\epsilon| < \boldsymbol{k}((G_m(g)))^{>0}$. We have to show that $f(g+\epsilon) = \sum_p \frac{f^{(p)}(g)}{p!} \epsilon^p$ in $L_{M,m+n+1}$.

Case 1. Suppose $f = \mathrm{E}(a)$ with $a \in A_m$.
It is a routine exercise to show that then for each $p \in \mathbf{N}$ we have

$$\frac{f^{(p)}}{p!} = F_p(a', \cdots, a^{(p)}) \, \mathrm{E}(a)$$

where $F_p(X_1, \cdots, X_p) \in \mathbf{Q}[X_1, \cdots, X_p]$ is the polynomial introduced in the proof of (I). Thus, arguing as in that proof and using $(2)_{m+1}$ we have

$$\begin{aligned} f(g+\epsilon) &= \mathrm{E}(a(g+\epsilon)) \\ &= \mathrm{E}(a(g)) \left(1 + \sum_{p=1}^{\infty} F_p(a'(g), \cdots, a^{(p)}(g)) \epsilon^p \right) \\ &= \sum_{p=0}^{\infty} \frac{f^{(p)}(g)}{p!} \epsilon^p. \end{aligned}$$

Case 2. Suppose $f = s \, \mathrm{E}(a)$ with $s \in K_m$ and $a \in A_m$.
By $(4)_m$ we have $s(g+\epsilon) = \sum_p \frac{s^{(p)}(g)}{p!} \epsilon^p$, which in combination with Case 1 easily implies the desired result for f.

General Case. Write $f = \sum f_a \, \mathrm{E}(a)$ in K_{m+1}.
Put $\phi_a = f_a \, \mathrm{E}(a)$ for $a \in A_m$. Then by $(1)_{m+1}$ and case 2

$$\begin{aligned}
f(g+\epsilon) &= \sum_a \phi_a(g+\epsilon) = \sum_a \left(\sum_p \frac{\phi_a^{(p)}(g)}{p!}\epsilon^p \right) \\
&= \sum_{a,p} \frac{\phi_a^{(p)}(g)}{p!}\epsilon^p \\
&= \sum_p \left(\sum_a \frac{\phi_a^{(p)}(g)}{p!}\epsilon^p \right) \\
&= \sum_p \frac{f^{(p)}(g)}{p!}\epsilon^p
\end{aligned}$$

provided the sum $\sum_{a,p} \frac{\phi_a^{(p)}(g)}{p!}\epsilon^p$ exists in $L_{M,m+n+1}$. To verify that this sum indeed exists we proceed as follows. Note that $\frac{\phi_a^{(p)}}{p!} = s_{a,p}\,\mathrm{E}(a)$ with $s_{a,p} \in K_m$ (and $s_{a,p} = 0$ if $f_a = 0$). Next write $g = h + \delta$ exactly as in the proof of (I). As in that proof we obtain for $a \in A_m$ and $p \in \mathbf{N}$

$$\frac{\phi_a^{(p)}(g)}{p!} = s_{a,p}(g)\,\mathrm{E}(a(g)) = \mathrm{E}(a(h)) \sum_{q=0}^{\infty} H_{a,p,q}\delta^q$$

with $H_{a,p,q} \in \boldsymbol{k}(\!(G_m(h))\!)$. Hence the above sum $\sum_{a,p}$ will exist in $L_{M,m+n+1}$ if the sum

$$\sum_{a,p,q} H_{a,p,q}\,\mathrm{E}(a(h))\delta^q\epsilon^p$$

exists in $L_{M,m+n+1}$. To see that the sum we just displayed indeed exists we argue, once again, as in the proof of (I). We let K, N, $\mathcal{A}$, u_a and y_a (for $a \in A_m$), A and C be exactly as in the latter half of that proof. We also put $V_1 := \mathrm{Supp}\ \epsilon$ and $V_2 := \mathrm{Supp}\ \delta$ and for $u = u_a \in U$ and $p, q \in \mathbf{N}$ we take

$$S_{u,p,q} := \mathrm{Supp}\ H_{a,p,q} \cdot \mathrm{Supp}\ y_a,$$

a reverse well ordered subset of C. Note that with these notations we have

$$\mathrm{Supp}\ (H_{a,p,q}\,\mathrm{E}(a(h))\delta^q\epsilon^p) \subseteq S_{u,p,q} \cdot u \cdot V_1^p \cdot V_2^q.$$

An easy extension of lemma 6.7 then tells us that the subset

$$W := \bigcup_{u \in U, p,q \in \mathbf{N}} S_{u,p,q} \cdot u \cdot V_1^p \cdot V_2^q$$

of $A = G_{M,m+n+1}$ is reverse well ordered and that for each element w of W there are only finitely many tuples $(u, p, q, s, v_{1,1}, \ldots, v_{1,p}, v_{2,1}, \ldots, v_{2,q})$ with $u \in U$, $p, q \in \mathbf{N}$, $s \in S_{u,p,q}$, $v_{1,1}, \ldots, v_{1,p} \in V_1$ and $v_{2,1}, \ldots, v_{2,q} \in V_2$ such that

$$w = s \cdot u \cdot v_{1,1} \cdots v_{1,p} \cdots v_{2,1} \cdots v_{2,q}.$$

But that is exactly what we need to conclude that the above sum $\sum_{a,p,q}$ exists. This finishes the proof of (II).

(6.14) We have now defined $f \circ g$—also written as $f(g)$—for $f \in \boldsymbol{k}((t))^{\mathrm{E}}$. We remark here that we had no choice in the matter: if $\circ$ is a composition operation as in theorem 6.2 (without assuming uniqueness), then it has to assign to elements f, g with $f \in \boldsymbol{k}((t))^{\mathrm{E}}$ the value that the above definitions assigned to it, as follows from Lemma 6.4 and the subsequent discussion in 6.5.

Let us record here some properties that follow easily from our inductive definition and proof. These properties will be useful in extending this composition to $f \in \boldsymbol{k}((t))^{\mathrm{LE}}$ and in deriving similar properties for this extended composition.

Lemma 6.15. *Let $f \in \boldsymbol{k}((t))^{\mathrm{E}}$. Then*

1. $r \circ g = r$ *for all* $r \in \boldsymbol{k}$,
2. $f \circ x = f$ *and* $x \circ g = g$,
3. *the map* $\phi \mapsto \phi \circ g : \boldsymbol{k}((t))^{\mathrm{E}} \to \boldsymbol{k}((t))^{\mathrm{LE}}$ *is an ordered field embedding,*
4. *if* $f = \sum f_i$ *in* $\boldsymbol{k}((t))^{\mathrm{E}}$, *then* $f \circ g = \sum f_i \circ g$ *in* $\boldsymbol{k}((t))^{\mathrm{LE}}$,
5. $\mathrm{E}(f) \circ g = \mathrm{E}(f \circ g)$, $\Phi(f) \circ g = f \circ \mathrm{E}(g)$, *and* $\Phi^{-n}(f) = f \circ L^n x$,
6. $f^s \circ g = (f \circ g)^s$ *for all* $s \in \boldsymbol{k}$, *where* $f > 0$,
7. $(f \circ g)' = (f' \circ g) \cdot g'$.

Proof. Items (1)-(4) have been established during the induction or are immediate consequences. For (5)-(7) we also argue by induction on f, using (1)-(4) freely.

Suppose first that $f \in K_0$ and write $f = \sum a_r x^r$. Put $f_\infty := \sum_{r>0} a_r x^r$ and $\epsilon(f) := \sum_{r<0} a_r x^r$, so that $f = f_\infty + a_0 + \epsilon(f)$. Then

$$\mathrm{E}(f) = \mathrm{E}(f_\infty) \cdot \exp(a_0) \cdot \sum_k \frac{\epsilon(f)^k}{k!} \ ,$$

hence by (3) and (4)

$$\mathrm{E}(f) \circ g = (\mathrm{E}(f_\infty) \circ g) \cdot \exp(a_0) \cdot \sum_k \frac{\epsilon(f)^k \circ g}{k!} \ .$$

Also $f \circ g = (f_\infty \circ g) + a_0 + (\epsilon(f) \circ g)$, and hence

$$\mathrm{E}(f \circ g) = \mathrm{E}(f_\infty \circ g) \cdot \exp(a_0) \cdot \sum_k \frac{(\epsilon(f) \circ g)^k}{k!} \ .$$

Now note that $f_\infty \in A_0$, so that $\mathrm{E}(f_\infty) \circ g = \mathrm{E}(f_\infty \circ g)$ by definition; moreover, by (3) we have $\epsilon(f)^k \circ g = (\epsilon(f) \circ g)^k$ for $k \in \mathbf{N}$. Hence $\mathrm{E}(f) \circ g = \mathrm{E}(f \circ g)$, which is the first identity of (5). For the second identity we note that $\Phi(f) = \sum a_r \mathrm{E}(rx)$ so that $\Phi(f) \circ g = \sum a_r \mathrm{E}(rg) = f \circ \mathrm{E}\, g$. For the third identity of (5) we first note that $\Phi^{-n}(x^r) = (\Phi^{-n} x)^r = (\mathrm{L}^n(x))^r$ by ... and an identity observed at the end of (2.7); thus, by (4) and an identity in (2.8): $\Phi^{-n} f = \sum a_r (\mathrm{L}^n x)^r = f \circ \mathrm{L}^n x$.

For (6) we assume $f > 0$. Write $f = ax^b(1+\epsilon)$ with $a, b \in \boldsymbol{k}$, $a > 0$ and $\epsilon \in \mathfrak{m}$. Then $f^s = a^s x^{bs} \sum_k \binom{s}{k} \epsilon^k$, so that $f^s \circ g = a^s g^{bs} \sum_k \binom{s}{k}(\epsilon^k \circ g)$. On the other hand, $f \circ g = ag^b(1 + (\epsilon \circ g))$, and hence $(f \circ g)^s = a^s g^{bs} \sum_k \binom{s}{k}(\epsilon \circ g)^k$. Thus $f^s \circ g = (f \circ g)^s$. Now (7):

$$(f \circ g)' = (\sum a_r g^r)' = \sum r a_r g^{r-1} g' = (\sum r a_r g^{r-1}) \cdot g' = (f' \circ g) \cdot g'.$$

Next we assume inductively that (5)-(7) hold for all $f \in K_m$. Suppose $f \in K_{m+1}$ and write $f = \sum f_a \operatorname{E}(a)$ in K_{m+1}. Put $f_\infty := \sum_{a>0} f_a \operatorname{E}(a)$ and $\epsilon(f) := \sum_{a<0} f_a \operatorname{E}(a)$, so that $f = f_\infty + f_0 + \epsilon(f)$. Then

$$\operatorname{E}(f) = \operatorname{E}(f_\infty) \cdot \operatorname{E}(f_0) \cdot \sum_k \frac{\epsilon(f)^k}{k!} ,$$

hence by (3) and (4)

$$\operatorname{E}(f) \circ g = (\operatorname{E}(f_\infty) \circ g) \cdot (\operatorname{E}(f_0) \circ g) \cdot \sum_k \frac{\epsilon(f)^k \circ g}{k!} .$$

Also $f \circ g = (f_\infty \circ g) + (f_0 \circ g) + (\epsilon(f) \circ g)$, and hence

$$\operatorname{E}(f \circ g) = \operatorname{E}(f_\infty \circ g) \cdot \operatorname{E}(f_0 \circ g) \cdot \sum_k \frac{(\epsilon(f) \circ g)^k}{k!} .$$

