# CLOSED ASYMPTOTIC COUPLES

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ABSTRACT. The derivation of a Hardy field induces on its value group a certain function $\psi$. If a Hardy field extends the real field and is closed under powers, then its value group is also a vector space over $\mathbb{R}$. These "ordered vector spaces with $\psi$-function" are two-sorted structures and are called **$H$-couples.** We show that their elementary theory has a model-companion which admits quantifier elimination in a natural language. The $H$-couple of a maximal Hardy field is a model of this model-companion, and this is also the case for the $H$-couple of the field of logarithmic-exponential series over $\mathbb{R}$. We analyse in detail the simple extensions of $H$-couples, and the corresponding types.

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## INTRODUCTION

We define in this introduction all the key concepts of this paper, and state the main theorem. We begin with motivating our subject via Hardy fields.

We assume familiarity with the basic theory of Hardy fields as developed by Bourbaki [4] and Rosenlicht [18], [19]. This theory is the modern incarnation of ideas on "Orders of Infinity" originating with Du Bois-Reymond [3] and put on a firm basis by Hardy [8]. Hardy fields are ordered differential fields of germs at $+\infty$ of real valued differentiable functions defined on half lines $(a,+\infty)$ with $a \in \mathbb{R}$. A Hardy field $F$ has valuation ring

$$\mathcal{O}(F) := \big\{ f \in F : |f| \leq n \text{ for some natural number } n \big\}$$

with associated valuation $v : F^{\times} \to V = v(F^{\times})$. This valuation measures the growth of functions at infinity: given $f, g \in F^{\times}$ we have

$$v(f) > v(g) \Longleftrightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

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A key fact [18] is that $v(f')$ only depends on $v(f)$ for $v(f) \neq 0$, $f \in F^\times$. Thus the value group $V$ comes equipped with a natural extra operation $\psi : V \setminus \{0\} \to V$ given by

$$\psi\big(v(f)\big) := v(f'/f) = v(f') - v(f) \quad \text{for } v(f) \neq 0,\ f \in F^\times.$$

The pair $(V, \psi)$ is an "asymptotic couple of Hardy type" in the sense of Rosenlicht [17] who studies especially the situation where the abelian group $V$ has finite rank; in that case $\psi$ takes only finitely many different values. Here we focus on the opposite situation where, roughly speaking, $(V, \psi)$ is existentially closed in a category of asymptotic couples. In addition we include a scalar multiplication $\mathbb{R} \times V \to V$ among the basic operations. Here is why.

Suppose our Hardy field $F$ extends $\mathbb{R}(x)$, and is closed under powers, that is, $0 < f \in F \implies f^r \in F$ for all $r \in \mathbb{R}$. (All maximal Hardy fields have these properties, see [4] or [18].) Then $V$ becomes an ordered vector space over $\mathbb{R}$ by setting $r \cdot v(f) := v(f^r)$ for $0 < f \in F$. Consider the two-sorted structure consisting of the ordered field $\mathbb{R}$ (first sort), the ordered abelian group $V$ equipped with the function $\psi : V \setminus \{0\} \to V$ as above (second sort), with the scalar multiplication $\mathbb{R} \times V \to V$ relating them. This two-sorted structure is interpretable in the ordered differential field $F$: $r \cdot v(f) = v(g)$ for $r \in \mathbb{R} = \{a \in F : a' = 0\}$ whenever $0 < f, g \in F$ and $rf'/f = g'/g$. (Note the contrast with henselian valued fields of equal characteristic 0 where the value group and residue field are also interpretable as definable quotient structures, but where one can show that there is no definable "interaction" between value group and residue field.)

In this paper we take a step towards a model theory of Hardy fields as ordered *differential* fields, by developing the model theory of these associated two-sorted structures. We now formally introduce the objects we shall be dealing with.

*Notation.* We put $S^{>a} := \{s \in S : s > a\}$ for an element $a$ of a linearly ordered set $S$; similarly for "$\geq$", "$<$" or "$\leq$" instead of "$>$".

Recall that an ordered vector space over an ordered field $\boldsymbol{k}$ is a vector space $V$ over $\boldsymbol{k}$ equipped with a linear ordering such that if $0 < v, w \in V$ and $0 < \lambda \in \boldsymbol{k}$, then $0 < v + w$ and $0 < \lambda v$. We then define an equivalence relation on $V$ by

$$v \sim w \ :\Longleftrightarrow\ \exists \lambda \in \boldsymbol{k}^{>1} : \frac{1}{\lambda}|v| \leq |w| \leq \lambda |v|.$$

The equivalence class of $v \in V$ is written as $[v]$ (or $[v]_{\boldsymbol{k}}$, if $\boldsymbol{k}$ is not clear from context), and is called its **$\boldsymbol{k}$-archimedean class.** We let $[V]$ (or $[V]_{\boldsymbol{k}}$) be the set of $\boldsymbol{k}$-archimedean classes, and linearly order $[V]$ by

$$\begin{aligned}[v] < [w] \ &:\Longleftrightarrow\ \forall \lambda \in \boldsymbol{k}^{>0} : \lambda |v| < |w| \\ &\Longleftrightarrow\ [v] \neq [w] \text{ and } |v| < |w|.\end{aligned}$$

Thus $[0] = \{0\}$ is the smallest $\boldsymbol{k}$-archimedean class. For ease of notation we put $V^* := V \setminus \{0\}$, and $[V^*] := \big\{[v] : v \in V^*\big\}$.

**Definition.** A **Hahn space** is an ordered vector space $V$ over an ordered field $\boldsymbol{k}$ such that for all vectors $v, w \in V^*$

$$[v] = [w] \implies \exists \lambda \in \boldsymbol{k} : [v - \lambda w] < [w].$$

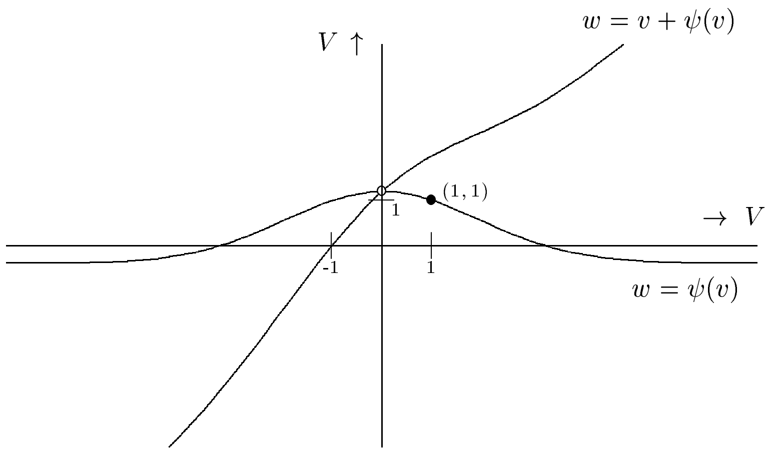
It is easy to see that any ordered vector space over $\mathbb{R}$ is a Hahn space. We have chosen the term "Hahn space" since these spaces satisfy an analogue of the Hahn embedding theorem as shown in section 1. There we also establish the good behaviour of Hahn spaces under scalar extension.

**Definition.** An $H$-**couple** $\mathcal{V} = (V, \psi)$ consists of a Hahn space $V$ over an ordered field $\boldsymbol{k}$, a distinguished positive element $1 \in V$, and a function $\psi : V^* \to V$ such that for all $v, w \in V^*$

1. $\psi(1) = 1$,
2. $\psi(v) \leq \psi(w) \iff [v] \geq [w] \quad$ (hence $\psi(v) = \psi(w) \iff [v] = [w]$),
3. $\psi(v) < \psi(w) + |w|$.

We refer to $\mathcal{V}$ as an "$H$-couple over $\boldsymbol{k}$" if we want to specify the scalar field $\boldsymbol{k}$.

The figure below shows the qualitative behavior of the functions $\psi$ and $\mathrm{id} + \psi$ on $V^*$. (In section 2 we will see that $\mathrm{id} + \psi$ is strictly increasing.) The picture is quite rough: it cannot show that $\psi$ is constant on $\boldsymbol{k}$-archimedean classes. But it has been a precious guide in our work.




*Examples.*

1. To every Hardy field $F \supseteq \mathbb{R}(x)$ closed under powers we associate the corresponding $H$-couple $\mathcal{V} = (V, \psi)$ over $\mathbb{R}$, as indicated above, with $1 := v(x^{-1})$. That we actually obtain an $H$-couple in this way is clear from remarks made above, and results in [16].
2. Let $\boldsymbol{k}$ be a logarithmic-exponential ordered field, and let $F$ be a differential subfield of $\boldsymbol{k}((t))^{\mathrm{LE}}$ containing $\boldsymbol{k}(x)$, and closed under powers, that is, if $0 < f \in F$ and $r \in \boldsymbol{k}$, then $f^r \in F$. (See [7] for the construction of $\boldsymbol{k}((t))^{\mathrm{LE}}$, the field of logarithmic-exponential series over $\boldsymbol{k}$.) Let $v$ be the valuation on $F$ with valuation ring $\{f \in F : |f| \leq r \text{ for some } r \in \boldsymbol{k}\}$, and associate to $F$ an $H$-couple just as we did for the Hardy fields above, with $1 := v(t) = v(x^{-1})$. (In section 2 we show this gives indeed an $H$-couple over $\boldsymbol{k}$.)

When dealing with $H$-couples $\mathcal{V} = (V, \psi)$ as model-theoretic objects we construe them as $\mathcal{L}_H$-structures, where $\mathcal{L}_H$ is the two-sorted language with

1. scalar variables ranging over the extended scalar field $\boldsymbol{k}_\infty := \boldsymbol{k} \cup \{\infty\}$,
2. vector variables ranging over the extended vector space $V_\infty := V \cup \{\infty\}$,

and with the following non-logical symbols:

3. $<$, 0, 1, $+$, $-$, $\cdot$, interpreted as usual in the ordered field $\boldsymbol{k}$ of scalars, with $\infty$ serving as a default value: the linear ordering on $\boldsymbol{k}$ is extended to a linear ordering on $\boldsymbol{k}_\infty$ by setting $\lambda < \infty$ for all $\lambda \in \boldsymbol{k}$, and $\lambda * \mu := \infty$ for $* \in \{+,-,\cdot\}$ and all $\lambda, \mu \in \boldsymbol{k}_\infty$ with $\lambda = \infty$ or $\mu = \infty$.
4. $<$, 0, 1, $+$, $-$, $\psi$, interpreted in the obvious way in $V$ and with $\infty$ serving as default value: the linear ordering on $V$ is extended to a linear order on $V_\infty$ by setting $a < \infty$ for all $a \in V$, and $a + \infty = \infty + a = a - \infty = \infty - a = \infty$ for all $a \in V_\infty$ , and $\psi(0) = \psi(\infty) = \infty$,
5. a symbol $\cdot$ for the map $\boldsymbol{k}_\infty \times V_\infty \to V_\infty$ that is the scalar multiplication on $\boldsymbol{k} \times V$ and with $\lambda \cdot v = \infty$ for all $(\lambda, v) \in \big(\boldsymbol{k}_\infty \times V_\infty\big) \setminus (\boldsymbol{k} \times V)$,
6. a symbol $:$ for the function $V_\infty^2 \to \boldsymbol{k}_\infty$ that assigns to each $(a,b) \in V^2$ with $[a] \le [b]$ and $b \ne 0$ the unique scalar $a : b = \lambda \in \boldsymbol{k}$ such that $[a - \lambda b] < [b]$, and that assigns to all other pairs $(a,b) \in V_\infty^2$ the default value $a : b = \infty$.

*Remarks.*

1. Despite overlap in how we write the symbols of (3), (4), and (5), we actually distinguish them: for example, the symbol $+$ in (3) is to be regarded as different from the symbol $+$ in (4). Similarly, the element $\infty \in \boldsymbol{k}_\infty$ is to be distinguished from the element $\infty \in V_\infty$.
2. The default values $\infty$ are included to make all basic operations totally defined, so that no ambiguities arise in the interpretation of terms.

It is easy to see that the $H$-couples in the model-theoretic sense are exactly the models of a *universal* theory in the language $\mathcal{L}_H$. Thus each substructure of an $H$-couple is also an $H$-couple, with possibly smaller scalar field. (That's why we included the division operation of (6).) We will keep writing $H$-couples as $(V,\psi)$, and so on, even when we regard them as $\mathcal{L}_H$-structures.

Let $(V,\psi)$ be an $H$-couple. Then clearly $\psi(v) < w + \psi(w)$ for all $v, w \in V^{>0}$. Thus $(V,\psi)$ has an "$H$-cut" in the following sense.

**Definition.** An $H$**-cut** of $(V,\psi)$ is a set $P \subseteq V$ such that:

1. For all $a, b \in V$, if $a < b \in P$, then $a \in P$.
2. $\psi(v) \in P$ and $w + \psi(w) \notin P$ for all $v, w \in V^{>0}$.

We then also call $(V,\psi,P)$ an $H$**-triple**, and we regard $(V,\psi,P)$ as an $\mathcal{L}_{H,P}$-structure, where $\mathcal{L}_{H,P}$ extends the language $\mathcal{L}_H$ by an extra unary predicate $P$, to be interpreted by the set $P \subseteq V$. Clearly the $H$-triples are then exactly the models of a universal theory in the language $\mathcal{L}_{H,P}$.

**Definition.** The $H$-couple $(V,\psi)$ is **closed** if $\psi(V^*)$ has no largest element, and

$$\psi(V^*) = \big\{a \in V : a < w + \psi(w) \text{ for all } w \in V^{>0}\big\}.$$

In that case $\Psi := \psi(V^*)$ is clearly the only $H$-cut of $(V,\psi)$; we call $(V,\psi,\Psi)$ a **closed** $H$**-triple**.

In section 2 we prove that each $H$-triple can be embedded (as $\mathcal{L}_{H,P}$-structure) into a closed $H$-triple with the same scalar field. We also show there that the $H$-couples associated to maximal Hardy fields and to the ordered differential field $\mathbb{R}((t))^{\mathrm{LE}}$ are closed.

We can now state the main result of this paper, to be established in section 3.

**Theorem.** *The theory of closed $H$-triples over real closed scalar fields is complete, eliminates quantifiers, and is the model-completion of the theory of $H$-triples.*

We actually obtain a relative quantifier elimination where the scalar field is not assumed to be real closed. In section 4 we show that no new structure is induced on the scalar field in closed $H$-triples and that the underlying ordered vector space in such triples is locally o-minimal. We also determine there the definable closure of a substructure in a closed $H$-triple, and study simple extensions of $H$-triples.

In section 5 we indicate a variant of the results above, where there is no scalar field. Here we have a model-completion that is even weakly o-minimal. In section 6 we discuss a connection to Kuhlmann's "contraction groups" in [9].

**Remarks on 1 and $P$.** The role of the distinguished positive element 1 with $\psi(1) = 1$ is to give a convenient normalization. This role is hardly essential, but does affect issues like completeness as stated in the Theorem above. To clarify this point further, consider an "$H$-couple without 1", that is, a couple $(V, \psi)$ consisting of a non-trivial Hahn space $V$ over an ordered field $\boldsymbol{k}$ and a function $\psi \colon V^* \to V$ satisfying axioms (2) and (3) for $H$-couples. (We do not distinguish a positive element 1 and omit the axiom $\psi(1) = 1$.) Then for each vector $a \in V$ the translate $(V, a + \psi)$ is clearly also an $H$-couple without 1. Choose any vector $b \in V^{>0}$, and put $a := b - \psi(b)$. Then $a + \psi(b) = b$, so by taking $b$ as our distinguished positive element 1 we make $(V, a + \psi)$ into an $H$-couple over $\boldsymbol{k}$.

Similarly, we include the predicate $P$ for an $H$-cut to get quantifier elimination as stated in the Theorem above: Using results from §2, it is easy to construct closed $H$-couples $(V_1, \psi_1)$, $(V_2, \psi_2)$ over $\mathbb{R}$, with common substructure $(V, \psi)$ containing a vector $v$, such that $v \in \psi_1(V_1^*)$, but $v \notin \psi_2(V_2^*)$.

**Notational conventions.** Let $(S, <)$ be a linearly ordered set. When $a \leq b$ in $S$ and $(S, <)$ is clear from context we use the following notations:

$$\begin{aligned} [a, b] &:= \{x \in S : a \leq x \leq b\}, \\ (-\infty, b] &:= \{x \in S : x \leq b\}, \\ [a, \infty) &:= \{x \in S : x \geq a\}. \end{aligned}$$

A set $X \subseteq S$ is called **convex** (in $S$) if $[a, b] \subseteq X$ for all $a, b \in X$ with $a < b$. For any subset $X$ of $S$ and $a \in X$, the **convex component of $a$ in $X$** is the (convex) set

$$\{x \in X^{\leq a} : [x, a] \subseteq X\} \cup \{x \in X^{\geq a} : [a, x] \subseteq X\}.$$

(This also depends on $S$.) The **convex components of** $X$ are by definition the convex components of the members of $X$ in $X$. They form a partition of $X$. The **convex hull of** $X$ (in $S$) is the smallest convex subset of $S$ containing $X$. We call a subset $X$ of $S$ **closed upward** (in $S$) if $a \in S$, $a > b \in X$ implies $a \in X$, and **closed downward** (in $S$) if $a \in S$, $a < b \in X$ implies $a \in X$.

Throughout the paper, we let $m, n$ range over the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers. If $G$ is an ordered abelian group and $g \in G$, we set $|g| := \max\{-g, g\}$, and let $\operatorname{sgn}(g) := -1$ if $g < 0$, $\operatorname{sgn}(0) := 0$, and $\operatorname{sgn}(g) := 1$ if $g > 0$.

## 1. Hahn Spaces

The notion of a Hahn space was already defined in the introduction. In this section we study embedding and scalar extension properties of Hahn spaces. We also include a very useful lemma about functions on ordered abelian groups. Throughout this section, let $\boldsymbol{k}$ be an ordered field and $V$ an ordered vector space over $\boldsymbol{k}$.

**Properties of $\boldsymbol{k}$-archimedean classes.** Let $v, w \in V$ and $\lambda \in \boldsymbol{k}^\times$. Then:

1. $[v] = \{0\} \iff v = 0$,
2. $[v] = [\lambda v]$,
3. $[v + w] \leq \max\bigl\{[v], [w]\bigr\}$,
4. $[v + w] = \max\bigl\{[v], [w]\bigr\}$, if $[v] \neq [w]$.

We say that $V$ is **$\boldsymbol{k}$-archimedean** if $[V^*]$ is a singleton. For example, $\boldsymbol{k}$ as an ordered vector space over itself is $\boldsymbol{k}$-archimedean.

**Jumps.** For $\gamma \in [V^*]$, we define the convex linear subspaces

$$V_{(\gamma)} := \bigl\{v \in V : [v] < \gamma\bigr\}, \qquad V^{(\gamma)} := \bigl\{v \in V : [v] \leq \gamma\bigr\}.$$

Note that the ordered vector space $V^{(\gamma)}/V_{(\gamma)}$ is $\boldsymbol{k}$-archimedean. It will be called the **jump** of $V$ corresponding to the $\boldsymbol{k}$-archimedean class $\gamma$.

*Remarks.* The following facts are easy consequences of the definitions:

1. $V$ is a Hahn space if and only if all jumps of $V$ are one-dimensional vector spaces.
2. Any linear subspace of a Hahn space over $\boldsymbol{k}$ is itself a Hahn space over $\boldsymbol{k}$ with respect to the induced ordering.
3. Any ordered vector space over the field of real numbers is a Hahn space.
4. $\mathbb{R}$ as an ordered vector space over $\mathbb{Q}$ is not a Hahn space.

**Hahn products.** Let $\Gamma$ be a totally ordered set and $(V_\gamma)_{\gamma\in\Gamma}$ a system of ordered vector spaces over $\boldsymbol{k}$. For each element $v = (v_\gamma)_{\gamma\in\Gamma}$ of the vector space $\prod_{\gamma\in\Gamma} V_\gamma$, we let

$$\operatorname{supp} v := \bigl\{\gamma \in \Gamma : v_\gamma \neq 0\bigr\}$$

denote the **support of $v$.** The subset $H\bigl(\Gamma, (V_\gamma)_{\gamma\in\Gamma}\bigr)$ of $\prod_{\gamma\in\Gamma} V_\gamma$ consisting of those elements with anti-wellordered support is a $\boldsymbol{k}$-linear subspace of $\prod_{\gamma\in\Gamma} V_\gamma$. It becomes an ordered vector space over $\boldsymbol{k}$ by setting, for $v \in H\bigl(\Gamma, (V_\gamma)_{\gamma\in\Gamma}\bigr)$, $v \neq 0$,

$$0 < v \ :\Longleftrightarrow\ 0 < v_{\mu(v)},$$

where $\mu(v) := \max(\operatorname{supp} v)$. We call $H\bigl(\Gamma, (V_\gamma)_{\gamma\in\Gamma}\bigr)$ the **Hahn product of $(V_\gamma)_{\gamma\in\Gamma}$.** Note that if all $V_\gamma$ are Hahn spaces over $\boldsymbol{k}$, then $H\bigl(\Gamma, (V_\gamma)_{\gamma\in\Gamma}\bigr)$ is a Hahn space over $\boldsymbol{k}$. If all $V_\gamma$ are equal to $V$, we also write $H(\Gamma, V)$. If $V$ is $\boldsymbol{k}$-archimedean, and we put $H := H(\Gamma, V)$, then we have a well defined map

$$[v] \mapsto \max(\operatorname{supp} v) : [H^*] \to \Gamma \qquad (v \in H^*),$$

and this map is an isomorphism of linearly ordered sets.

**Hahn embedding theorem.** Let $V'$ be an ordered vector space over an ordered field extension $\boldsymbol{k}'$ of $\boldsymbol{k}$. Then by an **embedding** $V \to V'$ we mean an injective order preserving $\boldsymbol{k}$-linear map $V \to V'$. Such an embedding $i : V \to V'$ induces a map $[v] \mapsto [i(v)] : [V] \to [V']$ from the set of $\boldsymbol{k}$-archimedean classes of $V$ into the set of $\boldsymbol{k}'$-archimedean classes of $V'$. This induced map is clearly injective if $\boldsymbol{k} = \boldsymbol{k}'$.

**Proposition 1.1.** *Let $\Gamma := [V^*]$. Then there exists an embedding*

$$V \to H := H\bigl(\Gamma, \bigl(V^{(\gamma)}/V_{(\gamma)}\bigr)\bigr)$$

*of ordered vector spaces over $\boldsymbol{k}$ with bijective induced map $[V] \to [H]$.*

*Proof.* The proof is an easy adaptation of Banaschewski's proof [2] of the Hahn embedding theorem as presented in [14], pp. 16–17. Let $S(V)$ be the collection of all $\boldsymbol{k}$-linear subspaces of $V$. We use the fact that there is a map $\sigma\colon S(V) \to S(V)$ with the property that $V = W \oplus \sigma(W)$ and $U \subseteq W \implies \sigma(U) \supseteq \sigma(W)$, for all $U, W \in S(V)$. For each $\gamma \in \Gamma$, we can decompose $V = V^{(\gamma)} \oplus \sigma\big(V^{(\gamma)}\big)$, and hence each $v \in V$ can be written $v = v_\gamma + v_\gamma^\sigma$ for certain $v_\gamma \in V^{(\gamma)}$, $v_\gamma^\sigma \in \sigma\big(V^{(\gamma)}\big)$. Consider the map

$$v \mapsto \big(v_\gamma + V_{(\gamma)}\big)_{\gamma\in\Gamma} : V \to H.$$

One verifies that this is an embedding with the desired property. □

**Corollary 1.2.** (Hahn Embedding Theorem for Hahn Spaces) *If $V$ is a Hahn space, then there exists an embedding $V \to H := H(\Gamma, \boldsymbol{k})$, where $\Gamma := [V^*]$, with bijective induced map $[V] \to [H]$.* □

Thus up to isomorphism of ordered vector spaces over $\boldsymbol{k}$ the Hahn spaces over $\boldsymbol{k}$ are exactly the ordered linear subspaces of Hahn products $H(\Gamma, \boldsymbol{k})$ for linearly ordered sets $\Gamma$.

