Valuation theoretic and model theoretic aspects of local uniformization

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1 Introduction

In this paper, I will take you on an excursion from Algebraic Geometry through Valuation Theory to Model Theoretic Algebra, and back. If our destination sounds too exotic for you, you may jump off at the Old World (Valuation Theory) and divert yourself with problems and examples until you catch our plane back home.

As a preparation for the foreign countries of the Old World, I recommend to read the first sections of the paper [V]. You may also look at the basic facts about valuations in the books [ZS], [R1], [EN], [WA2], [JA]. All other equipment will be distributed on the excursion. I do not recommend that you try to read about the exotic countries of model theory. An introduction will be given on the excursion. If you want to read more, you may ask me for a copy of the chapter "Introduction to model theoretic algebra" of the forthcoming book [K2], and later you may look at books like [CK], [CH1], [HO], [SA].

Since I want our excursion (and myself) to be as relaxed as possible, all varieties we meet will be assumed to be affine and irreducible (wherever it matters). But certainly, we do not assume them to be non-singular, nor do we assume that the base fields be algebraically closed or the characteristic be 0.

2 What does local uniformization mean?

Here is the bitter truth of mankind: In most cases we average human beings are too stupid to solve our problems globally. So we try to solve them locally. And if we are clever enough (and truly interested), we then may think of patching the local solutions together to obtain a global solution.

What is the problem we are considering here? It is the fact that an algebraic variety has singularities, and we want to get rid of them. That is, we are looking for a second variety having the same function field, and having no singularities. This would be the global solution of our problem. As we are too stupid for it, we are first looking for a local solution. Naively speaking, "local" means something like "at a point of the variety". So local solution would mean that we get rid of one singular point. We are looking for a new variety where our point becomes non-singular. But wait, this was nonsense. Because what is our old, singular point on the new variety? We cannot talk of the same points of two different varieties, unless we deal with subvarieties. But passing from varieties to subvarieties or vice versa will in general not provide the solution we are looking for. So do we have to forget about local solutions of our problem?

The answer is: no. Let us have a closer look at our notion of "point". Assume our variety V is given by polynomials $f_1,\ldots,f_n \in K[X_1,\ldots,X_\ell]$. Naively, by a point of V we then mean an ℓ-tupel $(a_1,\ldots,a_\ell)$ of elements in an arbitrary extension field L of K such that $f_i(a_1,\ldots,a_\ell) = 0$ for $1 \leq i \leq n$. This means that the kernel of the "evaluation homomorphism" $K[X_1,\ldots,X_\ell] \rightarrow L$ defined by $X_i \mapsto a_i$ contains the ideal $(f_1,\ldots,f_n)$. So it induces a homomorphism η from the coordinate ring $K[V] = K[X_1,\ldots,X_\ell]/(f_1,\ldots,f_n)$ into L over K. (The latter means that it leaves the elements of K fixed.) However, if $a'_1,\ldots,a'_\ell \in L'$ are such that $a_i \mapsto a'_i$ induces an isomorphism from $K(a_1,\ldots,a_\ell)$ onto $K(a'_1,\ldots,a'_\ell)$, then we would like to consider $(a_1,\ldots,a_\ell)$ and $(a'_1,\ldots,a'_\ell)$ as the same point of V. That is, we are only interested in η up to composition $\sigma\circ\eta$ with isomorphisms σ. This we can get by considering the kernel of η instead of η. This leads us to the modern approach: to view a point as a prime ideal of the coordinate ring.

But I wouldn't have told you all this if I intended to follow this modern approach. Instead, I want to build on the picture of homomorphisms. So I ask you to accept temporarily the convention that a point of V is a homomorphism of $K[V]$ over K (i.e., leaving K elementwise invariant), modulo composition with isomorphisms. Recall that $K[V] = K[x_1,\ldots,x_\ell]$, where x_i is the image of X_i under the canonical epimorphism $K[X_1,\ldots,X_\ell] \rightarrow K[X_1,\ldots,X_\ell]/(f_1,\ldots,f_n) = K[V]$. The function field $K(V)$ of V is the quotient field $K(x_1,\ldots,x_\ell)$ of $K[V]$. It is generated by $x_1,\ldots,x_\ell$ over K, hence it is finitely generated. Every finite extension of a field K of transcendence degree at least 1 is called an **algebraic function field** (over K), and it is in fact the function field of a suitable variety defined over K. When we talk of function fields in this paper, we will always mean algebraic function fields.