Now note that $f_\infty \in A_{m+1}$, so that $\operatorname{E}(f_\infty) \circ g = \operatorname{E}(f_\infty \circ g)$ by definition; moreover, by the inductive assumption we have $\operatorname{E}(f_0) \circ g = \operatorname{E}(f_0 \circ g)$, and by (3) we have $\epsilon(f)^k \circ g = (\epsilon(f) \circ g)^k$ for $k \in \mathbf{N}$. Hence $\operatorname{E}(f) \circ g = \operatorname{E}(f \circ g)$, which is the first formula of (5). The second formula of (5) is obtained as follows: $\Phi(f) = \sum \Phi(f_a) \operatorname{E}(\Phi a)$, hence

$$\begin{aligned}\Phi(f) \circ g &= \sum (\Phi(f_a) \circ g)(\operatorname{E}(\Phi a) \circ g) = \sum (f_a \circ \operatorname{E} g) \operatorname{E}(\Phi a \circ g) \\ &= \sum (f_a \circ \operatorname{E} g) \operatorname{E}(a \circ \operatorname{E} g) = f \circ \operatorname{E} g.\end{aligned}$$

The third formula of (5) follows from the various inductive assumptions and (4):

$$\begin{aligned}\Phi^{-n}(f) &= \sum \Phi^{-n}(f_a) \operatorname{E}(\Phi^{-n} a) = \sum (f_a \circ \operatorname{L}^n x) \operatorname{E}(a \circ \operatorname{L}^n x) \\ &= \sum (f_a \circ \operatorname{L}^n x)(\operatorname{E}(a) \circ \operatorname{L}^n x) \\ &= f \circ \operatorname{L}^n x.\end{aligned}$$

For (6) we assume $f > 0$. Write $f = f_\alpha \operatorname{E}(\alpha)(1+\epsilon)$ with $\alpha := \max\{a \in A_m : f_a \neq 0\}$, and $\epsilon \in \mathfrak{m}_{m+1}$. Then $f^s = f_\alpha^s \operatorname{E}(s\alpha) \sum_k \binom{s}{k} \epsilon^k$ and $f \circ g = (f_\alpha \circ g) \operatorname{E}(\alpha \circ g)(1 + \epsilon(g))$, so that

$$\begin{aligned}f^s \circ g &= (f_\alpha^s \circ g)(\operatorname{E}(s\alpha) \circ g) \sum_k \binom{s}{k} (\epsilon^k \circ g) \\ &= (f_\alpha \circ g)^s \operatorname{E}(s\alpha \circ g) \sum_k \binom{s}{k} (\epsilon \circ g)^k \\ &= (f_\alpha \circ g)^s (\operatorname{E}(\alpha \circ g))^s \sum_k \binom{s}{k} (\epsilon(g)^k \\ &= (f \circ g)^s .\end{aligned}$$

To obtain (7), note that $f' = \sum (f_a' + a' f_a)\,\mathrm{E}(a)$ and $f(g) = \sum f_a(g)\,\mathrm{E}(a(g))$. Thus

$$\begin{aligned} f(g)' &= \sum \big(f_a(g)' + a(g)' f_a(g)\big)\,\mathrm{E}(a(g)) \\ &= \sum \big(f_a'(g) g' + a'(g) g' f_a(g)\big)\,\mathrm{E}(a(g)) \\ &= f'(g) g'. \end{aligned}$$

This finishes the proof of the lemma.

(6.16) We are now ready to define $f \circ g$ for arbitrary f. Suppose that $f \in L_m$. Then $\widetilde{f} := \Phi^m(f) \in \boldsymbol{k}((t))^{\mathrm{E}}$, and we put

$$(**) \qquad f \circ g := \Phi^m f \circ \mathrm{L}^m g = \widetilde{f} \circ \mathrm{L}^m g.$$

Note that if $m = 0$ the right hand side of $(**)$ equals $f \circ g$ as previously defined. In fact, formula $(**)$ is forced on us by part (3) of Lemma 6.4 and the subsequent discussion in 6.5. In other words, we have established the part of theorem 6.2 that says there is *at most one* composition operation with the desired properties (1)-(6).

We have to show, of course, that the right hand side of $(**)$ is independent of the choice of m (for given f and g). This will follow if we prove that increasing m does not change the right hand side in $(**)$. Increasing m by 1 has the effect of replacing $\widetilde{f}$ by $\Phi(\widetilde{f})$. Thus the right hand side in $(**)$ gets replaced by

$$\Phi(\widetilde{f}) \circ \mathrm{L}^{m+1} g = \widetilde{f} \circ \mathrm{E}(\mathrm{L}^{m+1} g) = \widetilde{f} \circ \mathrm{L}^m g,$$

so this right hand side is indeed invariant when we increase m.

We have now (at last) defined unambiguously a composition operation

$$(f, g) \mapsto f \circ g : \boldsymbol{k}((t))^{\mathrm{LE}} \times \boldsymbol{k}((t))^{\mathrm{LE}}_{\infty} \longrightarrow \boldsymbol{k}((t))^{\mathrm{LE}} .$$

It remains to show that this composition satisfies properties (1)-(6) of theorem 6.2.

(6.17)

Proof of (1)-(6) *of Theorem 6.2.* Let $f \in L_m$ and define $\widetilde{f}$ as before. We will freely use Lemma 6.15, which already gives us (1) and the second identity of (2). To get the first identity of (2) we observe, using the third formula of Lemma 6.15(5):

$$\Phi^m(f \circ x) = \Phi^m(\widetilde{f} \circ \mathrm{L}^m x) = \Phi^m(\Phi^{-m} \widetilde{f}) = \Phi^m f,$$

hence $f \circ x = f$.

Properties (3) and (4) follow from parts (3) and (4) of lemma 4 and the fact that the powers of Φ are ordered field embeddings that preserve infinite sums. The first identity of property (5) is obtained from $\Phi^m(\mathrm{E} f) = \mathrm{E}(\Phi^m(f))$ as follows:

$$\mathrm{E}(f) \circ g = \mathrm{E}(\widetilde{f}) \circ \mathrm{L}^m g = \mathrm{E}(\widetilde{f} \circ \mathrm{L}^m g) = \mathrm{E}(f \circ g).$$

The second identity of (5) is obtained as follows:

$$\begin{aligned}\Phi(f) = \Phi(f) \circ x = \Phi^m(\Phi(f) \circ \mathrm{L}^m x = \Phi(\widetilde{f}) \circ \mathrm{L}^m x \\ = \widetilde{f} \circ \mathrm{E}(\mathrm{L}^m x) = \widetilde{f} \circ \mathrm{L}^m(\mathrm{E}(x)) = f \circ \mathrm{E}(x).\end{aligned}$$

To obtain associativity we need the following auxiliary identities:

$$(\log g) \circ h = \log(g \circ h), \text{ and } g^r \circ h = (g \circ h)^r \text{ for } r \in \boldsymbol{k}.$$

The first of these follows by applying E to both sides, and using (5), and for the second we write $g^r = \mathrm{E}(r \log g)$ and apply the first identity and (5). We are now ready to prove (6) for $f = \sum a_r x^r$ in K_0. Then $f \circ g = \sum a_r g^r$, hence by (4) and the second auxiliary identity:

$$(f \circ g) \circ h = \sum a_r (g^r \circ h) = \sum a_r (g \circ h)^r = f \circ (g \circ h).$$

Next, suppose (6) holds for all $f \in K_n$, and let $f = \sum f_a \mathrm{E}(a)$ in K_{n+1}. Then $f \circ g = \sum (f_a \circ g) \mathrm{E}(a \circ g))$, and thus

$$\begin{aligned}(f \circ g) \circ h &= \sum ((f_a \circ g) \circ h) \mathrm{E}((a \circ g) \circ h) \\ &= \sum (f_a \circ (g \circ h)) \mathrm{E}(a \circ (g \circ h)) \\ &= f \circ (g \circ h).\end{aligned}$$

We have now established associativity for $f \in \boldsymbol{k}((t))^{\mathrm{E}}$. Using this, and the first auxiliary identity above we obtain for $f \in L_m$:

$$\begin{aligned}(f \circ g) \circ h &= (\widetilde{f} \circ \mathrm{L}^m g) \circ h = \widetilde{f} \circ ((\mathrm{L}^m g) \circ h) \\ &= \widetilde{f} \circ (\mathrm{L}^m (g \circ h)) = f \circ (g \circ h).\end{aligned}$$

This finishes the proof of (6), and thereby the proof of the Theorem.

The following special case of the preservation of infinite sums under composition is worth mentioning.

Corollary 6.18. *Let $F \in \boldsymbol{k}[[Y_1, \ldots, Y_n]]$, and let $\epsilon_1, \ldots, \epsilon_n$ be infinitesimals of $\boldsymbol{k}((t))^{\mathrm{LE}}$. Then*

$$F(\epsilon_1, \ldots, \epsilon_n) \circ g = F(\epsilon_1(g), \ldots, \epsilon_n(g)) \ .$$

(6.19) We now verify the chain rule: $(f \circ g)' = (f' \circ g) g'$. Using previous notations this is seen as follows

$$\begin{aligned}
(f\circ g)' &= (\widetilde{f}\circ \mathrm{L}^m g)' \\
&= \big(\Phi^m(f)'\circ \mathrm{L}^m g\big)(\mathrm{L}^m g)' \quad \text{(by 6.15 (7))} \\
&= \big(\Phi^m(f)'\circ \mathrm{L}^m g\big)\frac{g'}{\prod_{i=1}^m \mathrm{L}^{i-1}g} \quad \text{(by 3.9 (3))} \\
&= \big(\Phi^m(f')\circ \mathrm{L}^m g\big)\Big(\prod_{i=1}^{m} \mathrm{E}^i(x)\circ \mathrm{L}^m g\Big)\frac{g'}{\prod_{i=1}^m \mathrm{L}^{i-1}g} \quad \text{(by 3.9 (5))} \\
&= \big(\Phi^m(f')\circ \mathrm{L}^m g\big)\prod_{i=1}^{m} \mathrm{L}^{i-1}(g)\frac{g'}{\prod_{i=1}^m \mathrm{L}^{i-1}g} \\
&= \big(\Phi^m(f')\circ \mathrm{L}^m g\big)g' \\
&= (f'\circ g)g'
\end{aligned}$$

Compositional Inverse

(6.20) By property (3) of Theorem 6.2 the map

$$\phi \mapsto \phi\circ g : \boldsymbol{k}((t))^{\mathrm{LE}} \to \boldsymbol{k}((t))^{\mathrm{LE}}$$

is injective and sends $\boldsymbol{k}((t))^{\mathrm{LE}}_\infty$ to itself. Thus $\boldsymbol{k}((t))^{\mathrm{LE}}_\infty$ is a monoid under composition with identity element x. It remains to show that this monoid is actually a group. First some easy observations.

(1) The elements $\mathrm{E}^n(x)$ and $\mathrm{L}^n(x)$ are each others inverse, namely

$$\mathrm{E}^n(x)\circ \mathrm{L}^n(x) = \mathrm{L}^n(x)\circ \mathrm{E}^n(x) = x\ .$$

(2) Whenever $f\circ g = x$, then $f > \boldsymbol{k}$ and $g\circ f = x$.

Observation (1) is clear. To see why (2) holds, let $f\circ g = x$. Since $x > \boldsymbol{k}$ it follows by property (3) of theorem 6.2 that $f > \boldsymbol{k}$. By associativity we get

$$(g\circ f)\circ g = g\circ (f\circ g) = g\circ x = x\circ g$$

and thus, by the injectivity of the map above, $x = g\circ f$, as desired.

The existence of compositional inverses is established in several lemmas which represent successive steps in constructing the inverse.