**Scalar extension.** Given a field extension $\boldsymbol{k}' \supseteq \boldsymbol{k}$, let $V_{\boldsymbol{k}'}$ be $V \otimes_{\boldsymbol{k}} \boldsymbol{k}'$ viewed as a vector space over $\boldsymbol{k}'$ in the usual way. We consider $V$ as $\boldsymbol{k}$-linear subspace of $V_{\boldsymbol{k}'}$ by identifying $v \in V$ with $v \otimes 1 \in V_{\boldsymbol{k}'}$. The following fact will be used several times:

**Lemma 1.3.** *Let $V$ be a Hahn space, and $\boldsymbol{k}' \supseteq \boldsymbol{k}$ a field extension. Then every non-zero vector $u \in V_{\boldsymbol{k}'}$ can be written as*

$$u = \sum_{i=1}^{m} \lambda_i u_i \qquad \text{with scalars } \lambda_i \in \boldsymbol{k}' \text{ and vectors } u_i \in V^{>0}, \tag{1.1}$$

*such that $[u_1] > [u_2] > \cdots > [u_m]$ in $[V^*]$.*

*Proof.* Let $0 \neq u = \sum_{j=1}^{n} \mu_j v_j$ ($\mu_j \in \boldsymbol{k}'$, $v_j \in V$). We show by induction on $n$ that $u$ can be rewritten as in the lemma, with $[u_1] \leq \max\big\{[v_j] : j = 1, \dots, n\big\}$. The case $n = 1$ is trivial; so assume $n > 1$. We can assume there is $k \in \{1, \dots, n\}$ such that $[v_1] = \cdots = [v_k] > [v_j]$, for $k < j \leq n$. For $j = 2, \dots, k$ we write $v_j = \rho_j v_1 + w_j$ with $\rho_j \in \boldsymbol{k}$, $w_j \in V$ , $[w_j] < [v_1]$. Put

$$w := \sum_{j=2}^{k} \mu_j w_j + \sum_{j=k+1}^{n} \mu_j v_j, \qquad \text{a sum of } n-1 \text{ terms.}$$

Then $u = (\mu_1 + \rho_2\mu_2 + \cdots + \rho_k\mu_k)v_1 + w$. Apply the induction hypothesis to $w$. □

**Proposition 1.4.** *Let $V$ be a Hahn space, and let $\boldsymbol{k} \subseteq \boldsymbol{k}'$ be an extension of ordered fields. Then there is a unique linear ordering on $V_{\boldsymbol{k}'}$ extending the ordering of $V$, making $V_{\boldsymbol{k}'}$ into an ordered vector space over $\boldsymbol{k}'$, such that the inclusion $V \hookrightarrow V_{\boldsymbol{k}'}$ is an embedding with injective induced map $[V] \to [V_{\boldsymbol{k}'}]$.*

*Proof.* Assume that we are given such an ordering on $V_{\boldsymbol{k}'}$. We can write each non-zero vector $u \in V_{\boldsymbol{k}'}$ as in the lemma, with $\lambda_1 \neq 0$. Then $u > 0$ if and only if $\lambda_1 > 0$. This shows uniqueness. Existence: By Corollary 1.2 above, we have an embedding $V \to H(\Gamma, \boldsymbol{k})$ of ordered vector spaces over $\boldsymbol{k}$ , where $\Gamma := [V^*]_{\boldsymbol{k}}$. Tensoring with $\boldsymbol{k}'$ gives a $\boldsymbol{k}'$-linear injective map $V_{\boldsymbol{k}'} \to H(\Gamma, \boldsymbol{k}')$. This induces an ordering on $V_{\boldsymbol{k}'}$, making it into an ordered vector space over $\boldsymbol{k}'$ with the desired properties, as one easily verifies. □

*Remarks.* Under the hypothesis of the proposition above we will consider $V_{\boldsymbol{k}'}$ as being equipped with the unique linear ordering of the proposition. Note that then $V_{\boldsymbol{k}'}$ is a Hahn space over $\boldsymbol{k}'$, and that the map $[V] \to [V_{\boldsymbol{k}'}]$ induced by the embedding $V \hookrightarrow V_{\boldsymbol{k}'}$ is a bijection.

**Corollary 1.5.** (Universal Property) *Let $V$ be a Hahn space and $\boldsymbol{k} \subseteq \boldsymbol{k}'$ be an extension of ordered fields. Any embedding $V \to V'$ into an ordered vector space $V'$ over $\boldsymbol{k}'$ with injective induced map $[V] \to [V']$ extends uniquely to an embedding $V_{\boldsymbol{k}'} \to V'$.* □

**A lemma about functions on ordered abelian groups.** We shall say that a function $f : X \to Y$ between linearly ordered sets $X$ and $Y$ has the **intermediate value property** if for all $x_1 < x_2$ in $X$ and all $y \in Y$ with $f(x_1) < y < f(x_2)$ or $f(x_2) < y < f(x_1)$ there is $x \in X$ such that $x_1 < x < x_2$ and $f(x) = y$.

Let $G$ be an ordered abelian group, and for $a, b \in G$ write $a = o(b)$ to indicate that $n|a| \leq |b|$ for each positive integer $n$.

**Lemma 1.6.** *Let $C \subseteq G$ be a convex subset, and let the function $\eta \colon C \to G$ have the following properties:*

1. *$\eta(x) - \eta(y) = o(x - y)$ for all distinct $x, y \in C$,*
2. *$\eta(y) = \eta(z)$ whenever $x, y, z \in C$ with $x < y < z$ and $z - y = o(z - x)$.*

*Then the function $x \mapsto x + \eta(x) \colon C \to G$ is strictly increasing and has the intermediate value property.*

*Proof.* That $x + \eta(x)$ is strictly increasing is an easy consequence of (1). To prove the intermediate value property, let $a, b \in C$ with $a < b$. Let $c := b - a$ and define $\eta_1 \colon [0, c] \to G$ by $\eta_1(x) := \eta(a + x) - \eta(a)$. Then properties (1) and (2) remain valid if $C$ is replaced by $[0, c]$ and $\eta$ by $\eta_1$, and it suffices to prove the intermediate value property for the corresponding function $x \mapsto x + \eta_1(x)$. So we can assume $C = [0, c]$ and $\eta(0) = 0$. Let $0 < v < c + \eta(c)$. It suffices to find $u \in (0, c)$ such that $u + \eta(u) = v$. We distinguish two cases:

1. $c - v = o(c)$. Then we put $u := v - \eta(c)$, so $0 < u < c$. By (1), we have $\eta(c) = o(c)$, hence $c - u = (c - v) + (v - u) = (c - v) + \eta(c) = o(c)$. Therefore, by (2), we have $\eta(u) = \eta(c)$, that is, $u + \eta(u) = v$.
2. $c - v \neq o(c)$. Since $v < c + \eta(c)$ and $\eta(c) = o(c)$ by (1), we get $0 < v < c$. Put $u := v - \eta(v)$. Since $\eta(v) = o(v)$ by (1), we have $0 < u, v < c$ and $v - u = o(v)$, hence $\eta(v) = \eta(u)$ by (2), that is, $u + \eta(u) = v$.

□

*Remark.* The lemma above remains of course valid when (2) is replaced by

2′. $\eta(y) = \eta(z)$ whenever $x, y, z \in C$ with $x > y > z$ and $z - y = o(z - x)$.

## 2. $H$-couples: Examples, and Embedding Properties

We refer to the introduction for various notions concerning $H$-couples. In this section $(V, \psi)$ is an $H$-couple over the ordered field $\boldsymbol{k}$, $\Psi := \psi(V^*)$, and $P$ is an $H$-cut of $(V, \psi)$. So $(V, \psi, P)$ is an $H$-triple over $\boldsymbol{k}$.

**Basic properties of $\psi$.** (See also [17].)

1. The map $v \mapsto \psi(v) : V \to V_\infty$ (with $\psi(0) = \infty > V$) is a valuation on the ordered group $V$, that is, $\psi(v + w) \geq \min\{\psi(v), \psi(w)\}$ for $v, w \in V$.

2. $\psi(v-w) > \min\{v,w\}$, for all $v,w \in P$. In particular, $\psi\big(\psi(v)-\psi(w)\big) > \min\big\{\psi(v),\psi(w)\big\}$, for all $v,w \in V^*$.
3. $[\psi(v)-\psi(w)] < [v-w]$ for $v,w \in V^*$, $v \neq w$.
4. The map $v \mapsto v+\psi(v) : V^* \to V$ is strictly increasing.

*Proof.* Property (1) follows easily from axiom (2) about $H$-couples. For (2), let $v,w \in P$, $v<w$. Then $\psi(w-v)+(w-v) > w$, hence $\psi(v-w) > v$. Property (3) follows from (1), (2) and axiom (2). Property (4) is now an immediate consequence of (3). ☐

Note that Lemma 1.6 and property (3) imply the intermediate value property for the function $x \mapsto x+\psi(x)$ on $V^{>0}$, and also the intermediate value property for $x \mapsto x+\psi(x)$ on $V^{<0}$. A consequence of this is:

**Lemma 2.1.** *The set*

$$(\mathrm{id}+\psi)\big(V^{>0}\big) = \big\{x+\psi(x) : x \in V^{>0}\big\}$$

*is closed upward. The set*

$$(-\mathrm{id}+\psi)\big(V^{>0}\big) = \big\{-x+\psi(x) : x \in V^{>0}\big\}$$

*is closed downward. Moreover,*

$$(-\mathrm{id}+\psi)\big(V^{>0}\big) = (\mathrm{id}+\psi)\big(V^{<0}\big) = \big\{a \in V : a < b \text{ for some } b \in \Psi\big\}.$$

*Proof.* Let $a>1$ in $V$. Then $|\psi(a)-1| = |\psi(a)-\psi(1)| < a-1$ by basic property (3), hence $|\psi(a)|<a$. Thus $2a+\psi(2a) = 2a+\psi(a) > a$, showing that $\mathrm{id}+\psi$ takes arbitrarily large values on $V^{>0}$. Now use the intermediate value property for $\mathrm{id}+\psi$ on $V^{>0}$ to deduce the first statement. For the second, note that $-2a+\psi(2a) < -a$, since $a>\psi(a)$. Since clearly $(-\mathrm{id}+\psi)\big(V^{>0}\big) = (\mathrm{id}+\psi)\big(V^{<0}\big)$ it follows as before that $(-\mathrm{id}+\psi)\big(V^{>0}\big)$ is closed downward. Let $a \in V$, $a<\psi(x)$ for some $x \in V^{>0}$. Set $y := \min\big\{x, \psi(x)-a\big\} > 0$; then $a \le \psi(x)-y \le \psi(y)-y \in (-\mathrm{id}+\psi)\big(V^{>0}\big)$. Thus $a \in (-\mathrm{id}+\psi)\big(V^{>0}\big)$. ☐

These facts will be tacitly used in the rest of the paper. Next we make the following easy but very useful observation.

**Proposition 2.2.** *There is at most one element $v \in V$ such that*

$$\Psi < v < (\mathrm{id}+\psi)\big(V^{>0}\big). \tag{2.1}$$

*Hence $(V,\psi)$ has at most two $H$-cuts, and $(V,\psi)$ has exactly two $H$-cuts if and only if there exists $v$ such that* (2.1) *holds. If $\Psi$ has a largest element, then $(V,\psi)$ has only one $H$-cut.*

*Proof.* If $v>v'$ are two elements satisfying (2.1), choosing $u := v-v' > 0$ yields $\psi(u) \le v' = v-u < (\psi(u)+u)-u = \psi(u)$, which is absurd. If $\Psi$ has a largest element $v'$, and $v$ is supposed to satisfy (2.1), then the same argument leads to absurdity. ☐

Closed $H$-couples have only one $H$-cut. In Lemma 2.4 we indicate a class of $H$-couples with two $H$-cuts. First a general fact that we shall use several times:

**Lemma 2.3.** *Let $\boldsymbol{k} \subseteq \boldsymbol{k}'$ be an ordered field extension, and $i\colon V \to V'$ be an embedding of $V$ into a Hahn space $V'$ over $\boldsymbol{k}'$, such that the induced map $[V] \to [V']$ is bijective. Then there is a unique function $\psi'\colon (V')^* \to V'$ such that $(V',\psi')$ is an*

*$H$-couple over $\boldsymbol{k}'$, with $1 \in V$ as its distinguished positive element, and $i\big(\psi(v)\big) = \psi'\big(i(v)\big)$ for all $v \in V^*$.*

*Proof.* Define $\psi'(v') := \psi(v)$ for $v' \in (V')^*$ and $v \in V^*$ such that $[v'] = [i(v)]$. Then $\psi'$ is well-defined, and $(V', \psi')$ is an $H$-couple. The main point to check here is axiom (3) for $H$-couples, which follows from the bijectivity of $[V] \to [V']$ and property (3) for $\psi$ stated at the beginning of this section. $\square$

Consider now an embedding of $V$ into the Hahn space $H := H(\Gamma, \boldsymbol{k})$ over $\boldsymbol{k}$ as in Corollary 1.2, with $\Gamma = [V^*]$, and identify $V$ with its image in $H$ via this embedding. Then the lemma above tells us that $\psi$ extends uniquely to a function $\psi_H : H^* \to H$ such that $(H, \psi_H)$ is an $H$-couple over $\boldsymbol{k}$ with distinguished element $1 \in V$. The next result shows that $(H, \psi_H)$ has always two $H$-cuts if $\Gamma$ has no least element.

**Lemma 2.4.** *Let $H = H(\Gamma, \boldsymbol{k})$ for some nonempty linearly ordered set $\Gamma$ without least element. Then each $H$-couple of the form $(H, \psi)$ has two $H$-cuts.*

*Proof.* Let $\kappa$ be the coinitality of $\Gamma$ and $(\gamma_\alpha)_{\alpha<\kappa}$ a coinital sequence in $\Gamma$. Choose $u_\alpha \in H^{>0}$ with $[u_\alpha] = \gamma_\alpha$ and set $w_\alpha := \psi(u_\alpha)$, for all $\alpha < \kappa$. Then $(w_\alpha)_{\alpha<\kappa}$ is cofinal in $\psi(H^*)$, and $(w_\alpha + u_\alpha)_{\alpha<\kappa}$ is coinitial in $(\mathrm{id} + \psi)\big(H^{>0}\big)$. Let $\gamma \in \Gamma$; take $\alpha_0 < \kappa$ such that $\gamma_{\alpha_0} < \gamma$. Then $\alpha_0 < \alpha, \beta < \kappa$ implies $[w_\alpha - w_\beta] < \gamma$, that is, $(w_\alpha)_{\gamma'} = (w_\beta)_{\gamma'}$ for all $\gamma' \geq \gamma$. So for each $\gamma \in \Gamma$, the sequence $\big((w_\alpha)_\gamma\big)_{\alpha<\kappa}$ is eventually constant. Let $v_\gamma \in \boldsymbol{k}$ be this constant, and set $v := (v_\gamma)_{\gamma\in\Gamma}$. One shows that $v \in H$, and that $w_\alpha < v < w_\alpha + u_\alpha$ for all $\alpha < \kappa$. Therefore $\psi\big(H^{>0}\big) < v < (\mathrm{id} + \psi)\big(H^{>0}\big)$. $\square$

**Examples of $H$-couples.**

**Example 1.** Consider $\boldsymbol{k}$ as an ordered vector space over itself. Then $(\boldsymbol{k}, \psi)$ with $\psi(v) = 1$ for all $v \in \boldsymbol{k}^\times$ and $1 \in \boldsymbol{k}$ as distinguished positive element, is an $H$-couple over $\boldsymbol{k}$. It has a unique embedding (as $\mathcal{L}_H$-structure) into any $H$-couple over $\boldsymbol{k}$. Here, $\boldsymbol{k}^{\leq 1}$ is the only $H$-cut of $(\boldsymbol{k}, \psi)$.

**Example 2.** Every finite dimensional Hahn space over $\boldsymbol{k}$ is isomorphic to the anti-lexicographically ordered vector space $\boldsymbol{k}^n$, for some $n$. (A non-zero vector $(\alpha_1, \ldots, \alpha_n) \in \boldsymbol{k}^n$ is positive in the anti-lexicographic ordering if and only if $\alpha_i > 0$, where $i = \max\{j : 1 \leq j \leq n, \alpha_j \neq 0\}$.)

Let us fix an $n > 0$, and some positive vector $(\lambda_1, \ldots, \lambda_n)$ in the anti-lexicographically ordered vector space $\boldsymbol{k}^n$ over $\boldsymbol{k}$. Let $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ be the standard basis vectors of $\boldsymbol{k}^n$, so $[e_1] < \cdots < [e_n]$ are the non-zero $\boldsymbol{k}$-archimedean classes of $\boldsymbol{k}^n$ . Let $(\boldsymbol{k}^n, \psi)$ be an $H$-couple with $(\lambda_1, \ldots, \lambda_n)$ as distinguished positive element. Define the $n \times n$-matrix $A = (\alpha_{ij}) \in \boldsymbol{k}^{n\times n}$ by

$$(\alpha_{i1}, \ldots, \alpha_{in}) := \psi(e_i) \quad \text{for } i = 1, \ldots, n.$$

Then $A$ has the following properties:

1. $(\alpha_{11}, \ldots, \alpha_{1n}) < \cdots < (\alpha_{n1}, \ldots, \alpha_{nn})$;
2. $\alpha_{ij} = \alpha_{jj}$ for all $1 \leq j < i \leq n$;
3. $(\lambda_1, \ldots, \lambda_n) = (\alpha_{i1}, \ldots, \alpha_{in}) > 0$, where $i := \max\{j : 1 \leq j \leq n, \lambda_j \neq 0\}$.

Here, (1) and (3) follow from axiom (2) and (1) for $H$-couples, respectively, whereas (2) is derived from $\psi(e_i) - \psi(e_j) < \varepsilon \cdot e_j$ for all $\varepsilon > 0$ and $1 \leq j < i \leq n$, which holds by axiom (3). Conversely, given a matrix $A = (\alpha_{ij}) \in \boldsymbol{k}^{n\times n}$ with properties (1)–(3), define $\psi$ by setting

$$\psi(\mu_1,\dots,\mu_n) := (\alpha_{i1},\dots,\alpha_{in}), \quad \text{where } i := \max\{j : 1 \le j \le n, \mu_j \neq 0\},$$

for $(\mu_1,\dots,\mu_n) \neq 0$ in $\boldsymbol{k}^n$, thus obtaining an $H$-couple $(\boldsymbol{k}^n,\psi)$ with distinguished positive element $(\lambda_1,\dots,\lambda_n)$. In this way, we get a one-to-one correspondence between $H$-couples $(\boldsymbol{k}^n,\psi)$ with distinguished positive element $(\lambda_1,\dots,\lambda_n)$ and matrices $A \in \boldsymbol{k}^{n\times n}$ with the three properties above.

**Example 3.** Given a Hahn space $V$ over $\boldsymbol{k}$, consider the embedding $V \hookrightarrow H(\Gamma,\boldsymbol{k})$ with $\Gamma = [V^*]$ as in Proposition 1.1 and Corollary 1.2. Suppose $V$ has countable dimension over $\boldsymbol{k}$. Then the proof of a theorem of Brown [5] shows that the image of this embedding is the direct sum

$$\boldsymbol{k}^{(\Gamma)} := \big\{v \in \boldsymbol{k}^\Gamma : \operatorname{supp} v \text{ finite}\big\}.$$

It is easy, for any linearly ordered set $\Gamma \neq \varnothing$, to give a description of $H$-couples $\big(\boldsymbol{k}^{(\Gamma)},\psi\big)$ with distinguished positive element $(\lambda_\gamma)_{\gamma\in\Gamma}$ in terms of certain infinite matrices in $\boldsymbol{k}^{\Gamma\times\Gamma}$, similar to the previous example. As an example, consider $\Gamma := \mathbb{Z}$, $V := \boldsymbol{k}^{(\Gamma)} = \bigoplus_{s\in\mathbb{Z}} \boldsymbol{k}e_s$ with $e_s > 0$ and $[e_s] < [e_{s+1}]$, for all $s \in \mathbb{Z}$. Define $\psi : V^* \to V$ by making $\psi$ constant on each $\boldsymbol{k}$-archimedean class, and setting

$$\psi(e_n) := -\big(e_1 + \cdots + e_{n-1}\big) \quad \text{if } n > 0$$

(hence $\psi(e_1) = 0$), and

$$\psi(e_{-n}) := e_{-n} + e_{-n+1} + \cdots + e_0 \quad \text{if } n \ge 0.$$

Then $(V,\psi)$, with distinguished positive element $e_0$, is an $H$-couple. In fact, if $\boldsymbol{k} = \mathbb{R}$, it is the $H$-couple associated to the smallest Hardy field which is closed under powers and contains $\mathbb{R}\big(\dots, e^x, x, \log x, \log\log x, \dots\big)$. (See [15], p. 263, [20], Cor. 2.) Note that $\Gamma$ has no smallest element, but that $(V,\psi)$ has only one $H$-cut. Modifying the definition of $\psi$ above by letting

$$\psi(e_{-n}) := e_0 + e_{-1} - e_{-n-1} \quad \text{for } n \ge 0,$$

we get an example of an $H$-couple with distinguished positive element $e_0$, and with two $H$-cuts, since in this case $\sup\Psi$ exists and equals $e_0 + e_{-1} \notin \Psi$.

**Example 4.** The $H$-couple associated with a maximal Hardy field is closed. More generally: Let $K$ be a Hardy field containing $\mathbb{R}(x)$ and closed under exponentiation (i.e. $f \in K \Longrightarrow e^f \in K$) and integration (i.e. $f \in K \Longrightarrow \exists g \in K : g' = f$). Then $K$ is also closed under powers, and the $H$-couple associated with $K$ is closed.

*Proof.* Note that if $f \in K^{>0}$, then $\log f \in K$, since $(\log f)' = f'/f \in K$. The ordered set $[v(K^\times)^*]$ has no least element since for any $f \in K^{>0}$ with $v(f) > 0$, we have $0 < r\cdot v(1/\log f) < v(f)$ for all $r \in \mathbb{R}^{>0}$. It remains to show that for $f \in K^\times$: either $v(f) = v(g'/g)$ for some $g \in K^\times$, $v(g) > 0$, or $v(f) = v(g')$ for some $g \in K^\times$, $v(g) > 0$. Take $g \in K^\times$ with $g' = f$. If $v(g) \ge 0$, then by subtracting a real constant from $g$ if necessary, we may assume $v(g) > 0$, and we are done. If $v(g) < 0$, then, changing $f$ to $-f$ and $g$ to $-g$ if necessary, we may assume $g$ is negative infinite, i.e. $g < \mathbb{R}$. Then $G := e^g$ satisfies $f = G'/G$, so $v(f) = v(G'/G)$ and $v(G) > 0$. □

In the next examples, we assume familiarity with [7].

**Example 5.** Let $\boldsymbol{k}$ be an ordered logarithmic-exponential field, and $F$ any differential subfield of $\boldsymbol{k}((t))^{\mathrm{LE}}$ containing $\boldsymbol{k}(x)$ and closed under powers. Then with $V := v(F^\times)$ and $\psi$ defined as in the introduction, we get an $H$-couple $(V,\psi)$ with distinguished positive element $1 = v(t)$. Moreover, if $F$ is also closed under exponentiation and integration, then $(V,\psi)$ is closed.

*Proof.* The valuation $v$ on $\boldsymbol{k}((t))^{\mathrm{LE}}$ is defined in terms of the leading monomial map Lm via $v(f) = -\log\bigl(\mathrm{Lm}(f)\bigr)$ for $f \in \boldsymbol{k}((t))^{\mathrm{LE}}$, $f \neq 0$, see [7], (2.9). In particular, the ordered $\boldsymbol{k}$-linear space $V$ (the value group) is an ordered $\boldsymbol{k}$-linear subspace of $\boldsymbol{k}((t))^{\mathrm{LE}}$ itself. It is easy to see that the ordered $\boldsymbol{k}$-linear space $\boldsymbol{k}((t))^{\mathrm{LE}}$ is a Hahn space, and thus $V$ is a Hahn space. It is clear that axiom (1) for $H$-couples holds, while axiom (2) is (4.5) in [7]. Axiom (3) is a consequence of Theorem 4, (c) in [16], since $v$ is a differential valuation by [7], (4.1). The last part of the statement follows by adapting the proof in the previous example. $\square$

**Example 6.** Let $\boldsymbol{k}$ be as in the previous example, and $F = \boldsymbol{k}((t))^{\mathrm{E}}$ the differential subfield of exponential series in $\boldsymbol{k}((t))^{\mathrm{LE}}$. One shows easily that $F$ is closed under powers. In the corresponding $H$-couple $(V,\psi)$, where $V := v(F^\times)$, the element $1 = v(t) = \psi(x)$ is the largest element of $\psi(V^*) = \Psi$, see (2.2) in [7]. It follows from results in [7], §5, that for each $f \in F^\times$, if $v(f) > 1$, then $f = g'$ for some $g \in F$ with $v(g) > 0$, while if $v(f) \leq 1$, then $f = g'/g$ for some $g \in F^\times$ with $v(g) > 0$. Thus $\Psi = V^{\leq 1}$ and $(\mathrm{id}+\psi)\bigl(V^{>0}\bigr) = V^{>1}$.