Lemma 6.21. *Given any g, there are m and n such that*

$$L^m(x)\circ g\circ \mathrm{E}^n(x) = x + \epsilon$$

where ϵ is an infinitesimal of $\boldsymbol{k}((t))^{\mathrm{E}}$.

Proof. If $g\in L_n$, then $g\circ \mathrm{E}^n(x) = \Phi^n(g)\in \boldsymbol{k}((t))^{\mathrm{E}}$, which reduces the proof of the lemma to the case that $g\in \boldsymbol{k}((t))^{\mathrm{E}}$. Actually, we will only assume that the leading monomial $\mathrm{Lm}(g)$ of g belongs to G_m for some m, as this is more convenient in the inductive argument below. We now proceed by induction on m.

Suppose $m = 0$. Then $g = cx^r(1+\delta)$ for some $c, r \in \boldsymbol{k}^{>0}$ and infinitesimal $\delta \in \boldsymbol{k}((t))^{\mathrm{LE}}$. Conjugating with $\mathrm{L}(x)$ has the following effect, as one easily checks:

$$\mathrm{L}(x) \circ g \circ \mathrm{E}(x) = rx + \log(c) + \delta_1 = rx(1+\delta_2)$$

for infinitesimals δ_1, δ_2. Conjugating again we obtain $\mathrm{L}^2(x) \circ g \circ \mathrm{E}^2(x) = x + \epsilon$ for some infinitesimal ϵ. Suppose $\epsilon \in L_k$ with $k > 0$. Then $\mathrm{L}^2(x) \circ g \circ \mathrm{E}^3(x) = \mathrm{E}(x)\big(1 + \frac{\epsilon(\mathrm{E}(x))}{\mathrm{E}(x)}\big)$, hence $\mathrm{L}^3(x) \circ g \circ \mathrm{E}^3(x) = x + \epsilon_1$ with infinitesimal $\epsilon_1 \in \mathrm{L}_{k-1}$. Continuing this way we obtain

$$\mathrm{L}^{2+k}(x) \circ g \circ \mathrm{E}^{2+k}(x) = x + \epsilon_k$$

with infinitesimal $\epsilon_k \in \mathrm{L}_0 = \boldsymbol{k}((t))^{\mathrm{E}}$. This finishes the case $m = 0$.

Suppose that the lemma holds for all g whose leading monomial belongs to G_m. Let g have leading monomial in $G_{m+1} \setminus G_m$. Write $g = \phi\,\mathrm{E}(a)(1+\delta)$ with $0 < \phi \in K_m$, $0 < a \in A_m$ and infinitesimal δ. Then

$$\mathrm{L}(x) \circ g = \log(g) = a + \log(\phi) + \epsilon$$

for some infinitesimal ϵ. Clearly $\mathrm{Lm}(\mathrm{L}(x) \circ g) = \mathrm{Lm}(a) \in G_m$. Now apply the inductive hypothesis to $\mathrm{L}(x) \circ g$.

Lemma 6.22. *Let $m \le n$, $f \in K_m$ and $\epsilon \in K_n$, such that $\mathrm{Lm}\,\epsilon < x$ in case $m = 0$, and $\mathrm{Lm}\,\epsilon < G_{m-1}$ for $m > 0$. Then $f(x+\epsilon) \in K_n$ and $\mathrm{Lm}\, f(x+\epsilon) = \mathrm{Lm}\, f$.*

Proof. By induction on m. For $m = 0$ this follows straight from the definition in 6.8. Assume the result holds for all $f \in K_m$. Let $m+1 \le n$, $f \in K_{m+1}$ and $\epsilon \in K_n$ with $\mathrm{Lm}\,\epsilon < G_m$. We have to show that then $f(x+\epsilon) \in K_n$. For this we simply consult the proof of $(4)_{m+1}$ in 6.13, with $g = x$. It is shown there that if $f = \mathrm{E}(a)$ with $a \in A_m$, then

$$f(x+\epsilon) = \mathrm{E}(a)\left(1 + \sum_{p=1}^{\infty} F_p(a', \cdots, a^{(p)})\epsilon^p\right) ,$$

and since the infinite sum on the right is an infinitesimal of K_n, we get $f(x+\epsilon) \in K_n$, and $\mathrm{Lm}\, f(x+\epsilon) = \mathrm{E}(a) = f = \mathrm{Lm}\, f$. The general case now follows easily from the inductive hypothesis and the fact that $\mathrm{Lm}\, f(x+\epsilon) = \mathrm{Lm}\big((\mathrm{Lm}\, f)(x+\epsilon)\big)$.

Lemma 6.23. *Let $\epsilon \in K_n$ and $\mathrm{Lm}\,\epsilon < x$. Then there exists $\delta \in K_0$ with $\mathrm{Lm}\,\delta < x$ such that $(x+\delta)\circ(x+\epsilon) = x+\epsilon^*$ with $\mathrm{Lm}\,\epsilon^* < G_0$. In particular, $(x+\delta)\circ(x+\epsilon) = x$ if $n = 0$.*

Remark. We do not need to assume here that ϵ is infinitesimal. But if ϵ is infinitesimal, then clearly any δ as in the lemma must also be infinitesimal.

Proof. Write $\epsilon = \epsilon_0 + \epsilon_1$ with $\epsilon_0 \in K_0$ and $\mathrm{Lm}\,\epsilon_1 < G_0$. For $f \in K_0$ we have by the Taylor expansion of $(4)_0$:

$$f(x+\epsilon) = f(x+\epsilon_0) + a\ ,$$

for some $a \in K_n$ with $\mathrm{Lm}\,a < G_0$. Hence, it suffices to find $\delta \in K_0$ with $\mathrm{Lm}\,\delta < x$ such that $f(x+\epsilon_0) = x$ for $f = x+\delta$. Thus replacing ϵ by ϵ_0, we assume from now on that $\epsilon \in K_0$. Put $D := \{\delta \in K_0 : \mathrm{Lm}\,\delta < x\}$. Then we have for $\delta \in D$

$$\begin{aligned}(x+\delta)\circ(x+\epsilon) &= x+\epsilon+\delta+\sum_{p=1}^{\infty}(\delta^{(p)}/p!)\epsilon^p \\ &= x+\epsilon+(\mathrm{I}+\mathrm{P}_\epsilon)(\delta)\end{aligned}$$

where $\mathrm{I}, \mathrm{P}_\epsilon : D \to D$ are the operators defined by $\mathrm{I}(\delta) = \delta$ and

$$\mathrm{P}_\epsilon(\delta) = \sum_{p=1}^{\infty}(\delta^{(p)}/p!)\epsilon^p\ .$$

Thus to find $\delta \in D$ such that $(x+\delta)\circ(x+\epsilon) = x$ amounts to solving the equation $(\mathrm{I}+\mathrm{P}_\epsilon)(\delta) = -\epsilon$ for the unknown $\delta \in D$. To do this we consider K_0 as the series field $\boldsymbol{k}((G_0))$ over $\boldsymbol{k}$. Then P_ϵ is a small operator, because

$$\mathrm{Supp}\ \mathrm{P}_\epsilon(\delta) \subseteq x^{-1}[\mathrm{Supp}\ \epsilon]\cdot \mathrm{Supp}\ \delta$$

for $\delta \in D$. Thus by 1.3 we have that $\delta := (\mathrm{I} - \mathrm{P}_\epsilon + \mathrm{P}_\epsilon^2 - \cdots)(-\epsilon)$ is a well defined element of D and satisfies $(x+\delta)\circ(x+\epsilon) = x$.

Lemma 6.24. *Let $0 < m \le n$, and let $\epsilon \in K_n$ with $\mathrm{Lm}\,\epsilon < G_{m-1}$. Then there exists $\delta \in K_m$ with $\mathrm{Lm}\,\delta < G_{m-1}$ such that $(x+\delta)\circ(x+\epsilon) = x+\epsilon^*$ with $\mathrm{Lm}\,\epsilon^* < G_m$. In particular, $(x+\delta)\circ(x+\epsilon) = x$ if $m = n$.*

Proof. Write $\epsilon = \epsilon_0 + \epsilon_1$ with $\epsilon_0 \in K_m$ and $\mathrm{Lm}\,\epsilon_1 < G_m$. For $f \in K_m$ we have by lemma 6.22 and the Taylor expansion of property $(4)_m$ in 6.8:

$$f(x+\epsilon) = f(x+\epsilon_0) + a\ ,$$

for some $a \in K_n$ with $\mathrm{Lm}\,a < G_m$. Hence, it suffices to find $\delta \in K_m$ with $\mathrm{Lm}\,\delta < G_{m-1}$ such that $f(x+\epsilon_0) = x$ for $f = x+\delta$. Thus replacing ϵ by ϵ_0, we assume from now on that $\epsilon \in K_m$. Put $D := \{\delta \in K_m : \mathrm{Lm}\,\delta < G_{m-1}\}$. By Taylor expansion we have for $\delta \in D$:

$$\begin{aligned}(x+\delta)\circ(x+\epsilon) &= x+\epsilon+\delta+\sum_{p=1}^{\infty}(\delta^{(p)}/p!)\epsilon^p \\ &= x+\epsilon+(\mathrm{I}+\mathrm{P}_\epsilon)(\delta)\end{aligned}$$

where $\mathrm{I}, \mathrm{P}_\epsilon : D \to D$ are the operators defined by $\mathrm{I}(\delta) = \delta$ and

$$\mathrm{P}_\epsilon(\delta) = \sum_{p=1}^{\infty}(\delta^{(p)}/p!)\epsilon^p\ .$$

Thus to find $\delta \in D$ such that $(x+\delta)\circ(x+\epsilon) = x$ amounts to solving the equation $(\mathrm{I}+\mathrm{P}_\epsilon)(\delta) = -\epsilon$ for the unknown $\delta \in D$. To do this we use once more the method of small operators.

Let $\mathrm{Supp}^*\ y$ denote the support of an element $y \in K_m$, where we consider K_m as the series field $K_{m-1}(\!(\mathrm{E}(A_{m-1}))\!)$ over K_{m-1}. Thus $\mathrm{Supp}^*\ y \subseteq \mathrm{E}(A_{m-1})$ and $\mathrm{Supp}^*\ y' \subseteq \mathrm{Supp}^*\ y$ for $y \in K_m$. When K_m is viewed in this way as a series field over K_{m-1}, then P_ϵ is a small operator, because

$$\mathrm{Supp}^*\ \mathrm{P}_\epsilon(\delta) \subseteq S \cdot \mathrm{Supp}^*\ \delta$$

for $\delta \in D$, where

$$S := \bigcup_{p=1}^{\infty} (\mathrm{Supp}^*\ \epsilon)^p \subseteq \mathrm{E}(A_{m-1})^{<1}\ .$$

Thus by 1.3 we have that $\delta := (\mathrm{I} - \mathrm{P}_\epsilon + \ \mathrm{P}_\epsilon^2 - \cdots)(-\epsilon)$ is a well defined element of D and satisfies $(x+\delta)\circ(x+\epsilon) = x$.

Corollary 6.25. *For each infinitesimal $\epsilon \in K_n$ there is an infinitesimal $\delta \in K_n$ such that $(x+\delta)\circ(x+\epsilon) = x$*

Proof. By one application of lemma 6.23 followed by n applications of lemma 6.24, for $m = 1, \ldots, m = n$ successively, we can compose on the left with suitable elements $(x+\delta_0), (x+\delta_1), \ldots, (x+\delta_n)$ with infinitesimals $\delta_0 \in K_0, \delta_1 \in K_1, \ldots, \delta_n \in K_n$ to obtain the identity x. Lemma 6.22 guarantees that we stay inside K_n during this process.

(6.26) Given any g we obtain now as follows an $f > \boldsymbol{k}$ with $f \circ g = x$. First, by lemma 6.21 we find m, n such that $\mathrm{L}^m(x)\circ g \circ \mathrm{E}^n(x) = x + \epsilon$ with infinitesimal $\epsilon \in K_N$ for some N. Then by 6.25 we have $(x+\delta)\circ \mathrm{L}^m(x)\circ g\circ \mathrm{E}^n(x) = x$ for some infinitesimal $\delta \in K_N$. Hence, by a remark in 6.20:

$$\mathrm{E}^n(x)\circ(x+\delta)\circ \mathrm{L}^m(x)\circ g = x\ .$$

(6.27) Levels

Level is a coarse notion of "growth rate" introduced by Rosenlicht in [24]. Marker and Miller [17] used LE-series to study levels for functions in $H(\mathbf{R}_{\mathrm{an,exp}})$. For completeness, we restate one of their results in the formalism of this paper.

For $f \in \boldsymbol{k}(\!(t)\!)^{\mathrm{LE}}_{\infty}$ and $k \in \mathbf{Z}$, we say that f has *level* k, if $\mathrm{Lm}(\mathrm{L}^{N+k}(f)) = \mathrm{L}^N(x))$ for some $N \in \mathbf{N}$ with $N + k \geq 0$. (Thus $\mathrm{E}^n(x)$ has level n, and $\mathrm{L}^n(x)$ has level $-n$.)

Proposition 6.28. *Every element of $\boldsymbol{k}((t))^{\mathrm{LE}}$ has a unique level. Indeed, if $f \in L_n$ and m is least such that $\mathrm{Lm}(\eta_n(f)) \in G_m$, then f has level $m-n$.*

This proposition is closely related to lemma 6.21. (The reader may even use this lemma to derive the above proposition as an exercise.)

Composition of germs

(6.29) The Hardy field $H(\mathbf{R}_{\mathrm{an,exp}})$ has a natural composition operation assigning to germs $\phi, \chi \in H(\mathbf{R}_{\mathrm{an,exp}})$ with $\chi > \mathbf{R}$ the germ $\phi \circ \chi \in H(\mathbf{R}_{\mathrm{an,exp}})$. Each $\chi \in H(\mathbf{R}_{\mathrm{an,exp}})$ with $\chi > \mathbf{R}$ has a compositional inverse $\phi \in H(\mathbf{R}_{\mathrm{an,exp}})$: $\phi > \mathbf{R}$ and $\phi \circ \chi = \chi \circ \phi =$ germ of the identity function on $\mathbf{R}$. Concerning these operations we have the following—obviously desirable—result.

Corollary 6.30. *The natural embedding of $H(\mathbf{R}_{\mathrm{an,exp}})$ into $\mathbf{R}((t))^{\mathrm{LE}}$ respects composition and compositional inverse.*

Proof. To show that compositions $\phi \circ \chi$ are preserved we use the fact from [5] that functions definable in $\mathbf{R}_{\mathrm{an,exp}}$ are given piecewise by terms in the language $\mathcal{L}_{\mathrm{an,exp,log}}$. This allows us to proceed by induction on terms representing ϕ, using 6.18 in the main inductive step. The preservation of compositional inverses follows from the preservation of composition.

(6.31) Remark. The series in the image of the natural embedding above are in some intuitive sense "convergent" LE-series. It would be desirable to make this more precise along the lines of [4] by introducing a suitable inductive system of normed subalgebras of $\mathbf{R}((t))^{\mathrm{LE}}$. The problem of composition for such a conjectural notion of convergent LE-series is already stated at the end of [4].

Relation to Écalle's transseries

(6.32) Écalle states as part of his "Lemme 4.1.2" [8, p.139] the stability of his "trigèbre" $\mathbf{R}[[[x]]]$ of transseries under composition and compositional inverse. He does not touch upon the difficulty—mentioned in 6.5 and the beginning of 6.12 above—of making sense of composition. As to the existence of compositional inverse, Écalle gives a half page sketch which provided the idea for the proof above. As the first stage in constructing the inverse he indicates something close to our lemma 6.21. The second stage consists in inverting an operator similar to what we do in the proof of lemma 6.24. Unfortunately, this operator only makes sense (to us) if Taylor expansion of $f(x+\epsilon)$ as a power series in the infinitesimal ϵ were valid more generally than is actually the case. It is for this reason that this second stage takes the more complicated form of 6.23, 6.24 and 6.25 in our proof.

Here is an example to show that a restriction as in 6.8, $(4)_m$ is necessary for Taylor expansion to be possible. Let $f(x) := \mathrm{E}(x^3) \in K_1$ and $\epsilon := x^{-1}$, an infinitesimal. Then $\sum_{p=0}^{\infty} \frac{f^{(p)}(x)}{p!}\epsilon^p$ does not exist, since $\mathrm{Lm}(\frac{f^{(p)}(x)}{p!}\epsilon^p) = x^p\,\mathrm{E}(x^3)$ increases with p.

§7 Interesting Subfields of $\mathbf{R}((t))^{\mathrm{LE}}$

(7.1) In this section we impose certain natural restrictions on the LE-series considered, and show that this leads to subfields of $\boldsymbol{k}((t))^{\mathrm{LE}}$ that share many properties of $\boldsymbol{k}((t))^{\mathrm{LE}}$ and are nicely contained in $\boldsymbol{k}((t))^{\mathrm{LE}}$, for example as truncation closed subfields. We pay special attention to the subfield $\boldsymbol{k}((t))^{\mathrm{LE,ft}}$ consisting of the LE-series with support of hereditarily finite type, as defined in (7.10) and (7.11) below. This subfield is in essence Écalle's trigèbre $\mathbf{R}[[[x]]]$ of transseries when $\boldsymbol{k}=\mathbf{R}$, and we prove here in detail some of its properties stated or hinted at in [8].

(7.2) For each m, let $\mathcal{F}_m$ be a subset of $\boldsymbol{k}[[X_1,\dots,X_m]]$ such that the subring $\boldsymbol{k}[X_1,\dots,X_m,\mathcal{F}_m]$ is closed under the operations $\frac{\partial}{\partial X_i}$ for $i=1,\dots,m$. Let $\mathcal{F}:=(\mathcal{F}_m)_{m\in\mathbf{N}}$. We say that a subfield K of $\boldsymbol{k}((t))^{\mathrm{LE}}$ is *weakly $\mathcal{F}$-closed* if $F(\epsilon_1,\dots,\epsilon_m)\in K$ whenever $F\in\mathcal{F}_m$ and $\epsilon_1,\dots,\epsilon_m$ are infinitesimals in K. (NB: we use the modifier "weakly" because in [6, 3.2] the definition of "$\mathcal{F}$-closed" also requires being real closed.) We define $\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$ to be the smallest weakly $\mathcal{F}$-closed subfield of $\boldsymbol{k}((t))^{\mathrm{LE}}$ that contains $\boldsymbol{k}(x)$, is closed under E, and contains $\log f$ whenever it contains f and $f>0$.

Proposition 7.3. *The field $\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$ is closed under the derivation, and under composition: if $f,g\in\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$ and $g>\boldsymbol{k}$, then $f'\in\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$ and $f\circ g\in\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$.*

Proof. Let K be the set of all $f\in\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$ such that $f'\in\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$. One checks immediately (using 3.9) that K is a weakly $\mathcal{F}$-closed subfield of $\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$, that K contains $\boldsymbol{k}(x)$, and that $\mathrm{E}(K)\subseteq K$ and $\log(K^{>0})\subseteq K$. Thus $K=\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$. This proves the assertion on the derivation. Next, let K_* be the set of all $f\in\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$ such that $f\circ g\in\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$ for all $g\in\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$ with $g>\boldsymbol{k}$. Again, the results of §6 imply easily that K_* is a weakly $\mathcal{F}$-closed subfield of $\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$ which contains $\boldsymbol{k}(x)$, and that $\mathrm{E}(K_*)\subseteq K_*$ and $\log(K_*^{>0})\subseteq K_*$. Hence $K_*=\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$, as desired.

(7.4) Remark. Note that each positive $u\in\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$ has an n^{th} root $u^{1/n}=\mathrm{E}(\frac{\log u}{n})$ in $\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$ for every $n>0$. When $\boldsymbol{k}=\mathbf{R}$ and $\mathcal{F}_m$ contains all $f\in\mathbf{R}[[X_1,\dots,X_m]]$ that converge on some neighborhood of the origin in $\mathbf{R}^m$, for all m, then the arguments of [5, 2.5 and 2.6] imply that $\mathbf{R}((t))^{\mathrm{LE},\mathcal{F}}$ is real closed.

Let now $\mathcal{F}_m^{conv}$ be the set of all $f\in\mathbf{R}[[X_1,\dots,X_m]]$ that converge on some neighborhood of the origin in $\mathbf{R}^m$, for all m. Then $\mathbf{R}((t))^{\mathrm{LE},\mathcal{F}^{conv}}$ is exactly the image of the natural embedding of $H(\mathbf{R}_{\mathrm{an,exp}})$ into $\mathbf{R}((t))^{\mathrm{LE}}$ that we considered in (3.11) and (6.29). (This is proved in the same way as corollary 3.12.) Note that by (6.29) and corollary 6.30, each series $>\mathbf{R}$ in $\mathbf{R}((t))^{\mathrm{LE},\mathcal{F}^{conv}}$ has its compositional inverse in this field as well.

(7.5) Subseries and Truncation

Let G be an ordered abelian group, put $K:=\boldsymbol{k}((G))$, and let $f=\sum_{g\in G}a_g g\in K$. Then any set $\Delta\subseteq G$ determines the *subseries* $\sum_{g\in\Delta}a_g g$ of f (an element of K), and any $h\in G$ determines the *truncation* $\sum_{g>h}a_g g$ of f (an element of K). We say that a set $F\subseteq K$ is *closed under subseries* (in K) if $f\in F$ whenever f is a subseries of some element of K. We say that $F \subseteq K$ is *truncation closed* if $f \in F$ whenever f is a truncation of some element of K. If $F \subseteq K$ is closed under subseries it is clearly also truncation closed. Note that for each set $S \subseteq G$ the additive subgroup $\boldsymbol{k}((S))$ of K is closed under subseries.

We now apply these notions to subsets of the series field $K = \boldsymbol{k}((G^{\mathrm{LE}}))$, and just write 'closed under subseries' and 'truncation closed' without referring every time to this ambient series field.

By construction all K_m are closed under subseries, and so is their union $\boldsymbol{k}((t))^{\mathrm{E}}$. Hence each L_n is closed under subseries, and thus their union $\boldsymbol{k}((t))^{\mathrm{LE}}$ also. The proofs in [6, §3] (where we work over $\boldsymbol{k} = \mathbf{R}$) give the following.

Proposition 7.6. *The field $\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$ is truncation closed.*

(7.7) LE-series with support of hereditarily finite type

In the case that $\mathcal{F}_m = \boldsymbol{k}[[X_1, \cdots, X_m]]$ for all m we shall explicitly describe $\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$ as the set of LE-series with support of "hereditarily finite type" (Proposition 7.12). We start with an excursion concerning subsets of finite type in ordered abelian groups.

Let G be an ordered abelian (multiplicative) group. A set $A \subseteq G$ is said to be *of finite type* (in G when we have to specify the ambient group) if there are $g, g_1, \dots, g_k \in G$ with $g_1, \dots, g_k < 1$ such that $A \subseteq g[g_1, \dots, g_k]$. In the following lemma we collect some basic facts on subsets of G of finite type.