**Embedding into closed $H$-triples.** Besides the $H$-triple $(V,\psi,P)$ over $\boldsymbol{k}$ we now let $(V',\psi',P')$ denote a second $H$-triple with ordered scalar field $\boldsymbol{k}'$. Since we are dealing here with (two-sorted) $\mathcal{L}_{H,P}$-structures there is a well-defined notion of embedding $i\colon (V,\psi,P) \to (V',\psi',P')$. Such an embedding $i$ is uniquely determined by its **scalar part** $i_{\mathrm{s}}\colon \boldsymbol{k} \to \boldsymbol{k}'$, an ordered field embedding, and its **vector part** $i_{\mathrm{v}}\colon V \to V'$, an ordered group embedding between the underlying ordered additive groups of $V$ and $V'$. Conversely, given an ordered field embedding $i_1 : \boldsymbol{k} \to \boldsymbol{k}'$ and an ordered group embedding $i_2 : V \to V'$ between the underlying ordered additive groups of $V$ and $V'$ there is an embedding $i : (V,\psi,P) \to (V',\psi',P')$ with $i_{\mathrm{s}} = i_1$ and $i_{\mathrm{v}} = i_2$ if and only if $i_1$ and $i_2$ satisfy the compatibility conditions

$$i_2(\lambda u) = i_1(\lambda) i_2(u), \quad i_2\bigl(\psi(v)\bigr) = \psi'\bigl(i_2(v)\bigr), \quad i_2(1_V) = 1_{V'},$$

for all $u, v \in V$, $v \neq 0$, $\lambda \in \boldsymbol{k}$, and

$$P' \cap i_2(V) = i_2(P).$$

Here $1_V$, $1_{V'}$ are the distinguished positive elements of $(V,\psi,P)$ and $(V',\psi',P')$, respectively. Given an embedding $i$ as above we usually write $i(\lambda)$ for $i_{\mathrm{s}}(\lambda)$ when $\lambda \in \boldsymbol{k}$, and $i(a)$ for $i_{\mathrm{v}}(a)$ when $a \in V$. Note that $i$ induces an embedding $[v] \mapsto \bigl[i(v)\bigr] : [V] \to [V']$ of linearly ordered sets.

We will now show that each $H$-triple $(V,\psi,P)$ has an **$H$-closure** $(V^{\mathrm{c}},\psi^{\mathrm{c}},P^{\mathrm{c}})$ in the following sense: $(V^{\mathrm{c}},\psi^{\mathrm{c}},P^{\mathrm{c}})$ is a closed $H$-triple over $\boldsymbol{k}$ extending $(V,\psi,P)$ such that any embedding $(V,\psi,P) \to (V',\psi',P')$ into a closed $H$-triple (not necessarily over $\boldsymbol{k}$) extends to an embedding $(V^{\mathrm{c}},\psi^{\mathrm{c}},P^{\mathrm{c}}) \to (V',\psi',P')$. (We do not require uniqueness.) Later, we will see that any two $H$-closures of $\mathcal{V} = (V,\psi,P)$ are isomorphic over $\mathcal{V}$, that is isomorphic by an isomorphism whose vector part is the identity on $V$ and whose scalar part is the identity on $\boldsymbol{k}$. (See Corollary 4.6.) Towards the existence proof we show three basic extension lemmas:

**Lemma 2.5.** *Suppose $a \in V$, $P < a < (\mathrm{id}+\psi)\big(V^{>0}\big)$. Then $(V,\psi,P)$ extends to an $H$-triple $(V \oplus \boldsymbol{k}\varepsilon, \psi^\varepsilon, P^\varepsilon)$ over $\boldsymbol{k}$ such that:*

1. *$\varepsilon > 0$, $a = \varepsilon + \psi^\varepsilon(\varepsilon)$.*
2. *Given any embedding $i\colon (V,\psi,P) \to (V',\psi',P')$ of $H$-triples and any $\varepsilon' \in V'$ with $\varepsilon' > 0$ and $i(a) = \varepsilon' + \psi'(\varepsilon')$, there is a unique extension of $i$ to an embedding $j\colon (V \oplus \boldsymbol{k}\varepsilon, \psi^\varepsilon, P^\varepsilon) \to (V',\psi',P')$ with $j(\varepsilon) = \varepsilon'$.*

*Proof.* Take an ordered vector space $V \oplus \boldsymbol{k}\varepsilon$ over $\boldsymbol{k}$ extending the ordered vector space $V$, such that $0 < \varepsilon < V^{>0}$. One verifies immediately that $V \oplus \boldsymbol{k}\varepsilon$ is a Hahn space. For a non-zero vector $w = v + \lambda\varepsilon$ ($v \in V$, $\lambda \in \boldsymbol{k}$), we put

$$\psi^\varepsilon(w) := \begin{cases} \psi(v), & \text{if } v \neq 0 \\ a - \varepsilon, & \text{otherwise.} \end{cases}$$

Also let $P^\varepsilon := \big\{w \in V \oplus \boldsymbol{k}\varepsilon : w \leq a - \varepsilon\big\}$. One verifies easily that $(V \oplus \boldsymbol{k}\varepsilon, \psi^\varepsilon, P^\varepsilon)$ is an $H$-triple extending $(V,\psi,P)$. Let $i\colon (V,\psi,P) \to (V',\psi',P')$ be an embedding of $H$-triples, and $\varepsilon' \in V'$, $\varepsilon' > 0$, with $i(a) = \varepsilon' + \psi'(\varepsilon')$. By making the usual identifications we may assume that $(V,\psi,P) \subseteq (V',\psi',P')$, and that $i$ is the natural inclusion. Then $0 < \varepsilon' < V^{>0}$, hence the inclusion $V \hookrightarrow V'$ extends to an embedding $V \oplus \boldsymbol{k}\varepsilon \to V'$ of ordered vector spaces over $\boldsymbol{k}$ sending $\varepsilon$ to $\varepsilon'$. It is easy to check that this embedding is the vector part of an embedding $(V \oplus \boldsymbol{k}\varepsilon, \psi^\varepsilon, P^\varepsilon) \to (V',\psi',P')$ that extends $i$. □

Note that $P^\varepsilon$ as in Lemma 2.5 has a maximum. In this situation we can apply the next lemma.

**Lemma 2.6.** *Suppose $P$ has a largest element. Then $(V,\psi,P)$ extends to an $H$-triple $(V \oplus \boldsymbol{k}\varepsilon, \psi^\varepsilon, P^\varepsilon)$ over $\boldsymbol{k}$ such that:*

1. *$\varepsilon > 0$, $\psi^\varepsilon(\varepsilon) = (\max P) + \varepsilon$.*
2. *Given any embedding $i\colon (V,\psi,P) \to (V',\psi',P')$ of $H$-triples and any $\varepsilon' \in V'$ with $\varepsilon' > 0$ and $\psi'(\varepsilon') = i(\max P) + \varepsilon'$, there is a unique extension of $i$ to an embedding $j\colon (V \oplus \boldsymbol{k}\varepsilon, \psi^\varepsilon, P^\varepsilon) \to (V',\psi',P')$ with $j(\varepsilon) = \varepsilon'$.*

*Proof.* We proceed exactly as in the proof of the previous lemma, except that in the definitions of $\psi^\varepsilon$ and $P^\varepsilon$ we put

$$\psi^\varepsilon(w) := \begin{cases} \psi(v), & \text{if } v \neq 0 \\ (\max P) + \varepsilon, & \text{otherwise,} \end{cases}$$

for non-zero $w = v + \lambda\varepsilon$ ($v \in V$, $\lambda \in \boldsymbol{k}$), and

$$P^\varepsilon := \big\{w \in V \oplus \boldsymbol{k}\varepsilon : w \leq (\max P) + \varepsilon\big\}.$$

Then $\psi^\varepsilon(\varepsilon) = (\max P) + \varepsilon > \max P$. □

We remark that $P^\varepsilon$ as in Lemma 2.6 still has a largest element (though larger than $\max P$), so that this lemma can be applied again to the extension $(V \oplus \boldsymbol{k}\varepsilon, \psi^\varepsilon, P^\varepsilon)$ of $(V,\psi,P)$.

**Lemma 2.7.** *Suppose $b \in P \setminus \Psi$. Then $(V,\psi,P)$ can be extended to an $H$-triple $(V \oplus \boldsymbol{k}a, \psi^a, P^a)$ over $\boldsymbol{k}$ such that:*

1. *$a > 0$, $\psi^a(a) = b$.*

2. *Given any embedding* $i\colon (V,\psi,P)\to(V',\psi',P')$ *of* $H$*-triples and any element* $a'>0$ *in* $V'$ *with* $\psi'(a')=i(b)$*, there is a unique extension of* $i$ *to an embedding* $j\colon (V\oplus \boldsymbol{k}a,\psi^a,P^a)\to(V',\psi',P')$ *with* $j(a)=a'$.

*Proof.* By Corollary 1.2, we may regard $V$ as an ordered linear subspace of $H\big([V^*],\boldsymbol{k}\big)$. We take an object $\gamma\notin[V^*]$ and extend the linear ordering on $[V^*]$ to a linear ordering of $\Gamma:=[V^*]\cup\{\gamma\}$ by setting $\gamma<[v]:\Longleftrightarrow b>\psi(v)$, for all $v\in V^*$. Next we view $H\big([V^*],\boldsymbol{k}\big)$ as an ordered linear subspace of $H(\Gamma,\boldsymbol{k})$ by identifying each function $f\colon [V^*]\to\boldsymbol{k}$ in $H\big([V^*],\boldsymbol{k}\big)$ with its extension to $\Gamma$ obtained by setting $f(\gamma):=0$. Thus $V\subseteq H(\Gamma,\boldsymbol{k})$. Choose $a>0$ in $H(\Gamma,\boldsymbol{k})$ with $\max(\operatorname{supp} a)=\gamma$. Note that $V\oplus\boldsymbol{k}a$ is a Hahn space, as an ordered linear subspace of the Hahn space $H(\Gamma,\boldsymbol{k})$. For non-zero $w=v+\lambda a$ ($v\in V$, $\lambda\in\boldsymbol{k}$), we set

$$\psi^a(w):=\begin{cases}\psi(v), & \text{if } [w]=[v]\\ b, & \text{otherwise, i.e. if } [w]=[a].\end{cases}$$

Also set $P^a:=\big\{w\in V\oplus\boldsymbol{k}a : w\le v \text{ for some } v\in P\big\}$. We have to check that then $(V\oplus\boldsymbol{k}a,\psi^a,P^a)$ is an $H$-triple over $\boldsymbol{k}$, and that $(V,\psi,P)\subseteq(V\oplus\boldsymbol{k}a,\psi^a,P^a)$. It is immediate that axioms (1) and (2) for $H$-couples are satisfied. The main point is axiom (3): Let $w=v+\lambda a$, $w'=v'+\lambda' a$ be positive elements of $V\oplus\boldsymbol{k}a$ ($v,v'\in V$, $\lambda,\lambda'\in\boldsymbol{k}$); we have to show that $\psi^a(w')<\psi^a(w)+w$. We can assume $[w']<[w]$, since otherwise $\psi^a(w')\le\psi^a(w)<\psi^a(w)+w$. We distinguish the following cases:

1. $[w']=[v']$, $[w]=[v]$. Then $\big[\psi(v')-\psi(v)\big]<[v'-v]=[v]=[w]$, hence $\psi^a(w')=\psi(v')<\psi(v)+w=\psi^a(w)+w$.
2. $[w']=[a]$, $[w]=[v]$. By basic properties (1) and (2) stated at the beginning of this section, we get $\big[b-\psi(v)\big]<[a-v]=[v]=[w]$, hence $\psi^a(w')=b<\psi(v)+w=\psi^a(w)+w$.
3. $[w']=[v']$, $[w]=[a]$. Similar to the second case, $\big[\psi(v')-b\big]<[v'-a]=[a]=[w]$, hence $\psi^a(w')=\psi(v')<b+w=\psi^a(w)+w$.

Moreover, $P^a$ is the unique $H$-cut of $(V\oplus\boldsymbol{k}a,\psi^a)$ such that $P^a\cap V=P$. To see this, let $P^a_0$ be any $H$-cut of $(V\oplus\boldsymbol{k}a,\psi^a)$ with $P^a_0\cap V=P$. Assume we are given $v\in V$ and $\lambda\in\boldsymbol{k}^\times$. To determine when $v+\lambda a\in P^a_0$, we distinguish several cases:

1. $\lambda>0$, $v\ge b$. Then $v+\lambda a\ge\psi^a(\lambda a)+\lambda a$, hence $v+\lambda a\notin P^a_0$.
2. $\lambda>0$, $b-v>a$. Then $\psi^a(a)>v+\lambda a$, hence $v+\lambda a\in P^a_0$.
3. $\lambda>0$, $0<b-v<a$. Choose $0<\mu<\lambda$. Then $b-v<(\lambda-\mu)a$, hence $v+\lambda a>b+\mu a=\psi^a(\mu a)+\mu a$, implying $v+\lambda a\notin P^a_0$.
4. $\lambda<0$, $v-b>a$. Then $v-b>(1-\lambda)a$, hence $v+\lambda a>\psi^a(\lambda a)+\lambda a$, so we get $v+\lambda a\notin P^a_0$.
5. $\lambda<0$, $v-b<a$. Then $v-b<-\lambda a$, hence $v+\lambda a<\psi^a(a)$, so $v+\lambda a\in P^a_0$.

Therefore $v+\lambda a\in P^a_0$ if and only if either $\lambda>0$ and $b-v>a$, or $\lambda<0$ and $v-b<a$. Hence $P^a=P^a_0$.

Now let $i\colon (V,\psi,P)\to(V',\psi',P')$ be an embedding of $H$-triples and $a'$ a positive element of $V'$ with $\psi'(a')=b$. We can assume that $(V,\psi,P)\subseteq(V',\psi',P')$, and that $i$ is the inclusion. Note that $a'\notin V$ determines the same cut in $V$ as $a$. Hence the inclusion $V\hookrightarrow V'$ extends to a unique embedding $V\oplus\boldsymbol{k}a\to V'$ of ordered vector spaces over $\boldsymbol{k}$ mapping $a$ to $a'$. This embedding is the vector part of an embedding $(V\oplus\boldsymbol{k}a,\psi^a,P^a)\to(V',\psi',P')$, by uniqueness of $P^a$ proved above. $\square$

*Remark.* Assume that in the last lemma there is no element $v \in V$ with $P < v < (\mathrm{id}+\psi)\big(V^{>0}\big)$, and that $P$ has no maximum. Then there is also no $w \in V \oplus \boldsymbol{k}a$ with $P^a < w < (\mathrm{id}+\psi^a)\big((V \oplus \boldsymbol{k}a)^{>0}\big)$, and $P^a$ has no largest element.

Starting with $(V, \psi, P)$ and suitably iterating and alternating the constructions of lemmas 2.5, 2.6 and 2.7 (possibly transfinitely often), we can build an increasing chain of $H$-triples over $\boldsymbol{k}$ whose union is an $H$-closure of $(V, \psi, P)$:

**Corollary 2.8.** *Every $H$-triple has an $H$-closure.* □

**Behavior under scalar extension.**

**Lemma 2.9.** *Let $\boldsymbol{k}' \supseteq \boldsymbol{k}$ be an extension of ordered fields. Then there are unique $\psi_{\boldsymbol{k}'}$ and $P_{\boldsymbol{k}'}$ such that $(V_{\boldsymbol{k}'}, \psi_{\boldsymbol{k}'}, P_{\boldsymbol{k}'})$ is an $H$-triple over $\boldsymbol{k}'$ extending $(V, \psi, P)$.*

*Proof.* Let $\psi_{\boldsymbol{k}'} := \psi'$ as in Lemma 2.3 (where $i : V \to V_{\boldsymbol{k}'}$ is the natural inclusion). We have to show that there is a unique $H$-cut $P_{\boldsymbol{k}'}$ for $(V_{\boldsymbol{k}'}, \psi_{\boldsymbol{k}'})$ with $P_{\boldsymbol{k}'} \cap V = P$. If $\Psi$ has a largest element, this is clear since $\Psi = \psi(V^*) = \psi_{\boldsymbol{k}'}(V^*_{\boldsymbol{k}'})$. So assume $\Psi$ has no largest element. Suppose $v \in V_{\boldsymbol{k}'}$ satisfies

$$\psi_{\boldsymbol{k}'}\big(V_{\boldsymbol{k}'}^{>0}\big) < v < (\mathrm{id}+\psi_{\boldsymbol{k}'})\big(V_{\boldsymbol{k}'}^{>0}\big).$$

It suffices to show that then $v \in V$. Let $H := H(\Gamma, \boldsymbol{k})$, $H' := H(\Gamma, \boldsymbol{k}')$, $\Gamma := [V^*]$. Consider the following commutative diagram of ordered vector spaces over $\boldsymbol{k}$ and embeddings between them:

$$\begin{tikzcd}
V_{\boldsymbol{k}'} \arrow[r, "\phi\otimes\mathrm{id}"] & H_{\boldsymbol{k}'} \arrow[r, "\mu"] & H' \\
V \arrow[u] \arrow[r, "\phi"] & H \arrow[u] \arrow[ur, "\iota"'] &
\end{tikzcd}$$

Here, $\phi$ is given by Corollary 1.2, the maps $V \to V_{\boldsymbol{k}'}$ and $H \to H_{\boldsymbol{k}'}$ are obtained from Proposition 1.4, $\iota$ is the natural inclusion $H(\Gamma, \boldsymbol{k}) \hookrightarrow H(\Gamma, \boldsymbol{k}')$, and $\mu$ is uniquely determined as an embedding by $\mu(h \otimes \lambda) = \lambda h$, for $\lambda \in \boldsymbol{k}'$, $h \in H$ (using Corollary 1.5). After identifying the $H$-couples $(V, \psi)$, $(V_{\boldsymbol{k}'}, \psi_{\boldsymbol{k}'})$ and $(H, \psi_H)$ via these embeddings with $\mathcal{L}_H$-substructures of $(H', \psi_{H'})$ we have $V_{\boldsymbol{k}'} \cap H = V$. By Lemma 2.4, $(H, \psi_H)$ and $(H', \psi_{H'})$ have two $H$-cuts, hence $v \in H$. Thus $v \in V$, as desired. □

In the next section we apply this last result as follows. Let $\mathcal{V} = (V, \psi, P)$ and $\mathcal{V}' = (V', \psi', P')$ be $H$-triples over ordered fields $\boldsymbol{k}$ and $\boldsymbol{k}'$, respectively. Let $\mathcal{V}_0 = (V_0, \psi_0, P_0)$ be a substructure of $\mathcal{V}$. Thus $\mathcal{V}_0$ is an $H$-triple over an ordered subfield $\boldsymbol{k}_0$ of $\boldsymbol{k}$. Let an embedding $i_0 \colon \mathcal{V}_0 \to \mathcal{V}'$ be given, and also an embedding $e \colon \boldsymbol{k} \to \boldsymbol{k}'$ of ordered fields, such that $e|\boldsymbol{k}_0 = (i_0)_{\mathrm{s}}$.

By the last lemma the Hahn space $(V_0)_{\boldsymbol{k}} := V_0 \otimes_{\boldsymbol{k}_0} \boldsymbol{k}$ over $\boldsymbol{k}$ expands uniquely to an $H$-triple $(\mathcal{V}_0)_{\boldsymbol{k}}$ over $\boldsymbol{k}$ such that $\mathcal{V}_0 \subseteq (\mathcal{V}_0)_{\boldsymbol{k}}$. With these notations we have:

**Lemma 2.10.** *The embedding $\mathcal{V}_0 \hookrightarrow \mathcal{V}$ extends uniquely to an embedding $(\mathcal{V}_0)_{\boldsymbol{k}} \to \mathcal{V}$ with scalar part $\mathrm{id}_{\boldsymbol{k}}$. The embedding $i_0 \colon \mathcal{V}_0 \to \mathcal{V}'$ extends uniquely to an embedding $(\mathcal{V}_0)_{\boldsymbol{k}} \to \mathcal{V}'$ with scalar part $e$.* □

*Proof.* By Corollary 1.5 the inclusion $V_0 \hookrightarrow V$ extends uniquely to an embedding $(V_0)_{\boldsymbol{k}} \to V$ of Hahn spaces over $\boldsymbol{k}$. This is actually an embedding $(\mathcal{V}_0)_{\boldsymbol{k}} \to \mathcal{V}$ of $H$-triples with scalar part $\mathrm{id}_{\boldsymbol{k}}$, by the uniqueness property in the last proposition. □

## 3. Elimination of Quantifiers

In this section we obtain the main results of this paper. Let $T_{H,P}$ denote the theory of closed $H$-triples in the language $\mathcal{L}_{H,P}$. By "formula" we shall mean "$\mathcal{L}_{H,P}$-formula". Let $x = (x_1,\dots,x_m)$ denote a tuple of distinct scalar variables, $y = (y_1,\dots,y_n)$ a tuple of distinct vector variables. We call a formula $\eta(x,y)$ a **scalar formula** if it is of the form $\zeta\big(s_1(x,y),\dots,s_N(x,y)\big)$ where $\zeta(z_1,\dots,z_N)$ is a formula in the language of ordered rings (as specified in part 3. of the description of $\mathcal{L}_H$ in the introduction), where $z_1,\dots,z_N$ are scalar variables, and $s_1(x,y),\dots,s_N(x,y)$ are scalar valued terms of $\mathcal{L}_{H,P}$.

**Theorem 3.1.** *Every formula $\varphi(x,y)$ is equivalent in $T_{H,P}$ to a boolean combination of scalar formulas $\eta(x,y)$ and of atomic formulas $\alpha(x,y)$.*

This elimination theorem says in particular that every formula is equivalent in $T_{H,P}$ to a formula that is free of quantifiers over vector variables. It will be derived from the following embedding result:

**Proposition 3.2.** *Let $(V,\psi,P)$ and $(V',\psi',P')$ be closed $H$-triples over $\boldsymbol{k}$ and $\boldsymbol{k}'$, respectively. Assume that $(V',\psi',P')$ is $\kappa$-saturated, where $\kappa := |V|^+$. Let $(V_0,\psi_0,P_0)$ be a substructure of $(V,\psi,P)$, and thus an $H$-triple over a subfield $\boldsymbol{k}_0$ of $\boldsymbol{k}$. Let an embedding $i_0\colon (V_0,\psi_0,P_0)\to(V',\psi',P')$ be given, and also an embedding $e\colon \boldsymbol{k}\to\boldsymbol{k}'$ of ordered fields, such that $e|\boldsymbol{k}_0 = (i_0)_{\mathrm{s}}$. Then $i_0$ can be extended to an embedding $i\colon (V,\psi,P)\to(V',\psi',P')$ such that $e = i_{\mathrm{s}}$.*

We postpone the proof of this proposition and first deduce Theorem 3.1 from it. To this end we use the following consequence of Proposition 3.2.