Lemma 7.8. *Subsets of G of finite type are reverse well ordered. The union of finitely many subsets of G of finite type is of finite type. Let $A \subseteq G$ be of finite type in G. Then the following hold:*

(1) *If $A \subseteq G^{\leq 1}$, then there are $a_1, \dots, a_k \in \langle A \rangle^{<1}$ such that $A \subseteq [a_1, \dots, a_k]$.*
(2) *If $A \neq \emptyset$, then A has a largest element $\max A$, and there are $a_1, \dots, a_k \in \langle A \rangle^{<1}$ such that $A \subseteq (\max A)[a_1, \dots, a_k]$.*
(3) *If H is a subgroup of G and $A \subseteq H$, then A is of finite type in H.*

Proof. We leave the first two parts of the lemma as easy exercises, and only prove the three numbered assertions about $A \subseteq G$ of finite type in G. Let $A \subseteq G^{\leq 1}$. Take $g, g_1, \dots, g_p \in G$ with $g_1, \dots, g_p < 1$ such that $A \subseteq g[g_1, \dots, g_p]$. We equip $\mathbf{N}^p$ with the product partial order:

$$(i_1, \dots, i_p) \leq (j_1, \dots, j_p) \Longleftrightarrow i_1 \leq j_1 \text{ and } \dots \text{ and } i_p \leq j_p \ .$$

It is wellknown that each subset of $\mathbf{N}^p$ has only finitely many minimal elements, in particular, the set I of minimal elements of $\{(i_1, \dots, i_p) \in \mathbf{N}^p : g g_1^{i_1} \cdots g_p^{i_p} \in A\}$ is finite. For each $i = (i_1, \dots, i_p) \in I$, put $h_i = g g_1^{i_1} \cdots g_p^{i_p}$, so that $h_i \leq 1$. Since $A \subseteq G^{\leq 1}$ it follows immediately that

$$A \subseteq \bigcup_{i \in I} h_i[g_1, \dots, g_p] \subseteq [g_1, \dots, g_p, (h_i)_{i \in I}] \ .$$

Simplifying notation, let $\{b_1, \ldots, b_q\} = \{g_1, \ldots, g_p\} \cup \{h_i : i \in I\}$, so that we have $A \subseteq [b_1, \ldots, b_q]$. Put

$$J := \text{ set of minimal elements of } \{(j_1, \ldots, j_q) \in \mathbf{N}^q : 1 \neq b_1^{j_1} \cdots b_q^{j_q} \in \langle A \rangle\} \ .$$

Thus the set J is finite. For $j = (j_1, \ldots, j_q) \in J$, put $a_j := b_1^{j_1} \cdots b_q^{j_q}$. It is easy to check that then $A \subseteq [(a_j)_{j \in J}] \subseteq \langle A \rangle^{\leq 1}$.

For (2), assume $A \neq \emptyset$. Then A has a largest element $\max A$ by the first part of the lemma. Then $(\max A)^{-1} A \subseteq G^{\leq 1}$ is of finite type, and the desired result now follows from (1) applied to $(\max A)^{-1} A$.

Property (3) is an immediate consequence of (2).

(7.9) Remark. Concerning the first statement of the lemma, one can easily make this more precise as follows: the reversed order type of a subset of G of the form $g[g_1, \ldots, g_k]$ with $g_1, \ldots, g_k \in G^{<1}$ is at most ω^k.

One might ask if, conversely, each subset of G with reversed order type ω is of finite type, assuming G is finitely generated to avoid trivial counterexamples. While this is clearly the case for cyclic G, it is false in general: let $G := x^{\mathbf{Z}}\,\mathrm{E}(\mathbf{Z}x) \subseteq G_1$ (generated by x and $\mathrm{E}(x)$), and note that the subset $\{x^{-m}\,\mathrm{E}(-nx) : n \leq m\sqrt{2}\}$ of $G^{\leq 1}$ has reversed order type ω, but is not of finite type.

(7.10) For any ordered multiplicative abelian group G we let $\boldsymbol{k}((G))^{\mathrm{ft}}$ consist of all series $f \in \boldsymbol{k}((G))$ such that $\mathrm{Supp}\, f$ is of finite type, that is $\mathrm{Supp}\, f \subseteq g[g_1, ..., g_k]$ for suitable $g, g_1, \ldots, g_k \in G$ with $g_1, ..., g_k < 1$. It is easy to see that $\boldsymbol{k}((G))^{\mathrm{ft}}$ is a subfield of $\boldsymbol{k}((G))$. The subfield $\boldsymbol{k}((t))^{\mathrm{E,ft}}$ of $\boldsymbol{k}((t))^{\mathrm{E}}$ consisting of the **E-series with support of hereditarily finite type** is obtained as a union $\bigcup_n K_n^{\mathrm{ft}}$ where $K_n^{\mathrm{ft}} := \boldsymbol{k}((G_n^{\mathrm{ft}}))^{\mathrm{ft}}$, and $(G_n^{\mathrm{ft}}))_{n \in \mathbf{N}}$ is a certain increasing sequence of ordered groups, with each G_n^{ft} an ordered subgroup of G_n.

We set $G_0^{\mathrm{ft}} := G_0 = x^{\boldsymbol{k}}$. For inductive reasons we also let $G_{-1}^{\mathrm{ft}} := \{1\} \subseteq G_0^{\mathrm{ft}}$. Assume that for a certain n we have defined G_{n-1}^{ft} and G_n^{ft} as ordered subgroups of G_{n-1} and G_n, respectively, and that $G_{n-1}^{\mathrm{ft}} = G_{n-1} \cap G_n^{\mathrm{ft}}$. (This is clearly the case for $n = 0$.) Thus G_{n-1}^{ft} is a convex subgroup of G_n^{ft}, and $G_n^{\mathrm{ft}} = G_n \cap K_n^{\mathrm{ft}}$.

Then we define G_{n+1}^{ft} as follows. Let

$$A_n^{\mathrm{ft}} := \{f \in K_n^{\mathrm{ft}} : \mathrm{Supp}\, f > G_{n-1}^{\mathrm{ft}}\} = A_n \cap K_n^{\mathrm{ft}},$$

an additive subgroup of K_n^{ft} with $G_n^{\mathrm{ft}} \cap \mathrm{E}(A_n^{\mathrm{ft}}) = \{1\}$. Now put $G_{n+1}^{\mathrm{ft}} := G_n^{\mathrm{ft}}\,\mathrm{E}(A_n^{\mathrm{ft}})$, ordered as a subgroup of G_{n+1}. It is easy to see that the inductive hypothesis above remains true when n is replaced by $n+1$. This finishes the construction of the sequences (G_n^{ft}) and (K_n^{ft}). It is easy to see that $\mathrm{E}(K_n^{\mathrm{ft}}) \subseteq K_{n+1}^{\mathrm{ft}}$ for all n, and thus $\boldsymbol{k}((t))^{\mathrm{E,ft}}$ is an ordered exponential subfield of $\boldsymbol{k}((t))^{\mathrm{E}}$. Note also that with $G^{\mathrm{E,ft}} := \bigcup G_n^{\mathrm{ft}}$, the **group of E-monomials of hereditarily finite type**, we have

$$\boldsymbol{k}((t))^{\mathrm{E,ft}} = \boldsymbol{k}((G^{\mathrm{E,ft}}))^{\mathrm{ft}} \subseteq \boldsymbol{k}((G^{\mathrm{E,ft}})) \ .$$

(7.11) Next we close off under logarithms. One verifies easily that $\Phi(G_n^{\mathrm{ft}}) \subseteq G_{n+1}^{\mathrm{ft}}$, $\Phi(A_n^{\mathrm{ft}}) \subseteq A_{n+1}^{\mathrm{ft}}$ and $\Phi(K_n^{\mathrm{ft}}) \subseteq K_{n+1}^{\mathrm{ft}}$. As in 2.6 this implies $\Phi(\boldsymbol{k}((t))^{\mathrm{E,ft}}) \subseteq \boldsymbol{k}((t))^{\mathrm{E,ft}}$, and each positive element of $\Phi(\boldsymbol{k}((t))^{\mathrm{E,ft}})$ has its logarithm in $\boldsymbol{k}((t))^{\mathrm{E,ft}}$. Because of these two facts we can now close off under logarithms as in 2.7. Put

$$L_i^{\mathrm{ft}} := \eta_i^{-1}\boldsymbol{k}((t))^{\mathrm{E,ft}} \ .$$

Then L_i^{ft} is an ordered exponential subfield of L_i, and the logarithm of each positive element of L_i^{ft} lies in L_{i+1}^{ft}. Finally, put

$$\boldsymbol{k}((t))^{\mathrm{LE,ft}} := \bigcup_i L_i^{\mathrm{ft}},$$

the field of **LE-series with support of hereditarily finite type**, an ordered logarithmic-exponential subfield of $\boldsymbol{k}((t))^{\mathrm{LE}}$. Let $G^{\mathrm{LE,ft}} := \bigcup \eta_i^{-1} G^{\mathrm{E,ft}}$ be the **group of LE-monomials of hereditarily finite type**. Then we have

$$\boldsymbol{k}((t))^{\mathrm{LE,ft}} = \boldsymbol{k}((G^{\mathrm{LE,ft}}))^{\mathrm{ft}} \subseteq \boldsymbol{k}((G^{\mathrm{LE,ft}})) \ .$$

Note that Φ maps $\boldsymbol{k}((t))^{\mathrm{LE,ft}}$ onto itself. It is also clear from the construction that $\boldsymbol{k}((t))^{\mathrm{LE,ft}}$ is closed under subseries.

We can now state a strong version of the fact that the support of each series in $\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$ is of finite type. (See [6, 5.13] for a much weaker version.)

Proposition 7.12. *Let $\mathcal{F}_m = \boldsymbol{k}[[X_1, \cdots, X_m]]$ for all m. Then*

$$\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}} = \boldsymbol{k}((t))^{\mathrm{LE,ft}} \ .$$

Thus for each $f \in \boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$ there are LE-monomials $g, g_1, \ldots, g_k$ of hereditarily finite type with $g_1, \ldots, g_k < 1$ such that Supp $f \subseteq g[g_1, \ldots, g_k]$.

Proof. We first show that each field $K_n^{\mathrm{ft}} = \boldsymbol{k}((G_n^{\mathrm{ft}}))^{\mathrm{ft}}$ is weakly $\mathcal{F}$-closed. Let $\epsilon_1, \cdots, \epsilon_m$ be infinitesimals of K_n^{ft} and let $F \in \mathcal{F}_m$. Then the sets Supp ϵ_i are subsets of $(G_n^{\mathrm{ft}})^{<1}$ of finite type, hence their union U is a subset of $(G_n^{\mathrm{ft}})^{<1}$ of finite type. Since Supp $F(\epsilon_1, \cdots, \epsilon_m) \subseteq [U]$ it follows that Supp $F(\epsilon_1, \cdots, \epsilon_m)$ is a subset of G_n^{ft} of finite type, that is, $F(\epsilon_1, \cdots, \epsilon_m) \in K_n^{\mathrm{ft}}$.

Therefore $\boldsymbol{k}((t))^{\mathrm{E,ft}}$ is weakly $\mathcal{F}$-closed, hence the fields L_i^{ft} are weakly $\mathcal{F}$-closed, and thus $\boldsymbol{k}((t))^{\mathrm{LE,ft}}$ is weakly $\mathcal{F}$-closed. This proves the inclusion

$$\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}} \subseteq \boldsymbol{k}((t))^{\mathrm{LE,ft}} \ .$$

For the reverse inclusion we show first by induction on n that $K_n^{\mathrm{ft}} \subseteq \boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$. To see why this is true for $n = 0$, first note that all powers $x^r = \mathrm{E}(r \log x)$ with $r \in \boldsymbol{k}$ belong to $\boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$. Now consider a series $f = \sum a_r x^r$ in $K_0^{\mathrm{ft}} = \boldsymbol{k}((x^{\boldsymbol{k}}))^{\mathrm{ft}}$. Then Supp $f \subseteq x^{r_0}[x^{r_1}, \ldots, x^{r_m}]$ with $r_1, \ldots, r_m < 0$. Thus each r with $a_r \neq 0$ can be written as $r = r_0 + i_{r,1} r_1 + \cdots + i_{r,m} r_m$ for suitable $i_{r,1}, \ldots, i_{r,m} \in \mathbf{N}$. Put $F(X_1, \ldots, X_m) := \sum a_r X_1^{i_{r,1}} \cdots X_m^{i_{r,m}} \in \boldsymbol{k}[[X_1, \ldots, X_m]]$, where the sum is over all $r \in \boldsymbol{k}$ such that $a_r \neq 0$. Then $f = x^{r_0} F(x^{r_1}, \ldots, x^{r_m}) \in \boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$.

Assume inductively that $K_n^{\mathrm{ft}} \subseteq \boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$. Then $G_{n+1}^{\mathrm{ft}} = G_n^{\mathrm{ft}} E(A_n^{\mathrm{ft}}) \subseteq \boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$. Consider a series $f = \sum a_\mu \mu$ in $K_{n+1}^{\mathrm{ft}} = \boldsymbol{k}((G_{n+1}^{\mathrm{ft}}))^{\mathrm{ft}}$. Thus Supp $f \subseteq \mu_0[\mu_1, \dots, \mu_m]$ with all $\mu_i \in G_{n+1}^{\mathrm{ft}}$ and $\mu_1, \dots, \mu_m < 1$. Write each μ with $a_\mu \neq 0$ as $\mu = \mu_0 \mu_1^{i_{\mu,1}} \cdots \mu_m^{i_{\mu,m}}$ for suitable $i_{\mu,1}, \dots, i_{\mu,m} \in \mathbf{N}$. Put

$$F(X_1, \dots, X_m) := \sum a_\mu X_1^{i_{\mu,1}} \cdots X_m^{i_{\mu,m}} \in \boldsymbol{k}[[X_1, \dots, X_m]],$$

where the sum is over all $\mu \in G_{n+1}^{\mathrm{ft}}$ such that $a_\mu \neq 0$. Then $f = \mu_0 F(\mu_1, \dots, \mu_m) \in \boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$. This inductive argument shows that $\boldsymbol{k}((t))^{\mathrm{E},\mathrm{ft}} \subseteq \boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$. Composing on the right with $\log x$ and its iterates gives $L_i^{\mathrm{ft}} \subseteq \boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$ for all i, and thus $\boldsymbol{k}((t))^{\mathrm{E},\mathrm{ft}} \subseteq \boldsymbol{k}((t))^{\mathrm{LE},\mathcal{F}}$, as required.

The proof of the next result depends heavily on lemma 7.8:

Proposition 7.13. *For all n we have*

$$K_n \cap \boldsymbol{k}((t))^{\mathrm{LE},\mathrm{ft}} = K_n^{\mathrm{ft}} \textit{ and } L_n \cap \boldsymbol{k}((t))^{\mathrm{LE},\mathrm{ft}} = L_n^{\mathrm{ft}} \ .$$

Proof. We first show that $K_n \cap K_{n+1}^{\mathrm{ft}} = K_n^{\mathrm{ft}}$. The inclusion $\supseteq$ is clear. Let $f \in K_n \cap K_{n+1}^{\mathrm{ft}}$. Then Supp f is of finite type in G_{n+1}^{ft} and is also contained in the subgroup $G_n \cap G_{n+1}^{\mathrm{ft}} = G_n^{\mathrm{ft}}$, hence is of finite type in G_n^{ft} by lemma 7.8 (3). Thus $f \in K_n^{\mathrm{ft}}$. This proves the equality $K_n \cap K_{n+1}^{\mathrm{ft}} = K_n^{\mathrm{ft}}$, from which one easily obtains

$$K_n \cap \boldsymbol{k}((t))^{\mathrm{E},\mathrm{ft}} = K_n^{\mathrm{ft}} \ .$$

To go from here to the first identity stated in the proposition we will next show by induction on n that

$$\Phi^p(G_n) \cap G_{n+p}^{\mathrm{ft}} = \Phi^p(G_n^{\mathrm{ft}}) \text{ and } \Phi^p(K_n) \cap K_{n+p}^{\mathrm{ft}} = \Phi^p(K_n^{\mathrm{ft}}) \quad , \qquad \text{for all } p > 0 \ .$$

For $n = 0$ the first identity is immediate since $G_0 = G_0^{\mathrm{ft}}$. For the second identity we consider an element $f \in K_0$ with $\Phi^p(f) \in K_p^{\mathrm{ft}}$, and we have to show that then $f \in K_0^{\mathrm{ft}}$. Write $f = \sum a_r x^r$ in K_0 where the sum is over the $r \in \boldsymbol{k}$ (all $a_r \in \boldsymbol{k}$). Then $\Phi^p(f) = \sum a_r \Phi^p(x^r) \in K_p^{\mathrm{ft}}$ implies that $\{\Phi^p(x^r) : a_r \neq 0\}$ is of finite type in G_p^{ft}, hence of finite type in its subgroup $\Phi^p(G_0)$ by lemma 7.8 (3). Therefore Supp $(f) = \{x^r : a_r \neq 0\}$ is of finite type in G_0. It follows that $f \in K_0^{\mathrm{ft}}$.

Let $n > 0$ and $p > 0$, and assume these identities hold for lower values of n. The two inclusions $\supseteq$ are clear from (7.10) and (7.11). Let $g \in G_n$ such that $\Phi^p(g) \in G_{n+p}^{\mathrm{ft}}$. For the first identity we have to show that then $g \in G_n^{\mathrm{ft}}$. Write $g = \mu \, \mathrm{E}(a)$ with $\mu \in G_{n-1}$ and $a \in A_{n-1}$. Then

$$\begin{aligned} \Phi^p(g) = \Phi^p(\mu)\, \mathrm{E}(\Phi^p a) \in G_{n+p}^{\mathrm{ft}} &= G_{(n-1)+p}^{\mathrm{ft}}\, \mathrm{E}(A_{(n-1)+p}^{\mathrm{ft}}) \\ &\subseteq G_{n+p} = G_{(n-1)+p}\, \mathrm{E}(A_{(n-1)+p}) \ . \end{aligned}$$

The uniqueness properties of the product decompositions of G_{n+p}^{ft} and G_{n+p} displayed here imply that $\Phi^p(\mu) \in G_{(n-1)+p}^{\mathrm{ft}}$ and $\Phi^p a \in A_{(n-1)+p}^{\mathrm{ft}} \subseteq K_{(n-1)+p}^{\mathrm{ft}}$. By the inductive assumption this gives $\mu \in G^{\mathrm{ft}}_{n-1}$ and $a \in K^{\mathrm{ft}}_{n-1} \cap A_{n-1} = A^{\mathrm{ft}}_{n-1}$, and thus $g \in G^{\mathrm{ft}}_n$. For the second identity we consider an element $f \in K_n$ with $\Phi^p(f) \in K^{\mathrm{ft}}_{n+p}$, and we have to show that then $f \in K^{\mathrm{ft}}_n$. Write $f = \sum c_g g$ in $\boldsymbol{k}((G_n))$ where the sum is over the $g \in G_n$ (all $c_g \in \boldsymbol{k}$). Then $\Phi^p(f) = \sum c_g \Phi^p(g) \in K^{\mathrm{ft}}_{n+p}$ implies that all $\Phi^p(g)$ with $c_g \neq 0$ belong to $G_{n+p} \cap K^{\mathrm{ft}}_{n+p} = G^{\mathrm{ft}}_{n+p}$, and thus by the first equality all g with $c_g \neq 0$ belong to G^{ft}_n, that is Supp $(f) \subseteq G^{\mathrm{ft}}_n$. Also, $\{\Phi^p(g) : c_g \neq 0\}$ is of finite type in G^{ft}_{n+p}, hence of finite type in the subgroup $\Phi^p(G^{\mathrm{ft}}_n)$ by lemma 7.8 (3). Therefore Supp $(f) = \{g : c_g \neq 0\}$ is of finite type in G^{ft}_n. It follows that $f \in K^{\mathrm{ft}}_n$. This finishes the proof of the two identities. The second of these identities implies (for $p > 0$) that

$$\begin{aligned}\Phi^p(K_n) \cap \boldsymbol{k}((t))^{\mathrm{LE,ft}} &= \Phi^p(K_n) \cap \left(K_{n+p} \cap \boldsymbol{k}((t))^{\mathrm{LE,ft}}\right) = \Phi^p(K_n) \cap K^{\mathrm{ft}}_{n+p} \\ &= \Phi^p(K^{\mathrm{ft}}_n)\end{aligned}$$

Applying Φ^{-p} this gives $K_n \cap L^{\mathrm{ft}}_p = K^{\mathrm{ft}}_n$, and by letting here p go to infinity we obtain the first identity of the proposition. By letting n go to infinity in this first identity we obtain

$$\boldsymbol{k}((t))^{\mathrm{E}} \cap \boldsymbol{k}((t))^{\mathrm{LE,ft}} = \boldsymbol{k}((t))^{\mathrm{E,ft}},$$

which is the special case $n = 0$ of the second identity of the proposition. Applying Φ^{-n} to both sides of this special case we obtain the second identity in general.

Integration and compositional inversion in $\boldsymbol{k}((t))^{\mathrm{LE,ft}}$

(7.14) It follows immediately from Propositions 7.3, 7.12 and 7.13 that the derivation on $\boldsymbol{k}((t))^{\mathrm{LE}}$ maps each K^{ft}_n and each L^{ft}_n into itself. But this is only a small step in the direction of showing that $\boldsymbol{k}((t))^{\mathrm{LE,ft}}$ is closed under integration and compositional inverses. In trying to adapt the proofs in sections 5 and 6 we run into problems, and to get around these we have to redo a number of earlier items "with finite type bounds on supports". That is, when a bound of finite type on the support of a series is given and an operation is applied to the series, the result should also have support contained in a set that is of finite type and entirely determined by the original bound. The following considerations will make this general idea more concrete. We start with redoing the material in (1.3) on small operators, by imposing everywhere bounds of finite type.

Let D be a $\boldsymbol{k}$-linear subspace of $\boldsymbol{k}((G))$ such that for each family $(a_i)_{i\in I}$ of elements in D for which $\sum a_i$ exists we have $\sum a_i \in D$. Put $D^{\mathrm{ft}} := D \cap \boldsymbol{k}((G))^{\mathrm{ft}}$. Let us say that an operator $\mathrm{P} : D \longrightarrow D$ is *small of finite type* if there exists $R \subseteq G^{<1}$ of finite type in G such that Supp $\mathrm{P}(s) \subseteq R \cdot$ Supp s for all $s \in D$. It follows that then P is a small operator, and that for all n and all $s \in D$:

$$\mathrm{Supp}\ \mathrm{P}^n(s) \subseteq [R]\mathrm{Supp}\ s\ .$$

Thus for each sequence (c_n) of coefficients in $\boldsymbol{k}$ the operator $\sum c_n \mathrm{P}^n : D \longrightarrow D$ maps D^{ft} into itself, and satisfies Supp $(\sum c_n \mathrm{P}^n(s)) \subseteq [R]$Supp s for all $s \in D$. In particular, the perturbation $\mathrm{I} - \mathrm{P}$ of the identity maps D^{ft} bijectively onto itself, since its inverse $\sum \mathrm{P}^n$ maps D^{ft} into itself.