**Lemma 3.3.** *Let $(V,\psi,P)\subseteq(V',\psi',P')$ be closed $H$-triples over $\boldsymbol{k}$ and $\boldsymbol{k}'$, respectively. Then $\boldsymbol{k}\preceq\boldsymbol{k}'$ (as ordered fields) if and only if $(V,\psi,P)\preceq(V',\psi',P')$.*

*Proof.* One direction being trivial, we assume $\boldsymbol{k}\preceq\boldsymbol{k}'$, and shall derive $(V,\psi,P)\preceq(V',\psi',P')$. Let $\varphi(x,y)$ be a formula. By induction on the complexity of $\varphi$, one shows, for all $(V,\psi,P)$ and $(V',\psi',P')$ as in the hypothesis of the lemma, and all $\lambda\in\boldsymbol{k}^m$, $v\in V^n$:

$$(V,\psi,P)\models\varphi(\lambda,v) \iff (V',\psi',P')\models\varphi(\lambda,v)$$

For the inductive step, let $\varphi=\exists z\theta$, where $\theta(x,y,z)$ is a formula and $z$ a single variable of the vector or scalar sort. Since $\theta$ is of lower complexity than $\varphi$ the direction "$\Rightarrow$" follows from the induction hypothesis. So assume $(V',\psi',P')\models\varphi(\lambda,v)$. Choose a $\kappa$-saturated elementary extension $(V'',\psi'',P'')$ of $(V,\psi,P)$, where $\kappa:=|V'|^+$. Let $\boldsymbol{k}''$ be the scalar field of $(V'',\psi'',P'')$. Then there is an elementary embedding of ordered fields $e\colon\boldsymbol{k}'\to\boldsymbol{k}''$ with $e|\boldsymbol{k}=\mathrm{id}$. By Proposition 3.2, there is an embedding $i\colon(V',\psi',P')\to(V'',\psi'',P'')$ with $i_{\mathrm{s}}=e$ and $i_{\mathrm{v}}|V=\mathrm{id}$. Using the induction hypothesis on $\theta$, it follows that $(V'',\psi'',P'')\models\varphi(\lambda,v)$. We conclude that $(V,\psi,P)\models\varphi(\lambda,v)$. $\square$

*Proof of Theorem 3.1 assuming Proposition 3.2.* Let $(V,\psi,P)$ and $(V',\psi',P')$ be closed $H$-triples over $\boldsymbol{k}$ and $\boldsymbol{k}'$, respectively, and let $(\lambda,v)\in\boldsymbol{k}^m\times V^n$ and $(\lambda',v')\in(\boldsymbol{k}')^m\times(V')^n$ satisfy the same scalar formulas and the same atomic formulas in $(V,\psi,P)$ and $(V',\psi',P')$, respectively. By a standard model-theoretic argument, it suffices to derive from these assumptions that $(\lambda,v)$ and $(\lambda',v')$ satisfy the same formulas in $(V,\psi,P)$ and $(V',\psi',P')$, respectively. We may assume $(V',\psi',P')$

is $\kappa$-saturated, where $\kappa := |V|^+$. Let $(V_0, \psi_0, P_0)$, with scalar field $\boldsymbol{k}_0$, be the substructure of $(V, \psi, P)$ generated by $(\lambda, v)$. Since $(\lambda, v)$ and $(\lambda', v')$ satisfy the same atomic formulas, there exists an embedding $i_0 \colon (V_0, \psi_0, P_0) \to (V', \psi', P')$ such that $i_0(\lambda_i) = \lambda_i'$ for $i = 1, \dots, m$ and $i_0(v_j) = v_j'$ for $j = 1, \dots, n$. Since $(\lambda, v)$ and and $(\lambda', v')$ satisfy the same scalar formulas, there exists an elementary embedding of ordered fields $e \colon \boldsymbol{k} \to \boldsymbol{k}'$ such that $e|\boldsymbol{k}_0 = (i_0)_{\mathrm{s}}$. By Proposition 3.2, there is an embedding $i \colon (V, \psi, P) \to (V', \psi', P')$ extending $i_0$ such that $e = i_{\mathrm{s}}$. By the previous lemma $i$ is an elementary embedding. Thus $(\lambda, v)$ and $(\lambda', v')$ satisfy the same $\mathcal{L}_{H,P}$-formulas. □

Let $T_{H,P,\mathsf{RCF}} \supseteq T_{H,P}$ be the theory of closed $H$-triples over real closed scalar fields. The following result was announced in the introduction:

**Theorem 3.4.** *The theory $T_{H,P,\mathsf{RCF}}$ is complete, decidable, and admits elimination of quantifiers. It is the model-completion of the theory of $H$-triples.*

The proof uses the following consequence of Proposition 3.2:

**Lemma 3.5.** *Let $(V, \psi, P)$ and $(V', \psi', P')$ be closed $H$-triples over scalar fields $\boldsymbol{k}$ and $\boldsymbol{k}'$, respectively. Then $\boldsymbol{k} \equiv \boldsymbol{k}'$ if and only if $(V, \psi, P) \equiv (V', \psi', P')$.*

*Proof.* One direction being trivial, we assume $\boldsymbol{k} \equiv \boldsymbol{k}'$, and shall derive $(V, \psi, P) \equiv (V', \psi', P')$. We can assume that $(V', \psi', P')$ is $\kappa$-saturated, where $\kappa := |V|^+$. We may further assume, by example 1 of section 2, that $(V, \psi, P)$ and $(V', \psi', P')$ have as common substructure an $H$-triple $(V_0, \psi_0, P_0)$ over the scalar field $\boldsymbol{k}_0 := \mathbb{Q}$. Since $\boldsymbol{k} \equiv \boldsymbol{k}'$ and $\boldsymbol{k}'$ is $|\boldsymbol{k}|^+$-saturated, there is an elementary embedding of ordered fields $e \colon \boldsymbol{k} \to \boldsymbol{k}'$. Since necessarily $e|\boldsymbol{k}_0 = \mathrm{id}$, Proposition 3.2 implies that $e$ is the scalar part of an embedding $(V, \psi, P) \to (V', \psi', P')$, which is elementary by Lemma 3.3. Thus $(V, \psi, P) \equiv (V', \psi', P')$. □

*Proof of Theorem 3.4.* The completeness of the theory $\mathsf{RCF}$ of real closed ordered fields, together with Lemma 3.5, implies the completeness of $T_{H,P,\mathsf{RCF}}$. By Corollary 2.8, every $H$-triple can be embedded into a closed $H$-triple over a real closed field. That $T_{H,P,\mathsf{RCF}}$ admits quantifier elimination follows from Theorem 3.1 and the fact that $\mathsf{RCF}$ admits quantifier elimination. □

The rest of this section is devoted to proving Proposition 3.2.

**The functions $\psi_a$.** We need a generalization of the intermediate value property of $\mathrm{id} + \psi$ on $V^{>0}$ and $V^{<0}$. Below, let $\mathcal{V} = (V, \psi)$ be an $H$-couple (not necessarily closed) over the scalar field $\boldsymbol{k}$. For $a = (a_1, \dots, a_n) \in V^n$, $n > 0$, we define a function $\psi_a \colon V_\infty \to V_\infty$. We proceed by induction:

1. For $n = 1$ (with $a \in V$) we put $\psi_a(v) := \psi(v - a)$.
2. For $n > 1$, we put $\psi_a(v) := \psi\big(\psi_{a'}(v) - a_n\big)$, where $a' := (a_1, \dots, a_{n-1})$.

We let $D_a := \big\{v \in V : \psi_a(v) \neq \infty\big\}$. Thus $D_a = V \setminus \{a\}$ for $n = 1$, and $D_a = \big\{v \in D_{a'} : \psi_{a'}(v) \neq a_n\big\}$ for $n > 1$. So given $a_1, a_2, a_3, \dots$ in $V$, we get

$$\psi_{(a_1,a_2)}(v) = \psi\big(\psi(v - a_1) - a_2\big),$$
$$\psi_{(a_1,a_2,a_3)}(v) = \psi\Big(\psi\big(\psi(v - a_1) - a_2\big) - a_3\Big),$$

and so on. One verifies easily by induction on $n$ that if $v, v' \in D_a$ with $v \neq v'$, then $\big[\psi_a(v) - \psi_a(v')\big] < [v - v']$.

**Lemma 3.6.** *Let $a=(a_1,\dots,a_n)\in V^n$, $\lambda_1,\dots,\lambda_n\in\boldsymbol{k}$, $n>0$. The function*

$$v\mapsto v+\lambda_1\psi_{a_1}(v)+\lambda_2\psi_{(a_1,a_2)}(v)+\cdots+\lambda_n\psi_a(v)\colon D_a\to V$$

*is strictly increasing, and has the intermediate value property on each convex component of $D_a$.*

*Proof.* Let $\eta\colon D_a\to V$ be the function given by

$$\eta(v):=\lambda_1\psi_{a_1}(v)+\lambda_2\psi_{(a_1,a_2)}(v)+\cdots+\lambda_n\psi_a(v).$$

Then $v\mapsto v+\eta(v)\colon D_a\to V$ is strictly increasing, since for distinct $v,v'\in D_a$ we have $[\eta(v)-\eta(v')]<[v-v']$. Let $C$ be a convex component of $D_a$ with $a_1<C$, and let $x<y<z$ be in $C$, with $z-y\le y-x$. Then

$$y-a_1<z-a_1=(z-y)+(y-a_1)\le 2(y-a_1),$$

so $\psi(y-a_1)=\psi(z-a_1)$; hence $\eta(y)=\eta(z)$ since $\eta(v)$ depends only on $\psi(v-a_1)$. By Lemma 1.6 the function $\eta|C$ has the intermediate value property. For the convex components $<a_1$ of $D_a$ we verify instead condition $(2')$ of the remark following Lemma 1.6. □

**Lemma 3.7.** *Suppose $\mathcal{V}$ is closed, and let $a=(a_1,\dots,a_n)\in V^n$, $n>0$. Then $D_a$ has at most $2^n$ convex components in $V$, and on each of these, $\psi_a$ is monotone and has the intermediate value property.*

*Proof.* We proceed by induction on $n$. For $n=1$, the two convex components of $D_a$ are $V^{<a}$, on which $\psi_a$ is increasing, and $V^{>a}$, on which $\psi_a$ is decreasing; on each of these, $\psi_a$ has the intermediate value property, since $\mathcal{V}$ is closed. Suppose the lemma holds for a certain $a=(a_1,\dots,a_n)\in V^n$, and let $\widehat{a}=(a_1,\dots,a_n,a_{n+1})\in V^{n+1}$. Consider a convex component $C$ of $D_a$. Then $\psi_a$ is monotone on $D_a$, say increasing on $C$, and has the intermediate value property on $C$. Put

$$\begin{aligned}C_1&:=\{v\in C:\psi_a(v)<a_{n+1}\},\\ C_2&:=\{v\in C:\psi_a(v)=a_{n+1}\},\\ C_3&:=\{v\in C:\psi_a(v)>a_{n+1}\}.\end{aligned}$$

Thus $C$ is the disjoint union of its convex subsets $C_1$, $C_2$ and $C_3$, and $C_1<C_2<C_3$. Also $\psi_{\widehat{a}}$ is clearly increasing on $C_1$, and decreasing on $C_3$. If both $C_1$ and $C_3$ are nonempty, then also $C_2$ is nonempty (because of the intermediate value property of $\psi_a$ on $C$), and thus $C_1$ and $C_3$ are the convex components of $D_{\widehat{a}}$ that are contained in $C$. Otherwise $C$ only contributes one convex component to $D_{\widehat{a}}$, or none at all, depending on whether one or both of $C_1$ and $C_3$ are empty. □

**Archimedean classes and coinitiality.**

**Lemma 3.8.** *Let $V\subseteq V'$ be an extension of ordered vector spaces over the ordered field $\boldsymbol{k}$, and let $b\in V'\setminus V$ be such that*

1. *for each $\varepsilon\in V^{>0}$ there are $a,c\in V$ with $a<b<c$ and $c-a<\varepsilon$,*
2. *$\{a\in V:a<b\}$ has no maximum, and $\{c\in V:c>b\}$ has no minimum.*

*Then $[V]=[V\oplus\boldsymbol{k}b]$ (as subsets of $[V']$).*

*Proof.* Assume not. Then there is $v\in V$ with $[b-v]\notin[V]$. Changing $b$ to $-b$ and $v$ to $-v$, if necessary, we may assume $b>v$. Let $\varepsilon\in V^{>0}$ be such that $v+\varepsilon<b$, by (2), and $a,c\in V$ with $v+\varepsilon\le a<b<c$ and $c-a<\varepsilon$, by (1). Then

$b-a<c-a<\varepsilon\leq a-v$, hence $[b-a]\leq[a-v]$. But $b-v=(b-a)+(a-v)$ and thus $[a-v]<[b-v]=[b-a]$, a contradiction. □

For the proof of Proposition 3.2, and also in §4, we shall need the following easy consequence of the lemma above:

**Corollary 3.9.** *Let $(V,\psi)\subseteq(V',\psi')$ be an extension of $H$-couples over $\boldsymbol{k}$ and over $\boldsymbol{k}'\supseteq\boldsymbol{k}$ respectively, such that $[V^*]$ has no minimum. If $x\in V'$ and $0<x<V^{>0}$, then $[V]_{\boldsymbol{k}}=\big[V\oplus\boldsymbol{k}\psi'(x)\big]_{\boldsymbol{k}}$ inside $[V']_{\boldsymbol{k}}$. In particular, if $[V]_{\boldsymbol{k}}\neq[V\oplus\boldsymbol{k}y]_{\boldsymbol{k}}$ for all $y\in V'\setminus V$, then $V^{>0}$ is coinitial in $(V')^{>0}$.*

*Proof.* Let $0<x<V^{>0}$, $x\in V'$. Then $b:=\psi'(x)$ satisfies the hypothesis of the previous lemma, where $V'$ is considered as an ordered vector space over $\boldsymbol{k}$. Thus we have $[V]_{\boldsymbol{k}}=[V\oplus\boldsymbol{k}b]_{\boldsymbol{k}}$. □

**Properties (A) and (B).** Given an extension $(V,\psi)\subseteq(V',\psi')$ of $H$-couples (not necessarily over the same scalar field), and $a=(a_1,\dots,a_n)\in V^n$, $n>0$, we have functions $\psi_a\colon V_\infty\to V_\infty$, with $D_a=\big\{v\in V:\psi_a(v)\neq\infty\big\}$, and $\psi'_a\colon V'_\infty\to V'_\infty$, with $D'_a=\big\{v'\in V':\psi'_a(v')\neq\infty\big\}$. Clearly $\psi_a$ is the restriction of $\psi'_a$ to $V_\infty$, and thus $D'_a\cap V=D_a$. Consider the following two properties of an extension $(V,\psi)\subseteq(V',\psi')$ of *closed* $H$-couples:

(A) For all $a\in V^n$, $n>0$, and convex components $C'$ of $D'_a$, $C'\cap V\neq\varnothing$.
(B) For all $x\in V'$, $a=(a_1,\dots,a_n)\in V^n$, $b\in V$, $\lambda_1,\dots,\lambda_n\in\boldsymbol{k}$, $n>0$, if

$$x+\lambda_1\psi'_{a_1}(x)+\lambda_2\psi'_{(a_1,a_2)}(x)+\cdots+\lambda_n\psi'_a(x)=b,$$

then $x\in V$.

(By Lemma 3.6 and Lemma 3.7 these conditions (A) and (B) are clearly satisfied for *elementary* extensions of closed $H$-couples.)

*Remark.* Let $(V,\psi)\subseteq(V',\psi')$ be an extension of closed $H$-couples satisfying (A) and (B), and let $a\in V^n$, $n>0$. If $v',w'\in D'_a\setminus D_a$ determine the same cut in the ordered set $V$, then they lie in the same convex component of $D'_a$, and $\operatorname{sgn}\big(v'-\psi'_a(v')\big)=\operatorname{sgn}\big(w'-\psi'_a(w')\big)$.

**Proof of Proposition 3.2 using (A) and (B).** The hardest part of the proof of Proposition 3.2 consists in showing that all extensions of closed $H$-couples satisfy (A) and (B). This was the last difficulty we overcame, and accordingly we postpone this part. Thus in this subsection we assume:

*All extensions of closed $H$-couples satisfy* (A) *and* (B).

Let the hypothesis in the statement of Proposition 3.2 hold. To simplify notation, we may as well assume that $(V_0,\psi_0,P_0)$ is a common substructure of $(V,\psi,P)$ and $(V',\psi',P')$, and that $\boldsymbol{k}$ is an ordered subfield of $\boldsymbol{k}'$, with $i_0$ and $e$ the natural inclusions. We want to extend $i_0$ to an embedding $i\colon(V,\psi,P)\to(V',\psi',P')$ such that $i_{\mathrm{s}}=e$. By scalar extension (Lemma 2.10), we can reduce to the case $\boldsymbol{k}_0=\boldsymbol{k}$. By Corollary 2.8, we can further reduce to the case that $(V_0,\psi_0,P_0)$ is closed. We may also assume that $V\neq V_0$. By a familiar Zorn's Lemma argument, it suffices to show that there is some $H$-triple $(V_1,\psi_1,P_1)\subseteq(V,\psi,P)$ strictly containing $(V_0,\psi_0,P_0)$ as a substructure, such that $i_0$ extends to an embedding $(V_1,\psi_1,P_1)\to(V',\psi',P')$.

*Case 1.* Assume that we have $v \in V \setminus V_0$ with $[V_0 \oplus \boldsymbol{k}v] = [V_0]$. Then $\psi(V_0 \oplus \boldsymbol{k}v) = \psi_0(V_0)$, in particular,

$$(V_1, \psi_1, P_1) := \big(V_0 \oplus \boldsymbol{k}v, \psi|(V_0 \oplus \boldsymbol{k}v)^*, P \cap (V_0 \oplus \boldsymbol{k}v)\big)$$

is a substructure of $(V, \psi, P)$. We claim that there is an embedding of this substructure into $(V', \psi', P')$ over $V_0$. To see this, we distinguish two cases:

1. No $u \in V_0 \oplus \boldsymbol{k}v$ satisfies $P_0 < u < V_0 \setminus P_0$. (Thus $\big(V_0 \oplus \boldsymbol{k}v, \psi|(V_0 \oplus \boldsymbol{k}v)^*\big)$ has only one cut, namely $P \cap (V_0 \oplus \boldsymbol{k}v)$.) By saturation, we can find $v' \in V'$ realizing the same cut in $V_0$ as $v$. It follows that we have an isomorphism $V_0 \oplus \boldsymbol{k}v \to V_0 \oplus \boldsymbol{k}v'$ of ordered vector spaces over $\boldsymbol{k}$ that sends $v$ to $v'$ and is the identity on $V_0$. Hence $\psi(v_0 + \lambda v) = \psi'(v_0 + \lambda v')$, for all $v_0 \in V_0$, $\lambda \in \boldsymbol{k}$, and there is no $u' \in V_0 \oplus \boldsymbol{k}v'$ with $P_0 < u' < V_0 \setminus P_0$. (Thus also $\big(V_0 \oplus \boldsymbol{k}v', \psi'|(V_0 \oplus \boldsymbol{k}v')^*\big)$ has only one cut.) So we have an embedding of $(V_1, \psi_1, P_1)$ into $(V', \psi', P')$ as desired.
2. There is $u \in V_0 \oplus \boldsymbol{k}v$ with $P_0 < u < V_0 \setminus P_0$. If $w \in V_0 \oplus \boldsymbol{k}v$ also satisfies $P_0 < w < V_0 \setminus P_0$, then $\psi(\delta) < u, w < \psi(\delta) + \delta$, hence $|u - w| < \delta$, for all $\delta \in V_0^{>0}$. Therefore $u = w$ because of $[V_0 \oplus \boldsymbol{k}v] = [V_0]$. After renaming, we may also assume $u = v$. By saturation, we can find $v' \in V$ such that $P_0 < v' < V_0 \setminus P_0$ and such that $v \in P \Leftrightarrow v' \in P'$. It follows as before that we get an embedding of $(V_1, \psi_1, P_1)$ into $(V', \psi', P')$ as desired.

*Case 2.* Assume that for every $v \in V \setminus V_0$ we have $[V_0 \oplus \boldsymbol{k}v] \neq [V_0]$. Fix some $v \in V \setminus V_0$. Then there is some $a_1 \in V_0$ such that $[v - a_1] \notin [V_0]$, hence $\psi(v - a_1) \notin V_0$. So for some $a_2 \in V_0$, $\big[\psi(v - a_1) - a_2\big] \notin [V_0]$, hence $\psi\big(\psi(v - a_1) - a_2\big) \notin V_0$. Continuing this way, we obtain elements $a_1, a_2, a_3, \dots$ in $V_0$ such that for all $n \geq 1$, $\psi_{(a_1,\dots,a_n)}(v) \notin V_0$. (We use the notation introduced earlier in this section.) Let

$$b_1 := v - a_1, \qquad b_n := \psi(b_{n-1}) - a_n \quad \text{for } n > 1.$$

Then $[b_n] \notin [V_0]$ and $\psi(b_n) = \psi_{(a_1,\dots,a_n)}(v)$, for all $n \geq 1$. We claim that $\{b_n\}_{n\geq 1}$ is a family of vectors linearly independent over $V_0$. Otherwise, we would have a linear relation among the $b_n$ and elements of $V_0$. By changing from $\{a_n\}_{n\geq 1}$ to $\{a_{n+k}\}_{n\geq 1}$ and from $v$ to $\psi_{(a_1,\dots,a_k)}(v)$, for some $k \geq 1$, if necessary, we can assume it to be of the form

$$v + \lambda_1 \psi_{a_1}(v) + \lambda_2 \psi_{(a_1,a_2)}(v) + \cdots + \lambda_n \psi_a(v) = v_0$$

for some $n > 0$, $a = (a_1, \dots, a_n)$, $\lambda_1, \dots, \lambda_n \in \boldsymbol{k}$, and $v_0 \in V_0$. But then condition (B) would imply $v \in V_0$, contrary to our assumption. Thus in particular,

$$[b_n] \neq [b_m] \quad \text{for all } m > n \geq 1, \tag{3.1}$$

since otherwise $b_{n+1} - b_{m+1} = a_{m+1} - a_{n+1} \in V_0$.

By saturation we can find $v' \in V' \setminus V_0$ realizing the same cut in the ordered set $V_0$ as $v$. Put

$$b'_1 := v' - a_1, \qquad b'_n := \psi'(b'_{n-1}) - a_n \quad \text{for } n > 1.$$

Inductively, using condition (B) and the proof of Lemma 3.7, one shows $v' \in D'_{(a_1,\dots,a_n)}$ for all $n \geq 1$; hence $b'_n \neq \infty$ for all $n \geq 1$. Note that, again by (B), $[b'_n] \notin [V_0]$ for all $n \geq 1$, and that $\{b'_n\}_{n\geq 1}$ is a family of linearly independent vectors over $V_0$.

We will now show, by induction on $n \geq 1$, that the cut $C(b_n)$ determined by $b_n$ in $V_0$ is the same as the cut $C(b'_n)$ determined by $b'_n$ in $V_0$. For $n = 1$, this follows by choice of $v'$. Now assume $n > 1$ and $C(b_{n-1}) = C(b'_{n-1})$. Let

$$C_1 := \{\psi_0(v_0) : v_0 \in V_0, [v_0] > [b_{n-1}]\},$$
$$C_2 := \{u_0 \in V_0 : u_0 \geq \psi_0(v_0) \text{ for some } v_0 \in V_0 \text{ with } [v_0] < [b_{n-1}]\}.$$

Then $C_1 < \psi(b_{n-1}) < C_2$ and $C_1 < \psi'(b'_{n-1}) < C_2$, $C_1 \cup C_2 = V_0$. Hence $C_1 - a_n < b_n < C_2 - a_n$, $C_1 - a_n < b'_n < C_2 - a_n$, thus $C(b_n) = C(b'_n)$, finishing the inductive step. This implies

$$\operatorname{sgn}(b_n) = \operatorname{sgn}(b'_n) \quad \text{for all } n \geq 1, \tag{3.2}$$

and, by the remark preceding this proof,

$$[b_n] < [b_m] \Leftrightarrow [b'_n] < [b'_m], \quad \text{for all } n, m \geq 1. \tag{3.3}$$

We set

$$V_1 := V_0 \oplus \bigoplus_{n=1}^{\infty} \boldsymbol{k} b_n \subseteq V, \quad \psi_1 := \psi|V_1^*, \quad P_1 := P \cap V_1.$$

Clearly $(V_1, \psi_1, P_1)$ is the $\mathcal{L}_{H,P}$-substructure of $(V, \psi, P)$ generated by $v$ over $(V_0, \psi_0, P_0)$. Consider the (injective) $\boldsymbol{k}$-linear map $V_1 \to V'$ that is the identity on $V_0$ and sends each $b_n$ to $b'_n$. Using the fact that $[b_n] \notin [V_0]$, $[b'_n] \notin [V_0]$, and $C(b_n) = C(b'_n)$, for all $n \geq 1$, together with (3.1)–(3.3), one sees that this map is also order-preserving. Moreover it is easily shown to be the vector part of an embedding $(V_1, \psi_1) \to (V', \psi')$ of $\mathcal{L}_H$-structures with the identity on $\boldsymbol{k}$ as scalar part. To show that we even have an embedding of $\mathcal{L}_{H,P}$-structures $(V_1, \psi_1, P_1) \to (V', \psi', P')$, it suffices to prove that $(V_1, \psi_1)$ has only one $H$-cut. For a contradiction, assume that there exists $v_1 \in V_1$ with $\psi_1(V_1^{>0}) < v_1 < (\mathrm{id} + \psi_1)(V_1^{>0})$. But $\psi_0(V_0^{>0})$ is cofinal in $\psi(V^{>0})$, and $(\mathrm{id} + \psi)(V_0^{>0})$ is coinitial in $(\mathrm{id} + \psi)(V^{>0})$, by Corollary 3.9. This implies $\psi(V^{>0}) < v_1 < (\mathrm{id} + \psi)(V^{>0})$, i.e. $(V, \psi)$ has two $H$-cuts, contradicting the closedness of $(V, \psi)$.

This finishes the proof of Proposition 3.2, except that we still have to prove properties (A) and (B) for all extensions $(V, \psi) \subseteq (V', \psi')$ of closed $H$-couples. The remainder of this section is devoted to this task. □

**Proof of (A) and (B).** We first make a more detailed study of the behavior of the functions $\psi_a$ on the convex components of $D_a$, in the case of a closed $H$-couple. In the remainder of this section we let $\mathcal{V} = (V, \psi)$ be a closed $H$-couple.

**Lemma 3.10.** *Let $p \in V$. Then there is $u \in V$ such that $\psi(x) = u$ for all sufficiently large $x \in \Psi + p$. Moreover, if $(V', \psi')$ is a closed $H$-couple extending $(V, \psi)$, the same $u \in V$ has the property that $\psi'(x) = u$ for all sufficiently large $x \in \Psi' + p$, where $\Psi' := \psi'((V')^*)$.*

*Proof.* First assume $-p \in \Psi$, so $\psi(x) + p > 0$ for all sufficiently small $x > 0$. Now take $x_0 > 0$ in $V$ such that $\psi(x_0) + p > 0$ and $[x_0] < [\psi(x_0) + p]$. (Decreasing $x_0$ makes $\psi(x_0) + p$ increase, so this is indeed possible.) We claim that

$$[\psi(x_0) + p] = [\psi'(x') + p] \quad \text{for all } 0 < x' < x_0 \text{ in } V'.$$

Otherwise,

$$[\psi(x_0) + p] < [\psi'(x') + p] \leq [\psi(x_0) + x_0 + p] = [\psi(x_0) + p],$$

a contradiction. Thus $u := \psi\big(\psi(x_0)+p\big)$ works. Now assume $-p \notin \Psi$. Then $-p = \psi(x_0) + x_0$ for some $x_0 > 0$ in $V$. We claim that

$$[x_0] = \big[\psi'(x') + p\big] \quad \text{for all } 0 < x' < x_0 \text{ in } V'.$$

Otherwise,

$$[x_0] = \big[\psi(x_0)+p\big] < \big[\psi'(x')+p\big] = \big[\psi(x_0) + x_0 - \psi'(x')\big] \le [x_0],$$

a contradiction. So $u := \psi(x_0)$ works in this case. □

*Notation.* We will denote the element $u$ in the lemma above by $\lim_{x\in\Psi+p}\psi(x)$. (Hence $\lim_{x\in\Psi+p}\psi(x) = \lim_{x\in\Psi'+p}\psi'(x)$.)

We fix some terminology. Let $f\colon V_\infty \to V_\infty$ be a function, and let $C$ be a non-empty convex subset of $V$ on which $f$ does not take the value $\infty$. Let $p, q \in V$, and let $S \subseteq V$ be downward closed. (We only use this for $f = \psi_a$, $C$ is a convex component of $D_a$, and $S = \Psi$.)

1. $f$ **increases on** $C$ **from** $p$ **to** $q$ if $f|C$ is increasing, $p \le q$, and
$$f(C) = [p,q] = \big\{v \in V : p \le v \le q\big\}.$$
(We allow $f|C$ constant and $p = q$.)
2. $f$ **increases on** $C$ **from** $-\infty$ **to** $q$ if $f|C$ is increasing and
$$f(C) = (-\infty, q] = \big\{v \in V : v \le q\big\}.$$
3. $f$ **increases on** $C$ **from** $p$ **to** $S$ if $f|C$ is increasing, and
$$f(C) = \{v \in S : v \ge p\}.$$
4. $f$ **increases on** $C$ **from** $-\infty$ **to** $S$ if $f|C$ is increasing, and $f(C) = S$.

Similarly, one defines what it means that $f$ **decreases on** $C$ **from** $p$ **to** $q$, $f$ **decreases on** $C$ **from** $p$ **to** $-\infty$, $f$ **decreases on** $C$ **from** $S$ **to** $q$, and $f$ **decreases on** $C$ **from** $S$ **to** $-\infty$.

Let now a second closed $H$-couple $\mathcal{V}' = (V', \psi')$ extending $\mathcal{V} = (V, \psi)$ be given, and let $a = (a_1, \dots, a_n) \in V^n$, $n > 0$.

Below, we write "component" instead of "convex component".

**Lemma 3.11.**

1. *Each component $C$ of $D_a$ is contained in a (necessarily unique) component $C'$ of $D'_a$, and the map $C \mapsto C'$ is a bijection between the set of components of $D_a$ and the set of components of $D'_a$, with $C' \cap V = C$ for each component $C$ of $D_a$.*
2. *$D_a$ has a (necessarily unique) component $C_\infty > a_1$ that is unbounded in $V$; the corresponding component $C'_\infty$ of $D'_a$ is unbounded in $V'$.*
3. *Let $C$ be a bounded component of $D_a$. Then there are $p, q \in V$ such that one of the following holds:*
   (a) *$\psi_a$ increases on $C$ from $p$ to $q$, and $\psi'_a$ increases on $C'$ from $p$ to $q$.*
   (b) *$\psi_a$ decreases on $C$ from $p$ to $q$, and $\psi'_a$ decreases on $C'$ from $p$ to $q$.*
   (c) *$\psi_a$ increases on $C$ from $p$ to $\Psi$, $\psi'_a$ increases on $C'$ from $p$ to $\Psi'$.*
   (d) *$\psi_a$ decreases on $C$ from $\Psi$ to $q$, $\psi'_a$ decreases on $C'$ from $\Psi'$ to $q$.*
4. *Let $C_\infty$ be the unbounded component $> a_1$ of $D_a$, $C'_\infty$ the corresponding component of $D'_a$. Then one of the following holds:*
   (a) *$\psi_a$ decreases on $C_\infty$ from $\Psi$ to $-\infty$, $\psi'_a$ decreases on $C'_\infty$ from $\Psi'$ to $-\infty$.*

(b) *There is* $p \in V$ *such that* $\psi_a$ *decreases on* $C_\infty$ *from* $p$ *to* $-\infty$, *and* $\psi'_a$ *decreases on* $C'_\infty$ *from* $p$ *to* $-\infty$.

*Proof.* We proceed by induction on $n$. The case $n = 1$ is easy to verify. Suppose the lemma holds for a certain $a = (a_1, \dots, a_n) \in V^n$. Let $\widehat{a} = (a_1, \dots, a_n, a_{n+1}) \in V^{n+1}$. Since the components $< a_1$ of $D_a$ are obtained from the components $> a_1$ of $D_a$ by reflection at the point $x = a_1$, about which the functions $\psi_a$ and $\psi'_a$ are symmetric, we only need to consider the case of components $> a_1$. So let $C > a_1$ be a component of $D_a$, and $C'$ the corresponding component of $D'_a$. Define

$$\begin{aligned} C_1 &:= \{v \in C : \psi_a(v) < a_{n+1}\}, \\ C_2 &:= \{v \in C : \psi_a(v) = a_{n+1}\}, \\ C_3 &:= \{v \in C : \psi_a(v) > a_{n+1}\}, \end{aligned}$$

and define the sets $C'_i$ for $i = 1, 2, 3$ in the same way, by replacing $C$ by $C'$ and $\psi_a$ by $\psi'_a$. Hence $C'_i \cap V = C_i$, for $i = 1, 2, 3$. The components of $D_{\widehat{a}}$ that are contained in $C$ are the nonempty sets among $C_1$ and $C_3$, and similarly, the components of $D'_{\widehat{a}}$ that are contained in $C'$ are the nonempty sets among $C'_1$ and $C'_3$.

Assume first $C$ is bounded in $V$ (and hence $C'$ is bounded in $V'$). We shall assume $\psi_a$ is increasing on $C$ (hence $\psi'_a$ increasing on $C'$). The case that $\psi_a$ is decreasing on $C$ is similar and left to the reader. We distinguish several cases:

1. There exist $p, q \in V$ such that $\psi_a$ increases on $C$ from $p$ to $q$, and $\psi'_a$ increases on $C'$ from $p$ to $q$.
   (a) $q \le a_{n+1}$. Then $C_3, C'_3 = \varnothing$. If $q < a_{n+1}$, then $C_1, C'_1 \neq \varnothing$, $C_2, C'_2 = \varnothing$, and $\psi_{\widehat{a}}$ increases on $C_1$ from $\psi(p - a_{n+1})$ to $\psi(q - a_{n+1})$, $\psi'_{\widehat{a}}$ increases on $C'_1$ from $\psi(p - a_{n+1})$ to $\psi(q - a_{n+1})$. If $a_{n+1} = q > p$, then $C_1, C'_1 \neq \varnothing$, $C_2, C'_2 \neq \varnothing$, and $\psi_{\widehat{a}}$ increases on $C_1$ from $\psi(p - a_{n+1})$ to $\Psi$, and $\psi'_{\widehat{a}}$ increases on $C'_1$ from $\psi(p - a_{n+1})$ to $\Psi'$. If $a_{n+1} = p = q$, then $C_1, C'_1 = \varnothing$.
   (b) $a_{n+1} \le p$, $q \neq a_{n+1}$. Then $C_1, C'_1 = \varnothing$ and $C_3, C'_3 \neq \varnothing$. If $a_{n+1} < p$, then $C_2, C'_2 = \varnothing$, and $\psi_{\widehat{a}}$ decreases on $C_3$ from $\psi(p - a_{n+1})$ to $\psi(q - a_{n+1})$, and $\psi'_{\widehat{a}}$ decreases on $C'_3$ from $\psi(p - a_{n+1})$ to $\psi(q - a_{n+1})$. If $a_{n+1} = p$, then $C_2, C'_2 \neq \varnothing$, and $\psi_{\widehat{a}}$ decreases on $C_3$ from $\Psi$ to $\psi(q - a_{n+1})$, and $\psi'_{\widehat{a}}$ decreases on $C'_3$ from $\Psi'$ to $\psi(q - a_{n+1})$.
   (c) $p < a_{n+1} < q$. Then $C_1, C'_1 \neq \varnothing$, $C_2, C'_2 \neq \varnothing$, $C_3, C'_3 \neq \varnothing$. Here, $\psi_{\widehat{a}}$ increases on $C_1$ from $\psi(p - a_{n+1})$ to $\Psi$, and $\psi'_{\widehat{a}}$ increases on $C'_1$ from $\psi(p - a_{n+1})$ to $\Psi'$. Similarly, $\psi_{\widehat{a}}$ decreases on $C_3$ from $\Psi$ to $\psi(q - a_{n+1})$, and $\psi'_{\widehat{a}}$ decreases on $C'_3$ from $\Psi'$ to $\psi(q - a_{n+1})$.
2. There exists $p \in V$ such that $\psi_a$ increases on $C$ from $p$ to $\Psi$, and $\psi'_a$ increases on $C'$ from $p$ to $\Psi'$. (Thus $p \in \Psi$.) This case is essentially treated as the first one, using Lemma 3.10. If, for example, $a_{n+1} < p$, so that $C_1, C'_1 = \varnothing$, $C_2, C_2 = \varnothing$ and $C_3, C'_3 \neq \varnothing$, then $\psi_{\widehat{a}}$ decreases on $C_3$ from $\psi(p - a_{n+1})$ to $\lim_{x \in \Psi - a_{n+1}} \psi(x)$, and $\psi'_{\widehat{a}}$ decreases on $C'_3$ from $\psi(p - a_{n+1})$ to $\lim_{x \in \Psi' - a_{n+1}} \psi'(x) = \lim_{x \in \Psi - a_{n+1}} \psi(x)$. We leave the details to the reader.

Now suppose $C = C_\infty$ is the unbounded component $> a_1$ of $D_a$, and hence $C' = C'_\infty$ the unbounded component $> a_1$ of $D'_a$. We have two cases again:

1. $\psi_a$ decreases on $C$ from $\Psi$ to $-\infty$, and $\psi'_a$ decreases on $C'$ from $\Psi'$ to $-\infty$. If $a_{n+1} > \Psi$, we have $C_1, C'_1 \neq \varnothing$ and $C_2, C'_2, C_3, C'_3 = \varnothing$. Hence $\psi_{\widehat{a}}$ decreases on $C_1$ from $\lim_{x \in \Psi - a_{n+1}} \psi(x)$ to $-\infty$, $\psi'_{\widehat{a}}$ decreases on $C'_1$ from $\lim_{x \in \Psi - a_{n+1}} \psi(x)$ to $-\infty$. In this case, $C_1$ is the unbounded component $> a_1$ of $D_{\widehat{a}}$, $C'_1$ is the

unbounded component $> a_1$ of $D'_{\widehat{a}}$. If, on the other hand, $a_{n+1} \in \Psi$, then $C_1, C'_1 \neq \emptyset$, $C_3, C'_3 \neq \emptyset$. So $\psi_{\widehat{a}}$ decreases on $C_1$ from $\Psi$ to $-\infty$, $\psi'_{\widehat{a}}$ decreases on $C'_1$ from $\Psi'$ to $-\infty$, and $\psi_{\widehat{a}}$ increases on $C_3$ from $\lim_{x\in\Psi-a_{n+1}}\psi(x)$ to $\Psi$, $\psi'_{\widehat{a}}$ increases on $C'_3$ from $\lim_{x\in\Psi-a_{n+1}}\psi(x)$ to $\Psi'$. The unbounded component $> a_1$ of $D_{\widehat{a}}$ is $C_1$, and the unbounded component $> a_1$ of $D'_{\widehat{a}}$ is $C'_1$.

2. There is $p \in V$ such that $\psi_a$ decreases on $C$ from $p$ to $-\infty$, and $\psi'_a$ decreases on $C$ from $p$ to $-\infty$. This case is treated similarly to the previous one, except that we now have three subcases, according to whether $a_{n+1} > p$, $a_{n+1} = p$, or $a_{n+1} < p$.

This finishes the inductive step, hence the proof of the lemma. □

**Corollary 3.12.** *$\psi'_a(C') \cap V = \psi_a(C)$, for each component $C$ of $D_a$.* □

We now also fix scalars $\lambda_1, \dots, \lambda_n \in \boldsymbol{k}$, so that we have functions $\theta\colon D_a \to V$ and $\theta'\colon D'_a \to V$ given by

$$\theta(v) := v + \lambda_1\psi_{a_1}(v) + \lambda_2\psi_{(a_1,a_2)}(v) + \cdots + \lambda_n\psi_a(v),$$
$$\theta'(v') := v' + \lambda_1\psi'_{a_1}(v') + \lambda_2\psi'_{(a_1,a_2)}(v') + \cdots + \lambda_n\psi'_a(v'),$$

for $v \in D_a$ and $v' \in D'_a$. Note that $\theta = \theta'|D_a$.

*Remark.* Let $C_\infty$ be the unbounded component $> a_1$ of $D_a$. Then $\theta$ is not bounded from above on $C_\infty$, that is, for any $b \in V$ there exists $x \in C_\infty$ with $\theta(x) > b$. To see this, note that $\big[\theta(x) - \theta(y)\big] = [x - y]$ for all $x, y \in D_a$, and that $[V]$ has no maximum, by closedness of $(V, \psi)$. Now choose any $y \in C_\infty$, and $x > y$ such that $[x] > [y], \big[b - \theta(y)\big]$. Then $\big[\theta(x) - \theta(y)\big] > \big[b - \theta(y)\big]$, in particular $\theta(x) > b$. Similarly, $\theta$ is not bounded from below on the unbounded component $< a_1$ of $D_a$.

**Lemma 3.13.** *Let $C$ be a component of $D_a$, with corresponding component $C'$ of $D'_a$. If $d \in C' \setminus C$, then $\theta'(d) \in V' \setminus V$.*

*Proof.* We proceed by induction on $n$. The case $n = 1$ is easily checked using Lemma 2.1. Assume the lemma holds for a certain $a = (a_1, \dots, a_n) \in V^n$ and certain scalars $\lambda_1, \dots, \lambda_n \in \boldsymbol{k}$. Let $\widehat{a} = (a_1, \dots, a_n, a_{n+1}) \in V^{n+1}$ and let a further scalar $\lambda_{n+1} \in \boldsymbol{k}$ be given. Then we have corresponding functions $\widehat{\theta}\colon D_{\widehat{a}} \to V$ and $\widehat{\theta}'\colon D'_{\widehat{a}} \to V$ given by

$$\widehat{\theta}(v) := \theta(v) + \lambda_{n+1}\psi_{\widehat{a}}(v),$$
$$\widehat{\theta}'(v') := \theta'(v') + \lambda_{n+1}\psi'_{\widehat{a}}(v').$$

Let $C$ be a component of $D_a$ with corresponding component $C'$ of $D'_a$. We define $C_i$ and $C'_i$ (for $i = 1, 2, 3$) as in the proof of Lemma 3.11. Then the components of $D_{\widehat{a}}$ that are contained in $C$ are the nonempty sets among $C_1$ and $C_3$, and the components of $D'_{\widehat{a}}$ that are contained in $C'$ are the nonempty sets among $C'_1$ and $C'_3$. We assume $d \in C'_i \setminus C_i$ for $i = 1$ or $i = 3$, and have to show that $\widehat{\theta}'(d) \notin V$. If $d$ lies in the convex hull of $C_i$ in $C'_i$, that is, if there are $p, q \in C_i$ such that $p < d < q$, then the injectivity of $\widehat{\theta}'$ and intermediate value property of $\widehat{\theta}|[p, q]$ already guarantee that $\widehat{\theta}'(d) \in V' \setminus V$, without use of the induction hypothesis. So from now on, we assume that $d$ does not lie in the convex hull of $C_i$ in $C'_i$.

Suppose there exists an element $c \in V$ lying strictly between $d$ and $a_1$, and set $\varepsilon := \frac{1}{2}|c - a_1| > 0$. Then $\psi'_{a_1}$ is constant on the segment

$$I = I_c := \big\{x \in V' : d - \varepsilon \leq x \leq d + \varepsilon\big\},$$

since $[x - a_1] = [d - a_1]$ for all $x \in I$. By an easy induction on $k$, one shows that $I \subseteq D'_{(a_1,\dots,a_k)}$ and that $\psi'_{(a_1,\dots,a_k)}$ is constant on $I$, for all $k = 1,\dots,n+1$. In particular, $I \subseteq C'_i$, and $\psi'_{\widehat{a}}$ is constant on $I$, and $\widehat{\theta}'(x) = \widehat{\theta}'(d) + x - d$ for all $x \in I$. If $I \cap C_i \neq \varnothing$, say $e \in I \cap C_i$, then

$$\widehat{\theta}'(d) = \theta'(d) + \lambda_{n+1}\psi'_{\widehat{a}}(d) = \theta'(d) + \lambda_{n+1}\psi_{\widehat{a}}(e) \notin V,$$

since $\theta'(d) \notin V$, by induction hypothesis. Thus for the rest of the proof we shall assume that whenever $c \in V$ lies strictly between $d$ and $a_1$, and $I = I_c$ is defined as above, then $I \cap C_i = \varnothing$.

Next we observe that the situation is symmetric about $a_1$, that is, the reflection $a_1 + x \mapsto a_1 - x : V' \to V'$ maps $D'_{\widehat{a}}$ onto itself, and $\widehat{\theta}'$ is invariant under this reflection. Therefore we shall assume in addition that $d$, $C$ and $C'$ are all $> a_1$.

We now first consider the case that $C$ is bounded in $V$, $\psi_a$ is increasing on $C$ (hence $\psi'_a$ increasing on $C'$), and $i = 1$. Then $C_1 < C_2 < C_3$ and $C'_1 < C'_2 < C'_3$. The following possibilities arise (see proof of Lemma 3.11):

1. $\psi_{\widehat{a}}$ increases on $C_1$ from $p$ to $\Psi$, and $\psi'_{\widehat{a}}$ increases on $C'_1$ from $p$ to $\Psi'$, for some $p \in V$. By the proof of Lemma 3.11, this implies $C_2 \neq \varnothing$. Since $d$ is not in the convex hull of $C_1$ in $C'_1$, either $d > C_1$ or $d < C_1$.
   (a) $d > C_1$. Then there exists an element $c \in V$ with $a_1 < c < d$ (take any $c \in C_1$), and thus $C_1 < I < C_2$, where $I = I_c$ as defined above. We can choose $b \in C_1$ so large that $|a_{n+1} - \psi_a(b)| \leq \varepsilon$, with $\varepsilon \in V^{>0}$ as above. Hence, in $[V']$,
   $$\big[\psi'_{\widehat{a}}(d) - \psi_{\widehat{a}}(b)\big] < \big[\psi'_a(d) - \psi_a(b)\big] \leq \big[a_{n+1} - \psi_a(b)\big] \leq [\varepsilon].$$
   Let $f(x) := \theta'(x) + \lambda_{n+1}\psi_{\widehat{a}}(b)$, for $x \in I$. Then
   $$\begin{aligned}\widehat{\theta}'(d) - f(d - \varepsilon) &= \varepsilon + \widehat{\theta}'(d - \varepsilon) - f(d - \varepsilon)\\ &= \varepsilon + \lambda_{n+1}\big(\psi'_{\widehat{a}}(d) - \psi_{\widehat{a}}(b)\big),\end{aligned}$$
   thus $\widehat{\theta}'(d) > f(d - \varepsilon)$, and similarly $\widehat{\theta}'(d) < f(d + \varepsilon)$. Hence, by the intermediate value property for $f$ on $I$ (Lemma 3.6), there exists $x \in I$ with $f(x) = \widehat{\theta}'(d)$. Since $I \cap C_1 = \varnothing$, we have $x \notin V$, hence $f(x) \notin V$ by induction hypothesis. Therefore $\widehat{\theta}'(d) \notin V$, as required.
   (b) $d < C_1$. Then $\psi'_{\widehat{a}}(x) = p$ for all $d \leq x < C_1$. In particular $\widehat{\theta}'(d) = \theta'(d) + \lambda_{n+1}p \notin V$, by induction hypothesis.
2. $\psi_{\widehat{a}}$ increases on $C_1$ from $p$ to $q$, and $\psi'_{\widehat{a}}$ increases on $C'_1$ from $p$ to $q$, for certain $p, q \in V$. Again either $d < C_1$ or $d > C_1$. Both subcases are treated as in (1), (b).

Next we consider the case that $C$ is bounded in $V$, $\psi_a$ is increasing on $C$ (hence $\psi'_a$ increasing on $C'$), and $i = 3$. Then either $\psi_{\widehat{a}}$ decreases on $C_3$ from $\Psi$ to $q$, and $\psi'_{\widehat{a}}$ decreases on $C'_3$ from $\Psi'$ to $q$, for some $q \in V$, or $\psi_{\widehat{a}}$ decreases on $C_3$ from $p$ to $q$, and $\psi'_{\widehat{a}}$ decreases on $C'_3$ from $p$ to $q$, for some $p, q \in V$. The latter subcase is treated as in (2) above. In the first subcase, suppose that $d < C_3$. Then $C_2 \neq \varnothing$, hence there exists $c \in V$ with $a_1 < c < d$, and thus $C_2 < I < C_3$, where $I = I_c$ as previously defined. Now for any $\varepsilon \in V^{>0}$, in particular for $\varepsilon = \frac{1}{2}(c - a_1)$, we can choose $b \in C_3$ so small that $|a_{n+1} - \psi_a(b)| \leq \varepsilon$. Now continue as in (1), (a) above. If $d > C_3$, argue as in (1), (b).

The case that $C$ is bounded in $V$ and $\psi_a$ is decreasing on $C$ can be handled in a similar way, and is left to the reader.

Now assume $C$ is unbounded in $V$, and $i = 1$. Then we have the following possibilities:

1. $\psi_{\widehat{a}}$ decreases on $C_1$ from $p$ to $-\infty$, and $\psi'_{\widehat{a}}$ decreases on $C'_1$ from $p$ to $-\infty$, for some $p \in V$. Again, either $d < C_1$ or $d > C_1$. The first option is treated as in (1), (b) above, whereas in the second case, $\widehat{\theta}'(d) \notin V$ follows from the remark preceding this lemma, and Lemma 3.6.
2. $\psi_{\widehat{a}}$ decreases on $C_1$ from $\Psi$ to $-\infty$, and $\psi'_{\widehat{a}}$ decreases on $C'_1$ from $\Psi'$ to $-\infty$. If $d < C_1$, we see, by inspection of the proof of Lemma 3.11, that necessarily $C_2 \neq \varnothing$. Hence there exists $c \in V$ with $a_1 < c < d$. Now adopt the argument in 1, (a) above. If $d > C_1$, we again apply the remark preceding the lemma.

Finally, consider the case that $C$ is unbounded and $i = 3$. Then $\psi_{\widehat{a}}$ increases on $C_3$ from $p$ to $\Psi$, and $\psi'_{\widehat{a}}$ increases on $C_3$ from $p$ to $\Psi'$, for some $p \in V$. If $d > C_3$, note that any $c \in C_3$ will satisfy $a_1 < c < d$, and continue as in 1, (a). If $d < C_3$, argue as in 1, (b). This finishes the induction. ☐

*Remark.* Property (A) now follows from Lemma 3.11, (1), and property (B) from the previous lemma.

## 4. Model-Theoretic Properties

Henceforth, "definable" will mean "definable with parameters". In this section we use the previous results to derive consequences for the induced structure on scalar field and vector space of a closed $H$-couple and for the definable closure of an $H$-triple in a closed extension. We also analyse simple extensions of $H$-couples, and use it to show that in a finitely generated $H$-couple the set $\Psi$ is well ordered.