Taking reciprocals preserves finite type bounds on supports as follows, where G is as usual an ordered abelian multiplicative group:

Lemma 7.15. *Let $S \subseteq G$ be of finite type, $a \in S$. Then there is a set $S(a) \subseteq G$ of finite type such that if $f \in \boldsymbol{k}((G))$ and Supp $f \subseteq S$, $\operatorname{Lm} f = a$, then Supp $f^{-1} \subseteq S(a)$.*

Proof. By 7.8(2) we have $S^{\geq a} \subseteq a[a_1, \dots, a_k]$ for certain $a_1, \dots, a_k < 1$ in G. Then one easily checks that if $f \in \boldsymbol{k}((G))$ and Supp $f \subseteq S$, $\operatorname{Lm} f = a$, then Supp $f^{-1} \subseteq a^{-1}[a_1, \dots, a_k]$.

(7.16) To integrate and take compositional inverses in $\boldsymbol{k}((t))^{\mathrm{LE,ft}}$ we need to know that taking derivatives preserves finite type bounds on supports in a rather strong way. For $f \in K_0$ we clearly have Supp $f^{(p)} \subseteq x^{-p}$Supp f for all $p \in \mathbf{N}$. For $f \in K_n^{\mathrm{ft}}$ with $n > 0$ some further preparation is necessary.

One complication in keeping track of supports is illustrated by the series

$$f := (1 - x\,\mathrm{E}(-x))^{-1} = \sum_{n=0}^{\infty} x^n\,\mathrm{E}(-nx) \in K_1^{\mathrm{ft}}.$$

Then Supp $f = \{x^n\,\mathrm{E}(-nx) : n \in \mathbf{N}\} \subseteq x^{\boldsymbol{k}}\,\mathrm{E}(A_0^{\mathrm{ft}})$ is of finite type, but Supp f is not contained in any set of the form $S_0 S_1$ where $S_0 \subseteq x^{\boldsymbol{k}}$ and $S_1 \subseteq \mathrm{E}(A_0^{\mathrm{ft}})$ are of finite type. It is for this reason that we introduce the notion of *nice subgroup of* G_n^{ft}, which allows us to state and prove the important lemma 7.20.

(7.17) Recall that the *weight* of a monomial $X_1^{e(1)} \cdots X_j^{e(j)}$ (where $e(1), \dots, e(j) \in \mathbf{N}$) is by definition the natural number $e(1) + 2e(2) + \cdots + je(j)$, and that a polynomial in $X_1, \dots, X_j$ over a field is said to have weight d (where $d \in \mathbf{N}$) if all monomials occurring in it have weight d. This is used below as follows: for $f, g \in \boldsymbol{k}((t))^{\mathrm{LE}}$ and $p \in \mathbf{N}$ the pth derivative of $f\,\mathrm{E}(g)$ is given by

$$(f\,\mathrm{E}\,g)^{(p)} = \Big(\sum_{i+j=p} f^{(i)} F_{i,j}(g', \dots, g^{(j)})\Big)\,\mathrm{E}(g),$$

where each $F_{i,j}(X_1, \dots, X_j) \in \mathbf{Q}[X_1, \dots, X_j]$ has weight j and does not depend on f and g. (This is easily shown by induction on p.)

(7.18) We now single out certain subgroups of G_n^{ft} as being *nice*. For $n = 0$ we declare the nice subgroups of $G_0^{\mathrm{ft}} = x^{\boldsymbol{k}}$ to be exactly its finitely generated subgroups that contain x. Assume the class of nice subgroups of G_n^{ft} has been defined as a subclass of the class of all subgroups of G_n^{ft}. Then we say that a subgroup H of $G_{n+1}^{\mathrm{ft}} = G_n^{\mathrm{ft}}\,\mathrm{E}(A_n^{\mathrm{ft}})$ is nice if $H = H' \cdot \mathrm{E}(C)$ for some nice subgroup H' of G_n^{ft} and some finitely generated (additive) subgroup C of A_n^{ft} such that Supp $a \subseteq H'$ for all $a \in C$. Note that then H' and C are uniquely determined by H, since $H' = H \cap G_n^{\mathrm{ft}}$ and $\mathrm{E}(C) = H \cap \mathrm{E}(A_n^{\mathrm{ft}})$.

Lemma 7.19. *Any finite subset of G_n^{ft}, and thus any subset of finite type in this group, is contained in some nice subgroup of G_n^{ft}.*

Proof. By induction on n. The case $n = 0$ is trivial. Let $f_1, \dots, f_k \in G_{n+1}^{\mathrm{ft}}$. We have to find a nice subgroup of G_{n+1}^{ft} containing $f_1, \dots, f_k$. Write $f_i = g_i\,\mathrm{E}(a_i)$

with $g_i \in G_n^{\mathrm{ft}}$ and $a_i \in A_n^{\mathrm{ft}}$, for $i = 1, \ldots, k$. Take elements $h_0, h_1, \ldots, h_l \in G_n^{\mathrm{ft}}$ with $h_1 \leq 1, \ldots, h_l \leq 1$ such that Supp $a_i \subseteq h_0[h_1, \ldots, h_l]$ for $i = 1, \ldots, k$. Inductively we may take a nice subgroup H' of G_n^{ft} such that $\{g_1, \ldots, g_k, h_0, h_1, \ldots, h_l\} \subseteq H'$. Let C be the additive subgroup of A_n^{ft} generated by $a_1, \ldots, a_k$. Then $H := H' \, \mathrm{E}(C)$ is a nice subgroup of G_{n+1}^{ft} containing $f_1, \ldots, f_k$.

Lemma 7.20. *Let H be a nice subgroup of G_{n+1}^{ft}, with $H = H' \, \mathrm{E}(C)$ as in (7.18) above. Then there are $h_0, h_1, \ldots, h_k \in H'$ with $h_1 \leq 1, \ldots, h_k \leq 1$ such that for all $f \in K_{n+1}$ with Supp $f \subseteq H$ we have Supp $f^{(p)} \subseteq h_0^p[h_1, \ldots, h_k]$Supp f for all p, and thus Supp $f^{(p)} \subseteq H$ for all p.*

Proof. Let $f \in K_{n+1}$ and Supp $f \subseteq H$. Write $f = \sum_a f_a \, \mathrm{E}(a)$ where the sum is over all $a \in C$, and all f_a belong to K_n and satisfy Supp $f_a \subseteq H'$. Taking the pth derivative of f gives

$$f^{(p)} = \sum_a \Big(\sum_{i+j=p} f_a^{(i)} F_{i,j}(a', \ldots, a^{(j)}) \Big) \, \mathrm{E}(a) \tag{1}$$

where the polynomials $F_{i,j} \in \mathbf{Q}[X_1, \ldots, X_j]$ are as in (7.17). The properties of H' and C imply that there are $g_0, g_1, \ldots, g_r \in H'$ with $g_0 \geq 1$ and $g_1 \leq 1, \ldots, g_r \leq 1$ such that Supp $a \subseteq g_0[g_1, \ldots, g_r]$ for all $a \in C$. We now proceed by induction on n. Consider first the case $n = 0$. For every monomial $X_1^{e(1)} \cdots X_j^{e(j)}$ occurring in $F_{i,j}$ with $i + j = p$ we have $e(1) + 2e(2) + \cdots + je(j) = j$, and thus, for all $a \in C$:

$$\begin{aligned} \mathrm{Supp}\ \big((a')^{e(1)} \cdots (a^{(j)})^{e(j)}\big) &\subseteq x^{-e(1)} g_0^{e(1)} x^{-2e(2)} g_0^{e(2)} \cdots x^{-me(m)} g_0^{e(m)} [g_1, \ldots, g_r] \\ &\subseteq x^{-j} g_0^{e}[g_1, \ldots, g_r], p \geq e := e(1) + e(2) + \cdots + e(m) \\ &\subseteq x^{-j} g_0^{p}[g_0^{-1}, g_1, \ldots, g_r]. \end{aligned} \tag{2}$$

Also Supp $f_a^{(i)} \subseteq x^{-i}$Supp f_a for all $a \in C$. In combination with (1) and (2) this gives Supp $f^{(p)} \subseteq x^{-p} g_0^p [g_0^{-1}, g_1, \ldots, g_r]$Supp f, and thus the lemma holds with $h_0 := x^{-1} g_0$ and $\{h_1, \ldots, h_k\} := \{g_0^{-1}, g_1, \ldots, g_r\}$. Suppose next that $n > 0$, and that the result holds for lower values of n. Thus there are elements $\lambda_0, \lambda_1, \ldots, \lambda_s \in H' \cap G_{n-1}^{\mathrm{ft}}$ with $\lambda_0 \geq 1$ and $\lambda_1 \leq 1, \ldots, \lambda_s \leq 1$ such that for all $g \in K_n$ with Supp $g \subseteq H'$ and for all m

$$\mathrm{Supp}\ g^{(m)} \subseteq \lambda_0^m [\lambda_1, \ldots, \lambda_s] \mathrm{Supp}\ g.$$

In particular for all $a \in C$ and $m > 0$:

$$\begin{aligned} \mathrm{Supp}\ a^{(m)} &\subseteq \lambda_0^m [\lambda_1, \ldots, \lambda_s] \mathrm{Supp}\ a \\ &\subseteq \lambda_0^m g_0 [g_1, \ldots, g_r, \lambda_1, \ldots, \lambda_s]. \end{aligned}$$

A computation as in (2) above then shows that for each monomial $X_1^{e(1)} \cdots X_j^{e(j)}$ occurring in $F_{i,j}$ with $i + j = p$ we have (for $a \in C$)

$$\begin{aligned} \mathrm{Supp}\ \big((a')^{e(1)} \cdots (a^{(j)})^{e(j)}\big) &\subseteq \lambda_0^j g_0^e [\lambda_1, \ldots, \lambda_s, g_1, \ldots, g_r] \\ &\subseteq \lambda_0^j g_0^p [g_0^{-1}, g_1, \ldots, g_r, \lambda_1, \ldots, \lambda_s]. \end{aligned} \tag{3}$$

Also $\mathrm{Supp}\, f_a \subseteq H'$ for $a \in C$, and thus for i, j with $i + j = p$

$$\mathrm{Supp}\, f_a^{(i)} \subseteq \lambda_0^i[\lambda_1, \dots, \lambda_s]\mathrm{Supp}\, f_a.$$

In combination with (1) and (3) this shows that the lemma holds with $h_0 := \lambda_0 g_0$ and $\{h_1, \dots, h_k\} := \{g_0^{-1}, g_1, \dots, g_r, \lambda_1, \dots, \lambda_s\}$.

We can now easily handle compositional inversion in $\boldsymbol{k}((t))^{\mathrm{LE,ft}}$:

Proposition 7.21. *Let $g \in \boldsymbol{k}((t))^{\mathrm{LE}}$, $g > \boldsymbol{k}$. Then there is $f \in \boldsymbol{k}((t))^{\mathrm{LE,ft}}$ such that $f \circ g = x$.*

Proof. This amounts to proving the "finite type" versions of 6.23, 6.24 and 6.25, followed by the same argument as in (6.26). These "finite type" versions are as follows:

(1) Let $\epsilon \in K_n^{\mathrm{ft}}$ and $\mathrm{Lm}\, \epsilon < x$. Then there exists $\delta \in K_0^{\mathrm{ft}}$ with $\mathrm{Lm}\, \delta < x$ such that $(x + \delta) \circ (x + \epsilon) = x + \epsilon^*$ with $\mathrm{Lm}\, \epsilon^* < G_0$.
(2) Let $0 < m \leq n$, and let $\epsilon \in K_n^{\mathrm{ft}}$ with $\mathrm{Lm}\, \epsilon < G_{m-1}$. Then there exists $\delta \in K_m^{\mathrm{ft}}$ with $\mathrm{Lm}\, \delta < G_{m-1}$ such that $(x + \delta) \circ (x + \epsilon) = x + \epsilon^*$ with $\mathrm{Lm}\, \epsilon^* < G_m$. In particular, $(x + \delta) \circ (x + \epsilon) = x$ if $m = n$.
(3) For each infinitesimal $\epsilon \in K_n^{\mathrm{ft}}$ there is an infinitesimal $\delta \in K_n^{\mathrm{ft}}$ such that $(x + \delta) \circ (x + \epsilon) = x$.