**Induced structure on the scalar field.** We first show that in a closed $H$-couple, no new structure is induced on the scalar field. More precisely:

**Corollary 4.1.** *Let $(V, \psi)$ be a closed $H$-couple over $\boldsymbol{k}$, and let $S \subseteq \boldsymbol{k}^n$ be definable in $(V, \psi)$. Then $S$ is already definable in the ordered field $\boldsymbol{k}$.*

*Proof.* Let the $H$-couples $(V_1, \psi_1)$ over $\boldsymbol{k}_1$ and $(V_2, \psi_2)$ over $\boldsymbol{k}_2$ be elementary extensions of $(V, \psi)$. (In particular, the ordered fields $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are elementary extensions of the ordered field $\boldsymbol{k}$.) Suppose $\lambda = (\lambda_1, \dots, \lambda_n) \in \boldsymbol{k}_1^n$ and $\mu = (\mu_1, \dots, \mu_n) \in \boldsymbol{k}_2^n$ realize the same type over $\boldsymbol{k}$ in $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, respectively. It suffices to show that then they realize the same type over $(V, \psi)$ in $(V_1, \psi_1)$ and $(V_2, \psi_2)$, respectively. We may assume that $(V_2, \psi_2)$ is $\kappa$-saturated, where $\kappa := |V_1|^+$. It follows that there is an elementary embedding $e\colon \boldsymbol{k}_1 \to \boldsymbol{k}_2$ that is the identity on $\boldsymbol{k}$ and that sends each $\lambda_i$ to $\mu_i$. By Proposition 3.2 and Lemma 3.3, $e$ is the scalar part of an elementary embedding $(V_1, \psi_1) \to (V_2, \psi_2)$ over $(V, \psi)$. Hence $\lambda$ and $\mu$ realize the same type over $(V, \psi)$ in $(V_1, \psi_1)$ and $(V_2, \psi_2)$. ☐

**Induced structure on the vector space.** Let $(V, \psi)$ be a closed $H$-couple over the scalar field $\boldsymbol{k}$. To discuss the induced structure on the underlying vector space $V$ we introduce the one-sorted language $\mathcal{L}_{\boldsymbol{k},\mathrm{v}}$ that extends the language $\{0, +, -, <\}$ of ordered abelian groups by an $n$-ary relation symbol $R_{\lambda,\varphi}$ for every $\lambda \in \boldsymbol{k}^m$ and $\mathcal{L}_H$-formula $\varphi = \varphi(x, y)$, where $x = (x_1, \dots, x_m)$ is a tuple of scalar variables and $y = (y_1, \dots, y_n)$ is a tuple of vector variables. We make $V$ into an $\mathcal{L}_{\boldsymbol{k},\mathrm{v}}$-structure by interpreting $0$, $+$, $-$, $<$ as usual, and $R_{\lambda,\varphi}$ as

$$\big\{v \in V^n : (V, \psi) \models \varphi(\lambda, v)\big\}.$$

Thus a set $S \subseteq V^n$ is definable in the one-sorted $\mathcal{L}_{\boldsymbol{k},\mathrm{v}}$-structure $V$ if and only if it is definable in the two-sorted $\mathcal{L}_H$-structure $(V,\psi)$.

Let $\mathbf{A} = (A,<,\dots)$ be a structure (in some one-sorted language $\mathcal{L}$ containing a binary relation symbol $<$) that expands a linearly ordered nonempty set $(A,<)$, dense without endpoints. Following Marker and Steinhorn we say that $\mathbf{A}$ is **locally o-minimal** if for each definable set $S \subseteq A$ and each $a \in A$ there exist $a_1, a_2 \in A$ such that $a_1 < a < a_2$, and $(a_1,a)$ is either disjoint from $S$ or contained in $S$, and $(a,a_2)$ is either disjoint from $S$ or contained in $S$. The structure $\mathbf{A}$ is called **weakly o-minimal** if every definable subset of $A$ is a finite union of convex subsets, see [12]. Clearly, if $\mathbf{A}$ is weakly o-minimal, then it is locally o-minimal.

For a closed $H$-couple $(V,\psi)$, the $\mathcal{L}_{\boldsymbol{k},\mathrm{v}}$-structure $V$ is never weakly o-minimal: Consider the definable subset $\boldsymbol{k}\cdot 1$ of $V$; it is not a finite union of convex subsets. However, we have:

**Proposition 4.2.** *Let $(V,\psi)$ be a closed $H$-couple over $\boldsymbol{k}$. Then $V$ is locally o-minimal as an $\mathcal{L}_{\boldsymbol{k},\mathrm{v}}$-structure.*

*Proof.* Below we consider $V$ as $\mathcal{L}_{\boldsymbol{k},\mathrm{v}}$-structure. Take a $\kappa$-saturated elementary extension $(V',\psi')$ of $(V,\psi)$ where $\kappa = |V|^+$. Thus $V'$ is then naturally a $\kappa$-saturated $\mathcal{L}_{\boldsymbol{k},\mathrm{v}}$-structure elementary extending $V$. Below we consider $V'$ as an $\mathcal{L}_{\boldsymbol{k},\mathrm{v}}$-structure in this way.

By familiar model-theoretic reasoning, it now suffices to show that, given $v \in V$, any two vectors $v_1, v_2 \in V'$ such that $v < v_i < v+\varepsilon$ for all $\varepsilon > 0$ in $V$, $i=1,2$, realize the same type over $V$ in $V'$. By translation over $-v$ we reduce to the case $v=0$. Then $\Psi < \psi'(v_i) < (\mathrm{id}+\psi)\big(V^{>0}\big)$, hence $\big[V \oplus \boldsymbol{k}\psi'(v_i)\big]_{\boldsymbol{k}} = [V]_{\boldsymbol{k}}$ inside $[V']_{\boldsymbol{k}}$, for $i=1,2$, by Corollary 3.9. After embedding $(V,\psi)$ into $(H,\psi_H)$ with $H = H\big([V^*],\boldsymbol{k}\big)$, cf. Lemma 2.4, we see that in *some* $H$-couple over $\boldsymbol{k}$ extending $(V,\psi)$, there is an element $u$ such that $\Psi < u < (\mathrm{id}+\psi)\big(V^{>0}\big)$. But $V \oplus \boldsymbol{k}u$ and $V \oplus \boldsymbol{k}\psi'(v_i)$ are isomorphic over $V$ as ordered vector spaces over $\boldsymbol{k}$. Since $V \oplus \boldsymbol{k}u$ is a Hahn space over $\boldsymbol{k}$, so is $V_i := V \oplus \boldsymbol{k}\psi'(v_i)$, for $i=1,2$. Let $P_i := V_i^{\le \psi'(v_i)}$, for $i=1,2$. Since $[V_i] = [V]$, we have $\psi'(V_i^*) \subseteq V$; let $\psi_i := \psi'|V_i^*$. As in the first part of the proof of Proposition 3.2 above, we obtain an isomorphism $(V_1,\psi_1,P_1) \to (V_2,\psi_2,P_2)$ over $(V,\psi,\Psi)$, mapping $\psi'(v_1)$ to $\psi'(v_2)$. Since $[v_i] < [V^*] = [V_i^*]$, the cut in $V_1$ realized by $v_1$ corresponds, under this isomorphism, to the cut in $V_2$ realized by $v_2$. Hence we can extend the vector part of this isomorphism to an isomorphism $V_1' := V_1 \oplus \boldsymbol{k}v_1 \to V_2 \oplus \boldsymbol{k}v_2 =: V_2'$ of ordered vector spaces over $\boldsymbol{k}$, mapping $v_1$ to $v_2$. Note that the image $\Psi \cup \{\psi'(v_i)\}$ of $\psi_i' := \psi'|(V_i')^*$ has a largest element $\psi'(v_i)$; hence $(V_i',\psi_i')$ has only one $H$-cut. Therefore the map under consideration is the vector part of an isomorphism $\big(V_1',\psi_1',\Psi'\cap V_1'\big) \to \big(V_2',\psi_2',\Psi'\cap V_2'\big)$ of $\mathcal{L}_{H,P}$-structures, whose scalar part is the identity on $\boldsymbol{k}$. Thus by relative quantifier elimination, $v_1$ and $v_2$ have the same type over $V$ in $V'$. $\square$

*Remark.* The $\mathcal{L}_{\boldsymbol{k},\mathrm{v}}$-structure $V$ is even"o-minimal at infinity": For any definable set $S \subseteq V$ there exists $a \in V$ such that either $V^{>a} \subseteq S$ or $V^{>a} \cap S = \varnothing$. This can be verified easily along the lines of the preceding proof.

**Definable closure.** Let $\mathcal{V} = (V,\psi,P)$ be an $H$-triple with scalar field $\boldsymbol{k}$, and let $\mathcal{V}' = (V',\psi',P')$ be a *closed* $H$-triple extending $\mathcal{V}$, with the same scalar field $\boldsymbol{k}$. An element $v' \in V'$ is said to be **definable over** $\mathcal{V}$ if there is an $\mathcal{L}_{H,P}$-formula $\varphi(x,y,z)$, where $x=(x_1,\dots,x_m)$ is a tuple of scalar variables, $y=(y_1,\dots,y_n)$ a tuple of vector variables and $z$ a vector variable, and there are $\lambda \in \boldsymbol{k}^m$, $v \in V^n$, such that $v'$ is the unique element in $V'$ with $\mathcal{V}' \models \varphi(\lambda, v, v')$. The **definable closure of $\mathcal{V}$ in $\mathcal{V}'$** is the substructure of $\mathcal{V}'$ that extends $\mathcal{V}$ and whose underlying vector space consists of all $v' \in V'$ that are definable over $\mathcal{V}$. If $\mathcal{V}'' \supseteq \mathcal{V}$ is another closed $H$-triple over $\boldsymbol{k}$, the definable closure of $\mathcal{V}$ in $\mathcal{V}'$ is isomorphic to the definable closure of $\mathcal{V}$ in $\mathcal{V}''$, by a unique isomorphism that is the identity on $\mathcal{V}$. We say that $\mathcal{V}$ is **definably closed** in $\mathcal{V}'$ if every $v' \in V'$ definable over $\mathcal{V}$ belongs to $V$. In that case $\mathcal{V}$ is definably closed in *every* closed $H$-triple over $\boldsymbol{k}$ extending $\mathcal{V}$, and we also just say then that $\mathcal{V}$ **is definably closed.** More generally, if $\mathcal{W} = (W, \dots) \supseteq \mathcal{V}$ is *any* $H$-triple over $\boldsymbol{k}$, we say that $\mathcal{V}$ is **definably closed in** $\mathcal{W}$ if $W \cap \overline{V} = V$, where $\overline{\mathcal{V}} = (\overline{V}, \dots)$ is the definable closure of $\mathcal{V}$ in an $H$-closure of $\mathcal{W}$.

**Lemma 4.3.** *Suppose there is no $a \in V$ with $P < a < (\mathrm{id} + \psi)\big(V^{>0}\big)$, and $P$ has no largest element. Then $\mathcal{V}$ is definably closed in $\mathcal{V}'$.*

*Proof.* By iterating the construction of Lemma 2.7 we obtain an increasing continuous chain $\big\{(V_\alpha, \psi_\alpha, P_\alpha)\big\}_{\alpha<\mu}$ ($\mu$ an ordinal) of $H$-triples contained in $\mathcal{V}'$ as substructures, with $(V_0, \psi_0, P_0) = (V, \psi, P)$, such that the union

$$\mathcal{V}^{\mathrm{c}} = (V^{\mathrm{c}}, \psi^{\mathrm{c}}, P^{\mathrm{c}}) = \bigcup_{\alpha<\mu} (V_\alpha, \psi_\alpha, P_\alpha),$$

is $H$-closed. The reference to Lemma 2.7 means that for $\alpha < \alpha + 1 < \mu$ we have $V_{\alpha+1} = V_\alpha \oplus \boldsymbol{k} a_\alpha$ with $a_\alpha > 0$ and $\psi_\alpha(a_\alpha) \in P_\alpha \setminus \psi_\alpha(V_\alpha^*)$. That the chain is continuous means that $(V_\delta, \psi_\delta, P_\delta) = \bigcup_{\alpha<\delta} (V_\alpha, \psi_\alpha, P_\alpha)$ for limit ordinals $\delta < \mu$. Since $\mathcal{V}^{\mathrm{c}} \preceq \mathcal{V}'$, it suffices to show: For any $v \in V^{\mathrm{c}} \setminus V$ there exists an element $w \neq v$ in $V^{\mathrm{c}}$ and an automorphism of $\mathcal{V}^{\mathrm{c}}$ that is the identity on $\mathcal{V}$ and sends $v$ to $w$. Now, given such $v$, take $\alpha$ with $0 \le \alpha < \alpha + 1 < \mu$ and $v \in V_{\alpha+1} \setminus V_\alpha$. Write

$$v = v_\alpha + \lambda a_\alpha \quad \text{with } v_\alpha \in V_\alpha,\ \lambda \in \boldsymbol{k}^\times.$$

Let $a \in V_{\alpha+1}^{>0}$ be any element $\neq a_\alpha$ with the same $\boldsymbol{k}$-archimedean class as $a_\alpha$, and let $w := v_\alpha + \lambda a$. By Lemma 2.7, there is a unique automorphism $\sigma$ of $(V_{\alpha+1}, \psi_{\alpha+1}, P_{\alpha+1})$ that is the identity on $(V_\alpha, \psi_\alpha, P_\alpha)$ and satisfies $\sigma(a_\alpha) = a$; hence $\sigma(v) = w$. Applying once more Lemma 2.7 iteratively, we can extend $\sigma$ to an automorphism of $(V^{\mathrm{c}}, \psi^{\mathrm{c}}, P^{\mathrm{c}})$ that sends $v$ to $w$, as desired. ☐

In general we define an $H$-triple $\overline{\mathcal{V}} = (\overline{V}, \overline{\psi}, \overline{P})$ with $\mathcal{V} \subseteq \overline{\mathcal{V}} \subseteq \mathcal{V}'$ as follows, distinguishing three mutually exclusive cases:

1. There is $a \in V$ with $P < a < (\mathrm{id} + \psi)\big(V^{>0}\big)$. This element $a$ determines a sequence $\{\varepsilon_n\}$ of positive elements of $V'$, with
$$[V^*] > [\varepsilon_0] > [\varepsilon_1] > [\varepsilon_2] > \cdots$$
and a corresponding sequence $\{V_n\}$ of linear subspaces of $V'$ with
$$V_n = V \oplus \boldsymbol{k}\varepsilon_0 \oplus \cdots \oplus \boldsymbol{k}\varepsilon_n$$
by requiring $a = \varepsilon_0 + \psi'(\varepsilon_0)$ and $\psi'(\varepsilon_{n+1}) = \max(P' \cap V_n) + \varepsilon_{n+1}$ (cf. Lemma 2.5 and 2.6). Then $\overline{V} := \bigcup_n V_n$, so that $\big[\,\overline{V}\,\big] = [V] \cup \big\{[\varepsilon_n] : n \ge 0\big\}$.
2. $P$ has a largest element. Proceeding as in case 1, except that we restrict to $n \ge 1$, we define a sequence $\{\varepsilon_n\}_{n\ge1}$ of positive elements of $V'$ with $[V^*] > [\varepsilon_1] > [\varepsilon_2] > \cdots$, and a corresponding sequence $\{V_n\}_{n\ge1}$ of linear subspaces of $V'$ with $V_n = V \oplus \boldsymbol{k}\varepsilon_1 \oplus \cdots \oplus \boldsymbol{k}\varepsilon_n$, by $\psi'(\varepsilon_1) = (\max P) + \varepsilon_1$, and $\psi'(\varepsilon_{n+1}) = \max(P' \cap V_n) + \varepsilon_{n+1}$. Then $\overline{V} := \bigcup_{n\ge1} V_n$, so that $\big[\,\overline{V}\,\big] = [V] \cup \big\{[\varepsilon_n] : n \ge 1\big\}$.

3. There is no $a \in V$ with $P < a < (\mathrm{id} + \psi)(V^{>0})$ and $P$ has no largest element. Then we put $\overline{V} := V$.

The previous lemma now easily implies:

**Corollary 4.4.** *The definable closure of $\mathcal{V}$ in $\mathcal{V}'$ is $\overline{\mathcal{V}}$.* □

In the next lemma we continue to use the notation introduced in the definition of $\overline{\mathcal{V}}$ above.

**Lemma 4.5.** *The only $H$-triples $\mathcal{W}$ with $\mathcal{V} \subseteq \mathcal{W} \subseteq \overline{\mathcal{V}}$ are $\mathcal{V}$, $\overline{\mathcal{V}}$ and*

1. $(V_n, \overline{\psi}|V_n^*, \overline{P} \cap V_n)$ *for $n \geq 0$, in case 1 above,*
2. $(V_n, \overline{\psi}|V_n^*, \overline{P} \cap V_n)$ *for $n \geq 1$, in case 2 above.*

*Proof.* First assume we are in case 1. Let $\mathcal{W}$ be an $H$-triple such that $\mathcal{V} \subseteq \mathcal{W} \subseteq \overline{\mathcal{V}}$ and let $W$ denote the underlying ordered vector space of $\mathcal{W}$. Let $w \in W \setminus V$. After subtracting from $w$ a vector in $V$ we have

$$w = \lambda_m \varepsilon_m + \cdots + \lambda_n \varepsilon_n \text{ with } n \geq m, \quad \lambda_m, \ldots, \lambda_n \in \boldsymbol{k}, \quad \lambda_m \neq 0, \lambda_n \neq 0.$$

By induction on $i$ we shall obtain $\varepsilon_i \in W$ for $i = 0, \ldots, n$, which immediately implies the lemma in case 1. Note that $\overline{\psi}(w) = \overline{\psi}(\varepsilon_m) = a - \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_m \in W$, hence $\varepsilon_0 = a - \overline{\psi}(\overline{\psi}(b) - a) \in W$, which proves our claim for $i = 0$. So assume $0 \leq i < n$, and $\varepsilon_0, \ldots, \varepsilon_i \in W$.

1. Suppose $i < m$. Then
$$\varepsilon_{i+1} + \cdots + \varepsilon_m = \overline{\psi}(w) - (a - \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_i) \in W,$$
hence
$$\varepsilon_{i+1} = \overline{\psi}(\varepsilon_{i+1} + \cdots + \varepsilon_m) - (a - \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_i) \in W.$$
2. Suppose $i \geq m$. Take $j$ minimal such that $i < j \leq n$ and $\lambda_j \neq 0$. Then
$$\lambda_j \varepsilon_j + \cdots + \lambda_n \varepsilon_n = w - (\lambda_m \varepsilon_m + \cdots + \lambda_i \varepsilon_i) \in W,$$
hence
$$\varepsilon_{i+1} + \cdots + \varepsilon_j = \overline{\psi}(\lambda_j \varepsilon_j + \cdots + \lambda_n \varepsilon_n) - (a - \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_i) \in W,$$
and therefore
$$\varepsilon_{i+1} = \overline{\psi}(\varepsilon_{i+1} + \cdots + \varepsilon_j) - (a - \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_i) \in W.$$

This finishes the induction step, and thus the proof of the lemma in case 1. For case 2 one argues similarly. □

**Uniqueness of $H$-closure.** Let $\mathcal{L}$ be a one-sorted language and $\mathbf{A} = (A, \ldots)$ an $\mathcal{L}$-structure. A **construction** of $\mathbf{A}$ is an enumeration $\{a_\alpha\}_{\alpha<\gamma}$ of $A$ ($\gamma$ an ordinal), such that, with $A_\alpha := \{a_\beta : \beta < \alpha\}$, the type of $a_\alpha$ over $A_\alpha$ in $\mathbf{A}$ is isolated, for each $\alpha < \gamma$. Let such a construction of $\mathbf{A}$ be given. Choose for each $\alpha < \gamma$ an $\mathcal{L}$-formula $\varphi_\alpha(y_\alpha, z)$, with $y_\alpha = (y_{\alpha 1}, \ldots, y_{\alpha n(\alpha)})$ a tuple of variables and $z$ a single variable, and a tuple $b_\alpha \in A_\alpha^{n(\alpha)}$, such that $\varphi_\alpha(b_\alpha, z)$ isolates the type of $a_\alpha$ over $A_\alpha$. We also choose by recursion on $\alpha$ a finite set $D_\alpha \subseteq A_\alpha$ as follows: $D_0 := \{a_0\}$, and for $0 < \alpha < \gamma$, put $D_\alpha := \{a_\alpha\} \cup D_{\beta_1} \cup \cdots \cup D_{\beta_{n(\alpha)}}$, where $b_\alpha = (a_{\beta_1}, \ldots, a_{\beta_{n(\alpha)}})$ for certain $\beta_1, \ldots, \beta_{n(\alpha)} < \alpha$. An elementary substructure $\mathbf{C} = (C, \ldots)$ of $\mathbf{A}$ is said to be **closed in A** (relative to the given construction and the further choices made) if for all $\alpha < \gamma$, $a_\alpha \in C$ implies $D_\alpha \subseteq C$. In that case a theorem of Ressayre ([13], Lemme 10.15, Théorème 10.18) implies that $\mathbf{A} \cong \mathbf{C}$.

**Corollary 4.6.** *Let $\mathcal{V}$ be an $H$-triple over $\boldsymbol{k}$. Then any two $H$-closures of $\mathcal{V}$ are isomorphic over $\mathcal{V}$.*

*Proof.* We may assume that $\mathcal{V}$ is definably closed. Then we build an $H$-closure $\mathcal{V}^{\mathrm{c}}$ of $\mathcal{V}$ as in the proof of Lemma 4.3. Let $\mathcal{W} \supseteq \mathcal{V}$ be another $H$-closure of $\mathcal{V}$. We have to show that $\mathcal{V}^{\mathrm{c}} \cong \mathcal{W}$ over $\mathcal{V}$. By the defining property of $H$-closure we can assume $\mathcal{V} \subseteq \mathcal{W} \subseteq \mathcal{V}^{\mathrm{c}}$. Write $\mathcal{V}^{\mathrm{c}} = \bigcup_{\alpha<\mu}(V_\alpha, \psi_\alpha, P_\alpha)$ as in the proof of Lemma 4.3. We now consider the underlying vector spaces $W$ and $V^{\mathrm{c}}$ of $\mathcal{W}$ and $\mathcal{V}^{\mathrm{c}}$ as structures for the language $\mathcal{L}_{\boldsymbol{k},\mathrm{v}}(V)$ obtained from $\mathcal{L}_{\boldsymbol{k},\mathrm{v}}$ by adding names for the vectors in $V$, see 4.2. By Lemma 2.7, the type of $a_\alpha$ over $V_\alpha$ in $V^{\mathrm{c}}$ (for $\alpha < \alpha+1 < \mu$) is isolated by the formula $\varphi\big(\psi_\alpha(a_\alpha), z\big)$, where $\varphi(y,z)$ is "$y = \psi(z)$ & $z > 0$". It follows easily that $V^{\mathrm{c}}$ has a construction. By Lemma 3.3 we have $W \preceq V^{\mathrm{c}}$. If $\alpha < \alpha+1 < \mu$ and $a_\alpha \in W$, then $\psi_\alpha(a_\alpha) \in W$, so $W$ is closed in $V^{\mathrm{c}}$ (relative to a suitable construction of $V^{\mathrm{c}}$ and associated choices of isolating formulas and so on). Thus by Ressayre's Theorem $V^{\mathrm{c}} \cong W$, which implies $\mathcal{V}^{\mathrm{c}} \cong \mathcal{W}$ over $\mathcal{V}$. □

*Remark.* We don't know whether the $H$-closure $\mathcal{V}^{\mathrm{c}}$ of an $H$-triple $\mathcal{V}$ is always minimal over $\mathcal{V}$, i.e. whether or not for some $\mathcal{V}$ there exists a closed $H$-triple $\mathcal{W} \supseteq \mathcal{V}$ strictly contained in $\mathcal{V}^{\mathrm{c}}$ as a substructure.

**Analysis of simple extensions.** Let an $H$-triple $\mathcal{V} = (V, \psi, P)$ with scalar field $\boldsymbol{k}$ be given, and a **simple extension** of $\mathcal{V}$, that is, an $H$-triple $\mathcal{V}' = (V', \psi', P')$ over the same scalar field $\boldsymbol{k}$ and extending $\mathcal{V}$ for which there exists a vector $c \in V'$ such that $\mathcal{V}'$ is generated as $\mathcal{L}_{H,P}$-structure over $\mathcal{V}$ by $c$. (This state of affairs is also indicated by writing $\mathcal{V}' = \mathcal{V}\langle c\rangle$, and we put $V' = V\langle c\rangle$ for the underlying ordered vector spaces in that case.)

Consider the following five properties that this simple extension with its distinguished generator $c$ may or may not have:

(I) $0 < c < V^{>0}$ and $c + \psi'(c) \in V$.
(II) $0 < c < V^{>0}$ and $-c + \psi'(c) \in V$.
(III) $\psi'(c) \in P \setminus \Psi$.
(IV) $c \notin V$ and $[V \oplus \boldsymbol{k}c] = [V]$.
(V) $V' = V \oplus \bigoplus_{n=1}^{\infty} \boldsymbol{k} b_n$ for vectors $b_n \in V'$ that are $\boldsymbol{k}$-linearly independent over $V$, with $[b_n] \notin [V]$ for all $n$, and such that there are vectors $a_n \in V$ with $b_1 = c - a_1$ and $b_{n+1} = \psi'(b_n) - a_{n+1}$ for all $n \geq 1$.

*Remarks.* If (I), respectively (II) holds, then $\mathcal{V}' \cong \big(V \oplus \boldsymbol{k}\varepsilon, \psi^\varepsilon, P^\varepsilon\big)$, as in Lemma 2.5, respectively Lemma 2.6, by an isomorphism that is the identity on $\mathcal{V}$ and sends $c$ to $\varepsilon$. If (III) holds, then $\mathcal{V}' \cong \big(V \oplus \boldsymbol{k}a, \psi^a, P^a\big)$, as in Lemma 2.7, by an isomorphism that is the identity on $\mathcal{V}$ and sends $c$ to $a$. If (IV) holds, then $V' = V \oplus \boldsymbol{k}c$. Note that if (V) holds, then $[b_n] \neq [b_m]$ for all $n \neq m$. (Otherwise $b_{n+1} - b_{m+1} = a_{m+1} - a_{n+1} \in V$, contradicting the linear independence of $\{b_i\}_{i\geq 1}$ over $V$.)

One sees easily that those properties are mutually exclusive. We call $\mathcal{V} \subseteq \mathcal{V}\langle c\rangle$ a **simple extension of type (I),** respectively (II), (III), (IV), (V), if (I), respectively (II), (III), (IV), (V) hold. Here the generator $c$ figuring in the definition of these properties has been specified. If we do not want to specify the generator we simply say that $\mathcal{V}'$ is a **simple extension of type (I),** respectively (II), (III), (IV), (V) to mean that for *some* $c \in V'$ we have $\mathcal{V}' = \mathcal{V}\langle c\rangle$ and $\mathcal{V} \subseteq \mathcal{V}\langle c\rangle$ is a simple extension of type (I), respectively (II), (III), (IV), (V).

We now show that if $\mathcal{V}$ is definably closed in $\mathcal{V}'$, then we can obtain $\mathcal{V}'$ by a finite number of simple extensions of types (I)–(V). More precisely:

**Proposition 4.7.** *Suppose $\mathcal{V}$ is definably closed in its simple extension $\mathcal{V}' = \mathcal{V}\langle c\rangle$, with $c \notin V$. Then either*

1. *$\mathcal{V} \subseteq \mathcal{V}\langle c\rangle$ is a simple extension of type* (V)*, or*
2. *there is a finite chain of $H$-triples*

$$\mathcal{V} = \mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \cdots \subseteq \mathcal{V}_n = \mathcal{V}' \quad (n \geq 1)$$

*such that $\mathcal{V}_1$ is a simple extension of $\mathcal{V}_0$ of type* (III) *or type* (IV)*, and each $\mathcal{V}_{i+1}$ is a simple extension of $\mathcal{V}_i$ of type* (III)*, for $i = 1, \ldots, n-1$.*

*Proof.* If $\mathcal{V} \subseteq \mathcal{V}\langle c\rangle$ is of type (IV), we are done. Suppose $\mathcal{V} \subseteq \mathcal{V}\langle c\rangle$ is not of type (IV). Then there is $a_1 \in V$ such that, with $b_1 := c - a_1$ we have $[b_1] \notin [V]$. This is the starting point for an inductive construction of elements $a_i \in V$ and $b_i \in V'$. Suppose we have already constructed $a_1, \ldots, a_n \in V$ and non-zero vectors $b_1, \ldots, b_n \in V'$ with $n \geq 1$, where $a_1$ and $b_1$ are as above, $b_{i+1} = \psi'(b_i) - a_{i+1}$ for $i = 1, \ldots, n-1$, such that $[b_i] \notin [V]$ for $i = 1, \ldots, n$.

We claim that then $[b_i] \neq [b_j]$ for $1 \leq i < j \leq n$ (hence $b_1, \ldots, b_n$ are linearly independent over $V$). Otherwise $[b_i] = [b_j]$, for certain $1 \leq i < j \leq n$, so $\psi'(b_j) = \psi'(b_i)$. But also $\psi'(b_j) = \psi'_{(a_{i+1},\ldots,a_j)}\big(\psi'(b_i)\big)$ (see §3), hence $\psi'(b_i) = \psi'_{(a_{i+1},\ldots,a_j)}\big(\psi'(b_i)\big)$. Thus by Lemma 3.6 the vector $\psi'(b_i)$ is definable over $\mathcal{V}$. Therefore $b_{i+1} \in V$, contradicting $b_{i+1} \notin [V]$.

If $\big[\psi'(b_n) - a_{n+1}\big] \notin [V]$ for some $a_{n+1} \in V$, we take such a vector $a_{n+1}$ and put $b_{n+1} := \psi'(b_n) - a_{n+1}$. If there is no such $a_{n+1}$, the construction breaks off, with $a_n$ and $b_n$ as the last vectors.

First assume that the construction goes on indefinitely, that is we obtain infinite sequences $\{a_i\}_{i\geq 1}$ in $V$ and $\{b_i\}_{i\geq 1}$ in $V'$ such that $b_1 = c - a_1$, $b_{i+1} = \psi'(b_i) - a_{i+1}$ and $[b_i] \notin [V]$ for all $i \geq 1$. Then one easily sees that $V' = V \oplus \bigoplus_{i=1}^{\infty} \boldsymbol{k} b_i$, and that $\mathcal{V} \subseteq \mathcal{V}\langle c\rangle$ is of type (V).

Now suppose our construction stops after the vectors $a_n$ and $b_n$ have been obtained. There are two ways in which this could happen:

1. $\psi'(b_n) \in V$,
2. $\psi'(b_n) \notin V$, but $\big[V \oplus \boldsymbol{k}\psi'(b_n)\big] = [V]$,

In the first case we put $\mathcal{V}_0 := \mathcal{V}$, and for $i = 1, \ldots, n$ we let $\mathcal{V}_i$ be the substructure of $\mathcal{V}'$ with underlying vector space

$$V_i := V \oplus \bigoplus_{j=n-i+1}^{n} \boldsymbol{k} b_j.$$

Then $\mathcal{V}_{i+1}$ is a simple extension of type (III) of $\mathcal{V}_i$, for $i = 0, \ldots, n-1$, and $\mathcal{V}_n = \mathcal{V}'$. In the second case, take $\mathcal{V}_0 := \mathcal{V}$, and for $i = 1, \ldots, n+1$ let $\mathcal{V}_i$ be the substructure of $\mathcal{V}'$ with underlying vector space

$$V_i := V \oplus \boldsymbol{k}\psi'(b_n) \oplus \bigoplus_{j=n-i+2}^{n} \boldsymbol{k} b_j.$$

Then $\mathcal{V}_0 \subseteq \mathcal{V}_1$ is a simple extension of type (IV), whereas $\mathcal{V}_i \subseteq \mathcal{V}_{i+1}$, for $i = 1, \ldots, n$, is a simple extension of type (III), and $V_{n+1} = V \oplus \boldsymbol{k}\psi'(b_n) \oplus \bigoplus_{j=1}^{n} \boldsymbol{k} b_j = V'$. $\square$

*Remark.* In the previous proposition, if $\mathcal{V}$ is a *closed* $H$-triple and $\mathcal{V} \subseteq \mathcal{V}'$ is not of type (V), the extension $\mathcal{V}_0 \subseteq \mathcal{V}_1$ will be of type (IV), since $\mathcal{V}$ admits no simple extensions of type (III).

Suppose that $\mathcal{V} \subseteq \mathcal{V}'$ is a simple extension such that $\mathcal{V}$ is not definably closed in $\mathcal{V}'$. To reduce to a situation where we can apply the last proposition, we let $\overline{\mathcal{V}} = (\overline{V}, \dots)$ be the definable closure of $\mathcal{V}$ in an $H$-closure of $\mathcal{V}'$, and let $\mathcal{W} = (W, \dots)$ be the $H$-triple with $\mathcal{V} \subset \mathcal{W} \subseteq \mathcal{V}'$ and $W = V' \cap \overline{V}$.

Then $\mathcal{W}$ is definably closed in $\mathcal{V}'$, so that Proposition 4.7 is applicable to the simple extension $\mathcal{W} \subseteq \mathcal{V}'$. The possibilities for the proper extension $\mathcal{V} \subset \mathcal{W}$ are described by Lemma 4.5, but can it actually happen that $\mathcal{W} = \overline{\mathcal{V}}$? The following example shows that this case indeed occurs, and also shows that there are simple extensions that cannot be obtained by a finite number of simple extensions of types (I)–(V).

*Example.* Let $\mathcal{V} = (V, \psi, P)$ the $H$-triple over $\boldsymbol{k}$ with $V := \boldsymbol{k}e_0$ ($e_0 > 0$), distinguished positive element $1 = e_0$, and $\max P = e_0$. (See §2, Example 1.) Let $\mathcal{V}' = (V', \psi', P')$ be the $H$-triple over $\boldsymbol{k}$, with $V' = \bigoplus_{n\in\mathbb{N}} \boldsymbol{k}e_{-n}$ and $\psi'(e_{-n}) := e_0 + e_{-1} - e_{-n-1}$ for all $n \in \mathbb{N}$, and $P' := \{v' \in V' : v' < e_0 + e_{-1}\}$, as in §2, end of Example 3. Note that $\mathcal{V}' = \mathcal{V}\langle e_{-1}\rangle$. Let $\overline{\mathcal{V}} = (\overline{V}, \dots)$ be the definable closure of $\mathcal{V}$ in an $H$-closure of $\mathcal{V}'$. Put $\varepsilon_n := e_{-n} - e_{-n-1}$ for $n \geq 1$. One sees easily (using Corollary 4.4) that then $\overline{\mathcal{V}} \subseteq \mathcal{V}\langle e_{-1}\rangle$ with

$$\overline{V} := \boldsymbol{k}e_0 \oplus \bigoplus_{n=1}^{\infty} \boldsymbol{k}\varepsilon_n.$$

It can be shown that $\mathcal{V}' = \mathcal{V}\langle e_{-1}\rangle$ can *not* be obtained from $\mathcal{V}$ by finitely many simple extensions of type (I)–(V). One proves that whenever $\mathcal{V}_1 = (V_1, \psi_1, P_1)$ is an $H$-triple with $\mathcal{V} \subseteq \mathcal{V}_1 \subset \overline{\mathcal{V}}$ and $c \in V\langle e_{-1}\rangle \setminus V_1$, then $\mathcal{V}_1 \subseteq \mathcal{V}_1\langle c\rangle$ is not of type (III), (IV) or (V). We leave the details to the reader.

**Well-orderedness of $\Psi$.** Our analysis of simple extensions was originally motivated by the question whether in a finitely generated $H$-couple $(V, \psi)$, the set $\Psi = \psi(V^*)$ is always well-ordered. To deduce this we need to take a closer look at type (V) extensions.

**Lemma 4.8.** *Let $V_0 \subseteq V$ be ordered vector spaces over the ordered field $\boldsymbol{k}$, and $v \in V \setminus V_0$ such that $[v] \notin [V_0]$. Then $[V_0 \oplus \boldsymbol{k}v] = [V_0] \cup \{[v]\}$ and $[v] \leq [w]$ for all $w \in (V_0 \oplus \boldsymbol{k}v) \setminus V_0$.*

This follows easily from the properties of $\boldsymbol{k}$-archimedean classes listed in the beginning of §1, especially property (4).

**Proposition 4.9.** *Let $\mathcal{V} \subseteq \mathcal{V}\langle c\rangle = \mathcal{V}'$ be a simple extension of $H$-triples of type* (V)*, with $V' = V \oplus \bigoplus_{n=1}^{\infty} \boldsymbol{k}b_n$ as in the definition of type* (V) *extensions. Then*

1. $[V'] = [V] \cup \{[b_n] : n = 1, 2, 3, \dots\}$.
2. *$V^{>0}$ is coinitial in $(V')^{>0}$.*
3. *$\mathcal{V}$ is definably closed in $\mathcal{V}'$.*
4. *The sequence $\{[b_n]\}$ is strictly decreasing.*

*Proof.* Using the lemma above, and the fact that $[b_i] \neq [b_j]$ for $i \neq j$, one shows by induction on $n$ that $\left[V \oplus \bigoplus_{i=1}^{n} \boldsymbol{k}b_i\right] = [V] \cup \{[b_i] : i = 1, \dots, n\}$. This proves (1).

For (2) we first note that if $y \in V' \setminus V$, then $[V] \neq [V \oplus \boldsymbol{k}y]$: write $y = v + \lambda_1 b_1 + \cdots + \lambda_n b_n$ with $v \in V$, $\lambda_1, \dots, \lambda_n \in \boldsymbol{k}$, and some $\lambda_i \neq 0$; hence $[y - v] = [b_i] \in [V] \setminus [V \oplus \boldsymbol{k}y]$ for some $i$, by (1). Thus by the last part of Corollary 3.9, if $[V^*]$ has no minimum, then (2) holds, hence (3) holds as well, by Lemma 4.5. We now prove (2) and (3) in the remaining case that $[V^*]$ has a minimum. Equivalently, we assume $\Psi$ has a maximum. We first show that then (3) holds. If it didn't, then by Lemma 4.5, there are $v \in V$, $\lambda_1, \dots, \lambda_n \in \boldsymbol{k}$ $(n > 0)$ such that $v' := v + \sum_{i=1}^n \lambda_i b_i > 0$ and $\max \Psi = \psi'(v') - v'$. If $[v] > [b_i]$ for all $i \in \{1, \dots, n\}$ with $\lambda_i \neq 0$, then $\psi'(v') = \psi(v)$, so $\max \Psi = \psi(v) -' v < \max \Psi$, a contradiction. Now assume $[v] < [b_i]$ for some $i \in \{1, \dots, n\}$ with $\lambda_i \neq 0$, and let $j \in \{1, \dots, n\}$ be such that $[b_j] = \max\bigl\{[b_i] : 1 \leq i \leq n, \lambda_i \neq 0\bigr\}$. Then

$$\max \Psi = \psi'(v') - v' = b_{j+1} + a_{j+1} - v - \sum_{i=1}^n \lambda_i b_i,$$

hence $\lambda_j = 0$, a contradiction. We have now established (3). To obtain (2), suppose that $\psi'(v') > \max \Psi$ for some $v' \in (V')^{>0}$. By Lemma 2.1, there is $w \in (V')^{>0}$ with $\psi'(w) - w = \max \Psi$. So $w$ is definable over $\mathcal{V}$, hence $w \in V$ and $\psi(w) = \max \Psi + w > \max \Psi$, which is impossible. This finishes the proof of (2).

As to (4), given any $n > 0$, we can choose by (2) an $a \in V^*$ with $|b_n| > |a|$. By Lemma 4.8 above and basic property (3) of $\psi$ listed at the beginning of §2, we have

$$[b_{n+1}] = \bigl[\psi'(b_n) - a_{n+1}\bigr] \leq \bigl[\psi'(b_n) - \psi(a)\bigr] < [b_n - a] = [b_n],$$

as required. $\square$

*Remark.* In the situation of this proposition the sequence $\bigl\{[b_n]\bigr\}$ enumerates the set $[V'] \setminus [V]$ in strictly decreasing order. Thus the sequence $\bigl\{[b_n]\bigr\}$ is independent of the choice of the sequence $\{b_n\}$. It also follows that $\Psi' \setminus \Psi$ is enumerated in strictly increasing order by the sequence $\bigl\{\psi'(b_n)\bigr\}$.

**Theorem 4.10.** *Let $\mathcal{V} \subseteq \mathcal{V}'$ be a finitely generated extension of $H$-couples over the same scalar field, such that $\Psi$ is well-ordered. Then $\Psi'$ is also well-ordered.*

*Proof.* We first equip $\mathcal{V}'$ and $\mathcal{V}$ with suitable $H$-cuts so that we are dealing with an extension of $H$-triples. By induction on the number of generators of $\mathcal{V}'$ over $\mathcal{V}$ we then reduce to the case that $\mathcal{V} \subseteq \mathcal{V}'$ is a simple extension.

Let $\overline{\mathcal{V}} = (\overline{V}, \dots)$ be the definable closure of $\mathcal{V}$ in an $H$-closure of $\mathcal{V}'$, and let $\mathcal{W} = (W, \dots)$ be the $H$-triple with $\mathcal{V} \subseteq \mathcal{W} \subseteq \mathcal{V}'$ and $W = V' \cap \overline{V}$. By Lemma 4.5, $\psi'(W^*)$ is well-ordered. Moreover, $\mathcal{W}$ is definably closed in $\mathcal{V}'$, and $\mathcal{W} \subseteq \mathcal{V}'$ is a simple extension. By Proposition 4.7 we then further reduce to the case that $\mathcal{W} \subseteq \mathcal{V}'$ is a simple extension of one of the types (III), (IV) or (V). If $\mathcal{W} \subseteq \mathcal{V}'$ is of type (III) or type (IV), $\Psi' \setminus \psi'(W^*)$ has at most one element, so $\Psi'$ is well-ordered. If $\mathcal{W} \subseteq \mathcal{V}'$ is of type (V), it follows from the remark preceding the theorem that $\Psi'$ is well-ordered. $\square$

**Corollary 4.11.** *For any $H$-couple $(V, \psi)$ over $\boldsymbol{k}$ that is finitely generated over its substructure with vector space $\boldsymbol{k} \cdot 1 \subseteq V$, the set $\Psi = \psi(V^*)$ is well-ordered.* $\square$

Another issue is whether in a finitely generated $H$-couple $(V, \psi)$ the set $\Psi = \psi(V^*)$ always has a supremum in $V$. This turns out to be false:

*Example.* We take $V = \bigoplus_{n \in \mathbb{N}} \boldsymbol{k} e_{-n}$ as in the example preceding Lemma 4.8, but define $\psi\colon V^* \to V$ by making it constant on $\boldsymbol{k}$-archimedean classes of $V$, and setting

$$\psi(e_0) := e_0, \qquad \psi(e_{-n}) := e_0 + e_{-1} + \cdots + e_{-n} - e_{-n-1} \quad \text{if } n > 0.$$

It is easy to check that $(V, \psi)$ is an $H$-couple with distinguished positive element $1 = e_0$. It is generated over its substructure with vector space $\boldsymbol{k} \cdot 1$ by its vector $e_{-1}$. The set $\Psi$ has no supremum in $V$, as is easily verified.

However, we note that if $\mathcal{V} \subseteq \mathcal{V}\langle c\rangle = \mathcal{V}'$ is a simple extension of $H$-triples of one of the types (I)–(V), and $\sup\Psi$ exists, so does $\sup\Psi'$. This is clear for simple extensions of types (I)–(IV), while for type (V) extensions, it follows from part (2) of Proposition 4.9.

## 5. Removing Scalars

In Proposition 5.7 below we show that closed $H$-couples become weakly o-minimal when we forget the scalar field. Apart from this attractive result, a more urgent motive for including this section is that in attempting to construct a model theory of Hardy fields, it appears useful to have analogues of the previous theorems in a setting where no scalar field is present.

**Definition 5.1.** An $H_0$**-couple** is a pair $\mathcal{V} = (V, \psi)$, consisting of a divisible ordered abelian group $V$, a distinguished positive element $1 \in V$, and a function $\psi\colon V^* \to V$, such that for all $v, w \in V^*$

1. $\psi(1) = 1$,
2. $\psi(nv) = \psi(v)$ for all $n > 0$,
3. $\psi(v) < \psi(w) + |w|$,
4. $|v| \leq |w| \Longrightarrow \psi(v) \geq \psi(w)$ (hence $\psi(v) = \psi(-v)$).

We consider a divisible ordered abelian group as an ordered vector space over $\mathbb{Q}$ in the usual way.

*Examples.*

1. Each $H$-couple becomes an $H_0$-couple by "forgetting" the scalar field.
2. If $F \supseteq \mathbb{R}(x)$ is a real closed Hardy field, $V := v(F^\times)$ its value group, $1 := v(x^{-1})$, and $\psi\colon V^* \to V$ is defined as in the introduction, then $(V, \psi)$ is an $H_0$-couple, with distinguished positive element 1.

**Definition 5.2.** An $H_0$**-cut** of an $H_0$-couple $(V, \psi)$ is a set $P \subseteq V$ which is closed downward, contains $\Psi := \psi(V^*)$, and is disjoint from $(\mathrm{id} + \psi)\big(V^{>0}\big)$. We then call $(V, \psi, P)$ an $H_0$**-triple.** An $H_0$-couple $(V, \psi)$ is **closed** if $\Psi$ has no maximum, and

$$\psi(V^*) = \big\{a \in V : a < w + \psi(w) \text{ for all } w \in V^{>0}\big\}.$$

In that case $\Psi = \psi(V^*)$ is the only $H_0$-cut of $(V, \psi)$, and we call $(V, \psi, \Psi)$ a **closed** $H_0$**-triple.** Note that a closed $H$-couple (closed $H$-triple) becomes a closed $H_0$-couple (closed $H_0$-triple) by forgetting the scalar field.

When dealing with $H_0$-couples $\mathcal{V} = (V, \psi)$ as model-theoretic objects we construe them as $\mathcal{L}_{H_0}$-structures, where $\mathcal{L}_{H_0}$ is the (one-sorted) language with (vector) variables ranging over the extended vector space $V_\infty := V \cup \{\infty\}$. The non-logical symbols of $\mathcal{L}_{H_0}$ are:

1. those listed under part (4) of the description of $\mathcal{L}_H$ in the introduction, to be interpreted as relations and functions on $V_\infty$, as indicated there;
2. a unary function symbol $\delta_n$ for each $n > 0$, to be interpreted on $V$ as the scalar multiplication by $1/n$ (and $\delta_n(\infty) := \infty$).

Adding to $\mathcal{L}_{H_0}$ a unary predicate symbol $P$ we obtain the language $\mathcal{L}_{H_0,P}$, and $H_0$-triples $(V, \psi, P)$ are then construed as $\mathcal{L}_{H_0,P}$-structures. The $H_0$-couples are easily seen to be the models of a universal theory in $\mathcal{L}_{H_0}$, and the same is true for the $H_0$-triples with respect to the language $\mathcal{L}_{H_0,P}$.

*Remark.* The division symbols $\delta_n$ are included to guarantee quantifier elimination for the theory of $H_0$-triples, see Corollary 5.6 below. Here is an example to show that if we omit them, then in the resulting smaller language the theory of $H_0$-triples would not eliminate quantifiers.

Let $(W,\psi)$ be a closed $H$-couple over $\mathbb{Q}$. Choose an element $b \notin W$ in an ordered vector space $W' := W \oplus \mathbb{Q}b$ over $\mathbb{Q}$ extending $W$, such that $\Psi < \frac{b}{2} < (\mathrm{id}+\psi)\big(W^{>0}\big)$. Then, by Lemma 3.8, $[W']_{\mathbb{Q}} = [W]_{\mathbb{Q}}$, hence $\psi$ extends uniquely to a map $\psi'\colon (W')^* \to W'$ such that $\mathcal{W}' = (W',\psi')$ is an $H$-couple over $\mathbb{Q}$ (Lemma 2.3). We consider $\mathcal{W}'$ as an $H_0$-couple. Note that $[W']_{\mathbb{Q}} = [W]_{\mathbb{Q}}$ implies

$$\Psi < \frac{b}{2} < (\mathrm{id}+\psi')\big((W')^{>0}\big).$$

Hence $\mathcal{W}'$ has two $H_0$-cuts. Now consider the ordered abelian group $V := W \oplus \mathbb{Z}b \subseteq W'$. Since $\Psi' = \Psi \subseteq W$, $(V,\psi'|V^*)$ is a substructure of $\mathcal{W}'$, for the language $\mathcal{L}_{H_0}$ with the division symbols $\delta_n$ removed. One checks easily that the two distinct $H_0$-cuts of $\mathcal{W}'$ have the same intersection with $V$, namely $\big\{v \in V : v \le \psi(w) \text{ for some } w \in W\big\}$.

*Notation.* If $\mathcal{V} = (V,\psi)$ is an $H_0$-couple, we set

$$[v] := \big\{w \in V : \psi(w) = \psi(v)\big\}, \quad \text{for } v \in V.$$

We let $[V] := \big\{[v] : v \in V\big\}$ and make it into a linearly ordered set by defining

$$\begin{aligned} [v] < [w] \ &:\Longleftrightarrow\ [v] \ne [w] \text{ and } |v| < |w| \\ &\Longleftrightarrow\ \psi(v) > \psi(w). \end{aligned}$$

In the case that $\mathcal{V}$ is obtained from an $H$-couple over $\boldsymbol{k}$ by "forgetting the scalar field", $[v]$ (or $[v]_{\boldsymbol{k}}$) also denotes the $\boldsymbol{k}$-archimedean class of a vector $v \in V$. Fortunately, this agrees with $[v]$ as just defined. Note also that the four properties of $\boldsymbol{k}$-archimedean classes stated in the beginning of §1 go through for the classes $[v] \subseteq V$ of an $H_0$-couple $\mathcal{V}$ as above, with $\lambda \in \mathbb{Q}^\times$ in property (2). For any $H_0$-couple $\mathcal{V}$ and vectors $v,w \in V^*$, $[v]_{\mathbb{Q}} \le [w]_{\mathbb{Q}}$ implies $[v] \le [w]$.

**Basic properties.** The beginning of section 2 up to and including Proposition 2.2 goes through for $H_0$-triples $(V,\psi,P)$, with $[v]$ interpreted according to the definition just given, and with $H_0$-cuts instead of $H$-cuts in Proposition 2.2. The proofs are the same.