To obtain (1) we imitate the proof of lemma 6.23, and first reduce to the case that $\epsilon \in K_0^{\mathrm{ft}}$. Then the operator $\mathrm{P}_\epsilon : D \to D$ defined in that proof is small of finite type. Thus δ as defined at the end of that proof belongs to $D^{\mathrm{ft}} \subseteq K_0^{\mathrm{ft}}$. To establish (2) we imitate the proof of lemma 6.24, and first reduce to the case that $\epsilon \in K_m^{\mathrm{ft}}$, $\epsilon \neq 0$, and consider the operator $\mathrm{P}_\epsilon : D \to D$ defined there. The key step is now to take a nice subgroup H of G_m^{ft} such that $\mathrm{Supp}\, \epsilon \subseteq H$, and to consider the restriction of P_ϵ to $D(H) := \{\delta \in D : \mathrm{Supp}\, \delta \subseteq H\}$. By lemma 7.20 there exist $h_0, h_1, \dots, h_k \in H'$ with $h_1 \leq 1, \dots, h_k \leq 1$ such that $\mathrm{Supp}\, \delta^{(p)} \subseteq h_0^p[h_1, \dots, h_k]\mathrm{Supp}\, \delta \subseteq H$ for all $\delta \in D(H)$ and all p. Write $\epsilon = \mathrm{Lm}(\epsilon)a$, with $\mathrm{Supp}\, a \subseteq H^{\leq 1}$. Thus for all $\delta \in D(H)$ and all $p > 0$

$$\begin{aligned} \mathrm{Supp}\, \delta^{(p)}\epsilon^p &\subseteq (h_0\, \mathrm{Lm}(\epsilon))^p[h_1, \dots, h_k][\mathrm{Supp}\, a]\mathrm{Supp}\, \delta \\ &\subseteq h_0\, \mathrm{Lm}(\epsilon)[(h_0\, \mathrm{Lm}(\epsilon), h_1, \dots, h_k][\mathrm{Supp}\, a]\mathrm{Supp}\, \delta \\ &\subseteq H. \end{aligned}$$

Since $h_0\, \mathrm{Lm}(\epsilon) < G_{m-1}$ it follows that $\mathrm{P}_\epsilon(D(H)) \subseteq D(H)$, and that the restriction $\mathrm{P}_\epsilon|D(H) : D(H) \to D(H)$ is small of finite type. Hence $\delta := (\mathrm{I}+\mathrm{P}_\epsilon)^{-1}(-\epsilon)$ belongs to $D(H)^{\mathrm{ft}}$, and therefore $\delta \in K_m^{\mathrm{ft}}$ as desired. The proof of (3) is just like that of corollary 6.25, appealing to (1) and (2) instead of 6.23 and 6.24.

Integrating in $\boldsymbol{k}((t))^{\mathrm{LE,ft}}$ requires much more care in keeping track of supports.

Proposition 7.22. *Each element of $\boldsymbol{k}((t))^{\mathrm{LE,ft}}$ has an integral in $\boldsymbol{k}((t))^{\mathrm{LE,ft}}$.*

Proof. We imitate Section 5, imposing suitable finite type bounds on supports. Given some n we assume inductively that for each nice subgroup H of G_n^{ft} and each set $S \subseteq H$ of finite type there is a set $\int S \subseteq H$ of finite type such that each $g \in K_n^{\text{ft}}$ with residue 0 and Supp $g \subseteq S$ has an integral $f \in K_n^{\text{ft}}$ with Supp $f \subseteq \int S$. (For $n = 0$ this is clearly the case by taking $\int S := xS$.) Now let H be a nice subgroup of G_{n+1}^{ft} and let $S \subseteq H$ be of finite type. Write $H = H' \, \mathrm{E}(C)$ as in (7.18), and let $g \in K_{n+1}^{\text{ft}}$ be such that Supp $g \subseteq S$. Write $g = \sum_a g_a \, \mathrm{E}(a)$ where a ranges over C and all $g_a \in K_n^{\text{ft}}$. As in 5.3 and 5.4 we find an integral $f = \sum_a f_a \, \mathrm{E}(a)$ in K_{n+1} of g. Here $f_0 \in K_n$ is an integral of g_0, and for each $a \neq 0$ the series $f_a \in K_n$ is the solution of the equation $y - \mu(a)y' = h(a)$ obtained by applying a certain operator which depends on a to $h(a)$ (where $\mu(a) := (-a')^{-1} \in K_n$ and $h(a) := (a')^{-1} g_a \in K_n$). In particular $f_a = 0$ whenever $g_a = 0$. Since Supp $g_0 \subseteq S \cap H'$, and $S \cap H'$ is of finite type, the inductive assumption implies that $f_0 \in K_n^{\text{ft}}$ with Supp $f_0 \subseteq \int(S \cap H') \subseteq H'$. For $a \neq 0$ we have the operator P_a on K_n defined by $\mathrm{P}_a(y) = \mu(a)y'$, so that $f_a = \sum \mathrm{P}_a^i(h(a))$. We shall restrict these operators to $D := \{y \in K_n : \text{Supp } y \subseteq H'\}$, and prove that these restricted operators are small of finite type "uniformly in a". Since Supp $(\mu(a)) \subseteq H'$ for $a \neq 0$ we clearly have $\mathrm{P}_a(D) \subseteq D$ and $h(a) \in D$ for $a \neq 0$. Next we show:

(*) There is $U \subseteq H'$ of finite type (depending only on S and H), with $U < x$ if $n = 0$ and $U < G_{n-1}$ if $n > 0$, such that for all $a \neq 0$ we have Supp $\mu(a) \subseteq U$.

To see this, let $C = \mathbf{Z}a_1 + \cdots + \mathbf{Z}a_k$. Then $\mathrm{Lm}(a')$ takes at most k values as a ranges over $C \setminus \{0\}$. (This follows from a purely valuation-theoretic fact: if $b_1, \ldots, b_N \in \boldsymbol{k}((t))^{\mathrm{LE}} \setminus \{0\}$ and $\mathrm{Lm}(b_i) \neq \mathrm{Lm}(b_j)$ for all $i \neq j$, then $b_1, \ldots, b_N$ are linearly independent over $\boldsymbol{k}$.) Now apply lemma 3.5, lemma 7.15, and the fact that Supp $a' \subseteq \bigcup_{i=1}^k \text{Supp } a_i' \subseteq H'$ for all $a \in C$.

With U as in (*) we may assume that $U = u_0[u_1, \ldots, u_r]$ with $u_1, \ldots, u_r \in H'$ and $u_1 \leq 1, \ldots, u_r \leq 1]$. Note that then $u_0 = \max U$, and thus $u_0 < x$ if $n = 0$ and $u_0 < G_{n-1}$ for $n > 0$. We now first consider the case $n = 0$. Then Supp $\mathrm{P}_a(y) \subseteq u_0 x^{-1}[u_1, \ldots, u_r]$Supp y for non-zero a and $y \in D$. Since $u_0 x^{-1} < 1$ it follows that $\mathrm{P}_a | D$ is a small operator of finite type, and

$$\begin{aligned}
\text{Supp } f_a = \text{Supp } \sum_{i=0}^{\infty} \mathrm{P}_a^i(h(a)) &\subseteq [u_0 x^{-1}, u_1, \ldots, u_r]\text{Supp } h(a) \\
&\subseteq u_0[u_0 x^{-1}, u_1, \ldots, u_r]\text{Supp } g_a.
\end{aligned}$$

Hence, with $S' := \int(S \cap H')$ the integral $f = \sum f_a \, \mathrm{E}(a)$ of $g = \sum g_a \, \mathrm{E}(a)$ satisfies

$$\begin{aligned}
\text{Supp } f &\subseteq S' \cup u_0[u_0 x^{-1}, u_1, \ldots, u_r]\text{Supp } g \\
&\subseteq S' \cup u_0[u_0 x^{-1}, u_1, \ldots, u_r]S \\
&\subseteq H.
\end{aligned}$$

This takes care of the case $n = 0$. Next, assume that $n > 0$. By lemma 7.20 there are $v_0, v_1, \ldots, v_s \in H' \cap G_{n-1}$ with $v_1 \leq 1, \ldots, v_s \leq 1$ such that Supp $y' \subseteq v_0[v_1, \ldots, v_s]$Supp y for all $y \in D$. Hence

$$\text{Supp } \mathrm{P}_a(y) \subseteq u_0 v_0[u_1, \ldots, u_r, v_1, \ldots, v_s]\text{Supp } y \text{ for } a \neq 0, \text{ and } y \in D.$$

Since $u_0 < G_{n-1}$ and $v_0 \in G_{n-1}$ we have $u_0 v_0 < 1$. Thus the considerations in (7.14) show that if $a \neq 0$, then $\mathrm{P}_a | D$ is a small operator of finite type, and

$$\begin{aligned}\operatorname{Supp} f_a = \operatorname{Supp} \sum_{i=0}^{\infty} \mathrm{P}_a^i(h(a)) &\subseteq [u_0 v_0, u_1, \ldots, u_r, v_1, \ldots, v_s] \operatorname{Supp} h(a) \\ &\subseteq u_0[u_0 v_0, u_1, \ldots, u_r, v_1, \ldots, v_s] \operatorname{Supp} g_a.\end{aligned}$$

Hence, with $S' := \int (S \cap H')$ the integral $f = \sum f_a \operatorname{E}(a)$ of $g = \sum g_a \operatorname{E}(a)$ satisfies

$$\begin{aligned}\operatorname{Supp} f &\subseteq S' \cup u_0[u_0 v_0, u_1, \ldots, u_r, v_1, \ldots, v_s] \operatorname{Supp} g \\ &\subseteq S' \cup u_0[u_0 v_0, u_1, \ldots, u_r, v_1, \ldots, v_s] S \\ &\subseteq H.\end{aligned}$$

This finishes the inductive proof of the "finite type versions" of 5.3 and 5.4. The rest of the argument imitates the proof of Theorem 5.6.

(7.23) Final Remark. In a late stage of our work we became aware of J. van der Hoeven's treatise "Asymptotique automatique" (École Polytechnique, Paris, 1997). Our paper (and parts of [6]) clearly have much in common with Chapters 1 and 2 of this remarkable *Thèse*, although notations and terminology are different. (For example, it seems that our "sets of finite type" are essentially his "grid-based sets", and that our "LE-series with support of hereditarily finite type" are his " grid-based transseries". With hindsight, the "grid" terminology may be preferable.) Van der Hoeven's thesis also seems to realize (in chapters 3-5) a large part of our good intentions in (5.10) as to solvability of algebraic ODE's in terms of LE-series.

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University of Illinois at Urbana-Champaign
E-mail address: vddries@math.uiuc.edu

Oxford University
E-mail address: ajm@maths.oxford.ac.uk

University of Illinois at Chicago
E-mail address: marker@math.uic.edu