**Embedding into closed $H_0$-triples.** An **$H_0$-closure** of the $H_0$-triple $\mathcal{V} = (V,\psi,P)$ is any closed $H_0$-triple $\mathcal{V}^{\mathrm{c}} = (V^{\mathrm{c}},\psi^{\mathrm{c}},P^{\mathrm{c}})$ extending $\mathcal{V}$, such that any embedding $\mathcal{V} \to \mathcal{V}'$ into a closed $H_0$-triple $\mathcal{V}'$ extends to an embedding $\mathcal{V}^{\mathrm{c}} \to \mathcal{V}'$. We want to show that each $H_0$-triple $\mathcal{V} = (V,\psi,P)$ has an $H_0$-closure. This will follow, just as for $H$-triples, by iterated application of three basic extension lemmas. The first two of these lemmas are exactly the Lemmas 2.5 and 2.6 modified as follows: $H$-triples become $H_0$-triples, $V \oplus \boldsymbol{k}\varepsilon$ becomes $V \oplus \mathbb{Q}\varepsilon$, and the phrase "over $\boldsymbol{k}$" should be omitted. The proofs go through, with similar trivial changes.

For the third extension lemma we have to modify the proof somewhat more, and therefore we will be more explicit.

**Lemma 5.3.** *Suppose $b \in P \setminus \Psi$. Then $(V,\psi,P)$ can be extended to an $H_0$-triple $(V \oplus \mathbb{Q}a, \psi^a, P^a)$ such that:*

1. *$a > 0$, $\psi^a(a) = b$.*

2. *Given any embedding $i\colon (V,\psi,P) \to (V',\psi',P')$ of $H_0$-triples and any element $a' > 0$ in $V'$ with $\psi'(a') = i(b)$, there is a unique extension of $i$ to an embedding $j\colon (V \oplus \mathbb{Q}a, \psi^a, P^a) \to (V',\psi',P')$ with $j(a) = a'$.*

*Proof.* By the Hahn embedding theorem for ordered abelian groups (see [14], I, §5, Satz 3), we can regard $V$ as ordered linear subspace of the ordered vector space $H\big([V^*]_{\mathbb{Q}},\mathbb{R}\big)$ over $\mathbb{Q}$. Take any object $\gamma \notin [V^*]_{\mathbb{Q}}$ and extend the ordering on $[V^*]_{\mathbb{Q}}$ to a linear ordering on $\Gamma := [V^*]_{\mathbb{Q}} \cup \{\gamma\}$ by defining $\gamma < [v]_{\mathbb{Q}} :\Longleftrightarrow b > \psi(v)$, for all $v \in V^*$. We can view $H\big([V^*]_{\mathbb{Q}},\mathbb{R}\big)$ as ordered subspace of $H(\Gamma,\mathbb{R})$, and thus $V$ as ordered subspace of $H(\Gamma,\mathbb{R})$. Choose any $a > 0$ in $H(\Gamma,\mathbb{R})$ with $\max(\operatorname{supp} a) = \gamma$. For non-zero $w = v + \lambda a$ ($v \in V$, $\lambda \in \mathbb{Q}$) we set

$$\psi^a(w) := \begin{cases} \psi(v), & \text{if } [w]_{\mathbb{Q}} = [v]_{\mathbb{Q}}, \\ b, & \text{if } [w]_{\mathbb{Q}} = [a]_{\mathbb{Q}}. \end{cases}$$

Also set $P^a := \big\{w \in V \oplus \mathbb{Q}a : w \le v \text{ for some } v \in P\big\}$. Then $(V \oplus \mathbb{Q}a, \psi^a, P^a)$ is an $H_0$-triple extending $(V,\psi,P)$ with the property stated in the lemma. This assertion can be verified much like the corresponding assertion in the proof of Lemma 2.7, in the case $\boldsymbol{k} = \mathbb{Q}$. □

**Corollary 5.4.** *Every $H_0$-triple has an $H_0$-closure.* □

**Elimination of quantifiers.** We have the following counterpart of Proposition 3.2:

**Proposition 5.5.** *Let $\mathcal{V} = (V,\psi,P)$ and $\mathcal{V}' = (V',\psi',P')$ be closed $H_0$-triples, where $\mathcal{V}'$ is $\kappa$-saturated, $\kappa := |V|^+$. Let $\mathcal{V}_0 = (V_0,\psi_0,P_0)$ be a substructure of $\mathcal{V}$, so again an $H_0$-triple. Any embedding $i_0$ of $\mathcal{V}_0$ into $\mathcal{V}'$ can be extended to an embedding of $\mathcal{V}$ into $\mathcal{V}'$.*

*Proof.* One can basically copy the proof in §3, changing $\boldsymbol{k}$ to $\mathbb{Q}$, and making other obvious modifications. □

Let $T_{H_0,P}$ be the theory of closed $H_0$-triples, in the language $\mathcal{L}_{H_0,P}$.

**Corollary 5.6.** *The theory $T_{H_0,P}$ is complete, decidable, and has elimination of quantifiers. It is the model completion of the theory of $H_0$-triples.*

*Proof.* Elimination of quantifiers follows from Proposition 5.5 and a variant of the well-known Robinson-Shoenfield-Blum criterion for quantifier elimination (see e.g. [21], Theorem 17.2). The $H_0$-triple $(V_0,\psi_0,P_0)$, with $V_0 := \mathbb{Q}$, $P_0 := \mathbb{Q}^{\le 1}$, $\psi_0(x) := 1$ for all $x \in \mathbb{Q}^*$, and $1 \in \mathbb{Q}^{>0}$ as distinguished element, can be embedded into any $H_0$-triple. This implies completeness of $T_{H_0,P}$. The rest now follows from Corollary 5.4. □

**Definable closure. Uniqueness of $H_0$-closure. Analysis of simple extensions. Well-orderedness of $\Psi$.** The correspondingly named subsections of §4 go through for $H_0$-triples and $H_0$-closures with the following changes: $H$-triples (over $\boldsymbol{k}$) become $H_0$-triples, $\mathcal{L}_{H,P}$-formulas become $\mathcal{L}_{H_0,P}$-formulas (without scalar variables $x_1,\dots,x_m$), $\boldsymbol{k}$-linear spaces $\boldsymbol{k}w$ become $\mathbb{Q}$-linear spaces $\mathbb{Q}w$, more generally, vector spaces over $\boldsymbol{k}$ become vector spaces over $\mathbb{Q}$, scalars from $\boldsymbol{k}$ (as in the proofs of lemmas 4.3 and 4.5) become scalars from $\mathbb{Q}$, and, finally, $\boldsymbol{k}$-linear independence (as in property (V) of the analysis of simple extensions) becomes $\mathbb{Q}$-linear independence. Also, the equivalence classes $[v]$ of vectors $v$ should of course be interpreted in the sense of the present section.

**Weak o-minimality.** We now use the $H_0$-version of Proposition 4.7 to show:

**Proposition 5.7.** *The theory $T_{H_0,P}$ of closed $H_0$-triples is weakly o-minimal, i.e. each closed $H_0$-triple is weakly o-minimal.*

(Compare with Proposition 4.2.) In the proof, we will apply a test for weak o-minimality stated in [1], Theorem 1.1, and attributed there to Kulpeshov [11].

**A criterion for weak o-minimality.** Let $\mathcal{L}$ be a language containing a binary relation symbol $<$, and let $\mathbf{A} = (A, <, \dots)$ be an $\mathcal{L}$-structure expanding a nonempty linearly ordered set $(A, <)$, dense without endpoints. A **cut** in $\mathbf{A}$ is just a downward closed set $C \subseteq A$. To such a cut $C$ we associate the set

$$\Phi_C(y) := \big\{c < y : c \in C\big\} \cup \big\{y < d : d \in A \setminus C\big\}$$

of $\mathcal{L}_A$-formulas in the variable $y$.

**Lemma 5.8.** (Kulpeshov) *An $\mathcal{L}$-structure $\mathbf{A} = (A, <, \dots)$ as above is weakly o-minimal if and only if for all cuts $C$ in $A$ there exist at most two complete $y$-types over $A$ extending $\Phi_C(y)$, and for each of these types, its set of realizations in any elementary extension $\mathbf{B} = (B, <, \dots)$ of $\mathbf{A}$ is convex in $B$.*

We may as well give here the short proof:

*Proof.* Suppose for each cut $C$ in $A$ there exist at most two complete $y$-types over $A$ extending $\Phi_C(y)$, and for each of those types its set of realizations in any elementary extension of $\mathbf{A}$ is convex. We claim that then $\mathbf{A}$ is weakly o-minimal.

Let $\mathcal{L}_{\mathrm{c}}$ be the language $\mathcal{L}$ augmented by a new unary relation symbol $C$ for each cut $C$ of $\mathbf{A}$. We naturally expand $\mathbf{A}$ to an $\mathcal{L}_{\mathrm{c}}$-structure $\mathbf{A}_{\mathrm{c}}$. Let $\mathbf{B}_{\mathrm{c}} \succeq \mathbf{A}_{\mathrm{c}}$ be an $|A|^+$-saturated elementary extension of $\mathbf{A}_{\mathrm{c}}$, $\mathbf{B} := \mathbf{B}_{\mathrm{c}}|\mathcal{L}$. Let $b_1 < b_2$ be elements of $B \setminus A$ such that $\mathbf{B}_{\mathrm{c}} \models C(b_1) \leftrightarrow C(b_2)$ for all these new relation symbols $C$. By a standard model-theoretic argument it suffices to show that $b_1$ and $b_2$ realize the same type (in $\mathcal{L}$) over $A$. Put

$$C := \{c \in A : c < b_1\} = \{c \in A : c < b_2\}.$$

Assume $C \neq \varnothing$, $C \neq A$. (The cases $C = \varnothing$ and $C = A$ are treated similarly.) Suppose $\varphi(y)$ is an $\mathcal{L}_A$-formula, and $\mathbf{B} \models \varphi(b_1) \wedge \neg\varphi(b_2)$. By hypothesis, $\mathbf{B} \models \varphi(e) \wedge \neg\varphi(f)$ whenever $C < e \leq b_1 < b_2 \leq f < A \setminus C$. By saturation there exist $c \in C$, $d \in A \setminus C$ such that

$$\mathbf{B} \models \forall y \forall z \big(c < y \leq b_1 < b_2 \leq z < d \rightarrow \varphi(y) \wedge \neg\varphi(z)\big).$$

Hence

$$\mathbf{A}_{\mathrm{c}} \models \forall y \big(c < y < d \rightarrow \big(\varphi(y) \leftrightarrow C(y)\big)\big).$$

Since $\mathbf{B}_{\mathrm{c}} \models \big(C(b_1) \leftrightarrow C(b_2)\big)$, this implies $\mathbf{B} \models \varphi(b_1) \leftrightarrow \varphi(b_2)$, a contradiction.

Assume conversely that $\mathbf{A}$ is weakly o-minimal, and let $C \subseteq A$ be a cut. We can assume $C \neq \varnothing$, $C \neq A$, the other two cases being similar. Let $\Psi(y)$ be the set of all $\mathcal{L}_A$-formulas $\psi(y)$ for which there exist $c \in C$, $d \in A \setminus C$ such that $\mathbf{A} \models \psi(a) \Leftrightarrow a \in C$, for all $a \in (c, d)$. Let $\mathbf{B} \succeq \mathbf{A}$, and let $D, E \subseteq B$ be the set of realizations of $\Psi(y) \cup \Phi_C(y)$ and $\Phi_C(y)$, respectively. Then $D \subseteq E$ is downward closed in $E$. By weak o-minimality of $\mathbf{A}$, all elements of $D$ have the same type over $A$. Note that for any $\psi(y), \psi'(y) \in \Psi(y)$, there exist $c \in C$, $d \in A \setminus C$ such that

$$\mathbf{A} \models \forall y \big(c < y < d \rightarrow \big(\psi(y) \leftrightarrow \psi'(y)\big)\big).$$

Hence all elements of $E \setminus D$ realize $\neg\Psi(y) = \{\neg\psi(y) : \psi(y) \in \Psi(y)\}$, and therefore have the same type over $A$. $\square$

**Proof of Proposition 5.7.** Let $\mathcal{V} = (V, \psi, P)$ be a closed $H_0$-triple, $\mathcal{V}' = (V', \psi', P')$ an elementary extension of $\mathcal{V}$, and $C$ a cut in $V$. By quantifier elimination, the complete $y$-types over $V$ extending $\Phi_C(y)$ correspond bijectively to isomorphism classes over $\mathcal{V}$ of simple extensions $\mathcal{V}\langle c\rangle$ of $\mathcal{V}$ with distinguished generator $c$ such that $C < c < V \setminus C$. We claim:

1. Up to isomorphism over $\mathcal{V}$, there exist at most two simple extensions $\mathcal{V}\langle c\rangle$ of $\mathcal{V}$ with distinguished generator $c$ such that $C < c < V \setminus C$.
2. If $c$ is an element of $V'$ with $C < c < V \setminus C$, then the set of all $d \in V'$ such that $\mathcal{V}\langle c\rangle \cong \mathcal{V}\langle d\rangle$ by an isomorphism over $\mathcal{V}$ that maps $c$ to $d$, is a convex subset of $V'$.

By Kulpeshov's criterion it will then follow that $\mathcal{V}$ is weakly o-minimal. So assume $\mathcal{V} \subseteq \mathcal{V}\langle c\rangle$ is a simple extension with $C < c < V \setminus C$. We may (for our purpose) assume that $c \in V'$. By our analysis of simple extensions, either

1. $\mathcal{V} \subseteq \mathcal{V}\langle c\rangle$ is of type (IV) or type (V), or
2. there are $n \geq 1$, $a_1, \dots, a_n \in V$, and non-zero $b_1, \dots, b_n \in V'$ such that $b_1 = c - a_1$, $b_{j+1} = \psi'(b_j) - a_{j+1}$ for $1 \leq j < n$, the vectors $\psi'(b_n), b_1, \dots, b_n$ are $\mathbb{Q}$-linearly independent over $V$, $[b_j] \notin [V]$ for $1 \leq j < n$, $[b_i] \neq [b_j]$ for $1 \leq i < j \leq n$ and $V\langle c\rangle = V \oplus \mathbb{Q}\psi'(b_n) \oplus \bigoplus_{j=1}^n \mathbb{Q}b_j$.

In all three cases, an argument as in the proof of Proposition 3.2 shows that then for any simple extension $\mathcal{V}\langle d\rangle$ of $\mathcal{V}$ with $C < d < V \setminus C$, the $H_0$-couples $(V\langle c\rangle, \psi'|V\langle c\rangle^*)$ and $(V\langle d\rangle, \psi'|V\langle d\rangle^*)$ are isomorphic over $(V, \psi)$ by an isomorphism mapping $c$ to $d$. Also, $(V\langle c\rangle, \psi'|V\langle c\rangle^*)$ has at most two $H_0$-cuts, and hence can be expanded in at most two ways to an $H_0$-triple. This proves the first part of the claim.

For the second part we need some notation: Let $n \in \mathbb{N}$, $b \in V$, $a = (a_1, \dots, a_n) \in V^n$, and $\lambda = (\lambda_0, \dots, \lambda_n) \in \mathbb{Q}^{n+1}$. Consider the map

$$y \mapsto \theta_{b,a,\lambda}(y) := b + \lambda_0 y + \sum_{j=1}^{n} \lambda_j \psi'_{(a_1,\dots,a_j)}(y) \colon D'_a \to V'.$$

It is monotone on each convex component of $D'_a$. (By the analogues of Lemmas 3.6 and 3.7 for $H_0$-couples.) In particular, for each convex component $D$ of $D'_a$, the set $D \cap \theta_{b,a,\lambda}^{-1}(P')$ is downward or upward closed in $D$.

Now assume first that $(V\langle c\rangle, \psi'|V\langle c\rangle^*)$ has only one $H_0$-cut. Then each $H_0$-triple $\mathcal{V}\langle d\rangle$, where $C < d < V \setminus C$, is isomorphic to $\mathcal{V}\langle c\rangle$ by an isomorphism over $\mathcal{V}$ mapping $c$ to $d$. So assume that $(V\langle c\rangle, \psi'|V\langle c\rangle^*)$ has two $H_0$-cuts. This means that there exists $w \in V\langle c\rangle$ such that

$$\psi'\big(V\langle c\rangle^*\big) < w < (\mathrm{id} + \psi')\big(V\langle c\rangle^{>0}\big).$$

In all three cases for $\mathcal{V} \subseteq \mathcal{V}\langle c\rangle$, we find $n \in \mathbb{N}$, $b \in V$, $a \in V^n$ and $\lambda \in \mathbb{Q}^{n+1}$ such that $w = \theta_{b,a,\lambda}(c)$. Observe that for any $d \in V'$ with $C < d < V \setminus C$, $d$ lies in $D'_a$. In fact, it lies in the same convex component $D$ of $D'_a$ as $c$, and $\mathcal{V}\langle c\rangle \cong \mathcal{V}\langle d\rangle$ by an isomorphism over $\mathcal{V}$ with $c \mapsto d$ if and only if either both $c$ and $d$ are in $D \cap \theta_{b,a,\lambda}^{-1}(P')$, or both are not in $D \cap \theta_{b,a,\lambda}^{-1}(P')$. Thus also in this case, the second part of the claim follows. $\square$

## 6. Relation to Contraction Groups

Our couples resemble the contraction groups of Kuhlmann [9], [10], and there is indeed a formal connection as indicated below. (A difference is that contraction groups have nothing like our cut $P$.)

Contraction groups arise as follows: let $F$ be a Hardy field closed under taking logarithms (i.e. $f \in F^{>0} \Rightarrow \log f \in F$), with its valuation $v : F^\times \to V = v(F^\times)$. The logarithm map then induces a so-called contraction map $\chi\colon V^{<0} \to V^{<0}$ by

$$\chi\big(v(f)\big) := v(\log f) \quad \text{for all } f \in F^{>0} \text{ with } v(f) < 0,$$

which we extend to a map $V \to V$ by requiring $\chi(-y) = -\chi(y)$. If $F$ is also closed under exponentiation, then $V$ is divisible, and $\chi$ is surjective ($\chi(V) = V$). This means that the pair $(V, \chi)$ (ordered group with contraction map) is a **divisible centripetal contraction group**, as axiomatized in [9], where it was shown that the elementary theory of non-trivial divisible centripetal contraction groups is complete and has quantifier elimination in its natural language. The weak o-minimality of this theory is proved in [10] (and also follows from its completeness and (3.16) in [6]).

In the example above, we have for $f \in F^{>0}$, with $y = v(f) < 0$:

$$\psi(y) = v\big((\log f)'\big) = v\big((\log f)'/\log f\big) + v(\log f) = \psi\big(\chi(y)\big) + \chi(y), \tag{6.1}$$

Let now $(V, \psi)$ be any closed $H_0$-couple. For $y < 0$ in $V$, let $\chi(y) = z$ be the unique solution in $V^*$ of the equation

$$z + \psi(z) = \psi(y).$$

For $y > 0$, set $\chi(y) := -\chi(-y)$, and $\chi(0) := 0$. It is easily seen that then $(V, \chi)$ is a divisible centripetal contraction group; clearly $\chi$ is definable (without parameters) in $(V, \psi)$. However, we cannot definably reconstruct $\psi$ in $(V, \chi)$:

**Proposition 6.1.** *In* no *divisible centripetal contraction group $(V, \chi)$ can one define, even allowing parameters, a function $\psi\colon V^* \to V$ such that $(V, \psi)$ is a closed $H_0$-couple (for some choice of $1 > 0$) and $\chi + \psi \circ \chi = \psi$ on $V^{<0}$.*

Before we can prove this we need some preparations. In the rest of this subsection we let $(V, \psi)$ denote a closed $H_0$-couple.

**Iterates of $\psi$.** For $n > 0$, let $\psi^n : V_\infty \to V_\infty$ be the $n$-fold composition $\psi \circ \psi \circ \cdots \circ \psi$. Put

$$D_n := \big\{v \in V : \psi^n(v) \neq \infty\big\}.$$

So $D_n = D_a$ for $a = (0, \dots, 0) \in V^n$. For example $D_1 = V^*$, $D_2 = V^* \setminus \psi^{-1}(0)$, etc. By induction on $n$ one shows easily that $\psi^n(D_n) = \Psi$.

**Lemma 6.2.** *Let $n > 0$ and $\psi^n(v) < 0$. Then $\psi^i(v) < 0$ for $i = 1, \dots, n$, and*

$$\big[\psi^n(v)\big] < \big[\psi^{n-1}(v)\big] < \cdots < \big[\psi(v)\big] < [v].$$

*Proof.* Fix a vector $v_0 > 0$ in $V$ such that $\psi(v_0) = 0$. For $n = 1$, note that $\psi(v) < 0 = \psi(v_0)$ implies $[v] > [v_0]$, hence $\big[\psi(v)\big] = \big[\psi(v) - \psi(v_0)\big] < [v - v_0] = [v]$.

Assume inductively that the lemma holds a certain $n > 0$. Let $v \in D_{n+1}$ with $\psi^{n+1}(v) < 0$. Applying the case $n = 1$ to $\psi^n(v)$ instead of $v$ gives $\big[\psi^{n+1}(v)\big] < \big[\psi^n(v)\big]$. By the inductive assumption the remaining inequalities will follow from $\psi^n(v) < 0$. Suppose $\psi^n(v) \geq 0$. Then $\psi^n(v) \in \Psi^{>0}$, thus $\big[\psi^n(v)\big] \leq [1]$. On the other hand, $\psi^{n+1}(v) < 0$ implies $\big[\psi^n(v)\big] > [v_0]$, a contradiction. □

Let $D_\infty := \bigcap_{n>0} D_n$ and

$$V_{\inf} := \{v \in D_\infty : \psi^n(v) < 0 \text{ for all } n > 0\},$$
$$V_{\text{fin}} := V \setminus V_{\inf}.$$

Note that $[v_0] < [v]$ for all $v \in V_{\inf}$, and that $V_{\inf} \cap V^{>0}$ is closed upward and $V_{\inf} \cap V^{<0}$ is closed downward.

*Remark.* The previous lemma, together with $\psi^n(D_n) = \Psi$, implies that for all $n > 0$, we can find an element $v \in D_n$ such that all iterates

$$\psi(v), \psi^2(v), \dots, \psi^n(v)$$

are negative. Hence if $(V, \psi)$ is $\aleph_0$-saturated, then $V_{\inf} \neq \varnothing$.

The proof of the next lemma is easy and left to the reader.

**Lemma 6.3.** *$V_{\text{fin}}$ is a convex subspace of $V$, and $(V_{\text{fin}}, \psi|V_{\text{fin}}^*)$ is a closed $H_0$-couple. Moreover, $\psi(V_{\inf}) = V_{\inf} \cap V^{<0}$.* □

Let $\chi$ be the contraction map defined by $\psi(v) = \chi(v) + \psi(\chi(v))$ for all $v < 0$.

**Lemma 6.4.** *Let $v \in V^{<0}$ and $\psi^2(v) < 0$. Then $\chi(v) = \psi(v) - \psi^2(v)$.*

*Proof.* We have $[v] > [\psi(v)]$, so $\psi(v) - \psi^2(v) < 0$. We compute:

$$\big(\psi(v) - \psi^2(v)\big) + \psi\big(\psi(v) - \psi^2(v)\big) = \big(\psi(v) - \psi^2(v)\big) + \psi^2(v) = \psi(v).$$

By the defining equation (6.1) of $\chi$, it follows that $\chi(v) = \psi(v) - \psi^2(v)$. □

**Proof of Proposition 6.1.** Suppose $(V, \psi)$ is a closed $H_0$-couple such that we can define $\psi$ in $(V, \chi)$. We may assume that $(V, \psi)$ is $\aleph_0$-saturated. For ease of notation we shall also assume that $\psi$ is actually defined without parameters in $(V, \chi)$. (In the general case the role of $V_{\text{fin}}$ below is taken over by the convex hull in $V$ of an $H_0$-closure inside $(V, \psi, P)$ of the substructure of $(V, \psi, P)$ generated by the finitely many parameters used to define $\psi$.) We modify $\psi$ to a function $\widetilde{\psi} : V^* \to V$ by putting

$$\widetilde{\psi}(v) := \begin{cases} \psi(v), & \text{if } v \in V_{\text{fin}}^* \\ \psi(v) + 1, & \text{if } v \in V \setminus V_{\text{fin}}. \end{cases}$$

Then $(V, \widetilde{\psi})$ is still an $H_0$-couple, and $\widetilde{\psi}(V_{\inf} \setminus V_{\text{fin}}) = \psi(V_{\inf} \setminus V_{\text{fin}})$, as is easily checked. Thus $\Psi = \widetilde{\psi}(V^*)$, so $(V, \widetilde{\psi})$ is even a *closed* $H_0$-couple. Let $\widetilde{\chi}$ be the contraction map associated to $(V, \widetilde{\psi})$. By completeness of the theory of closed $H_0$-couples, the same formula of $\mathcal{L}_{H_0}$ that defines $\psi$ in $(V, \chi)$ will define $\widetilde{\psi}$ in $(V, \widetilde{\chi})$. By Lemma 6.4, $\chi = \widetilde{\chi}$, hence $\psi = \widetilde{\psi}$, contradiction. □

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