Now recall what it means to look for another variety V' having the same function field $F := K(V)$ as

V (i.e., being birationally equivalent to V). It just means to look for another set of generators $y_1, \ldots, y_k$ of F over K. Now the points of V' are the homomorphisms of $K[y_1, \ldots, y_k]$ over K, modulo composition with isomorphisms. But in general, $y_1, \ldots, y_k$ will not lie in $K[x_1, \ldots, x_\ell]$, hence we do not see how a given homomorphism of $K[x_1, \ldots, x_\ell]$ could determine a homomorphism of $K[y_1, \ldots, y_k]$. But if we could extend the homomorphism of $K[x_1, \ldots, x_\ell]$ to all of $K(x_1, \ldots, x_\ell)$, then this extension would assign values to every element of $K[y_1, \ldots, y_k]$. Let us give a very simple example.

Example 2.1 Consider the coordinate ring $K[x]$ of $V = \mathbb{A}^1_K$. That is, x is transcendental over K, and the function field $K(V)$ is just the rational function field $K(x)$ over K. A homomorphism of the polynomial ring $K[V] = K[x]$ is just given by "evaluating" every polynomial $g(x)$ at $x = a$. I have seen many people who suffered in school from the fact that one can also try to evaluate rational functions $g(x)/h(x)$. The obstruction is that a could be a zero of h, and what do we get then by evaluating $1/h(x)$ at a? (In fact, if our homomorphism is not an embedding, i.e., if a is not transcendental over K, then there will always be a polynomial h over K having a as a root.) So we have to accept that the evaluation will not only render elements in $K(a)$, but also the element ∞, in which case we say that the evaluated rational function has a pole at a. So we can extend our homomorphism to a map P on all of $K(x)$, taking into the bargain that it may not always render finite values. But on the subring $\mathcal{O}_P = \{g(x)/h(x) \mid h(a) \neq 0\}$ of $K(x)$ on which P is finite, it is still a homomorphism. $\diamondsuit$

What we have in front of our eyes in this example is one of the two basic classical examples for the concept of a **place**. (The other one, the p-adic place, comes from number theory.) Traditionally, the application of a place P is written in the form $g \mapsto gP$, where instead of gP also $g(P)$ was used in the beginning, reminding of the fact that P originated from an evaluation homomorphism. If you translate the German "g an der Stelle a auswerten" literally, you get "evaluate g at the place a", which explains the origin of the word "place".

Associated to a place P is its **valuation ring** $\mathcal{O}_P$, the maximal subring on which P is finite, and a valuation v_P. In our case, the value $v_P(g/h)$ is determined by computing the zero or pole order of g/h (pole orders taken to be negative integers). In this way, we obtain values in $\mathbb{Z}$, which is the value group of v_P. In general, given a field L with place P and associated valuation v_P, the valuation ring $\mathcal{O}_P = \{b \in L \mid bP \neq \infty\} = \{b \in L \mid v_P b \geq 0\}$ has a unique maximal ideal $\mathcal{M}_P = \{b \in L \mid bP = 0\} = \{b \in L \mid v_P b > 0\}$. The **residue field** is $LP := \mathcal{O}_P/\mathcal{M}_P$ so that P restricted to $\mathcal{O}_P$ is just the canonical epimorphism $\mathcal{O}_P \to LP$. The characteristic of LP is called the **residue characteristic** of (L, P). If P is the identity on $K \subseteq L$, then $K \subseteq LP$ canonically. The valuation v_P can be defined to be the homomorphism $L^\times \to L^\times/\mathcal{O}_P^\times$. The latter is an ordered abelian group, the **value group** of (L, v_P). We denote it by $v_P L$ and write it additively. Note that $bP \neq \infty \Leftrightarrow b \in \mathcal{O}_P \Leftrightarrow v_P b \geq 0$, and $bP = 0 \Leftrightarrow b \in \mathcal{M}_P \Leftrightarrow v_P b > 0$.

Instead of (L, P), we will often write (L, v) if we talk of valued fields in general. Then we will write av and Lv instead of aP and LP. If we talk of an **extension of valued fields** and write $(L|K, v)$ then we mean that v is a valuation on L and K is endowed with its restriction. If we only have to consider a single extension of v from K to L, then we will use the symbol v for both the valuation on K and that on L. Similarly, we use "$(L|K, P)$".

Observe that in Example 2.1, P is uniquely determined by the homomorphism on $K[x]$. Indeed, we can always write g/h in a form such that a is not a zero of both g and h. If then a is not a zero of h, we have that $(g/h)P = g(a)/h(a) \in K(a)$. If a is a zero of h, we have that $(g/h)P = \infty$. Thus, the residue field of P is $K(a)$, and the value group is $\mathbb{Z}$. On the other hand, we have the same non-uniqueness for places as we had for homomorphisms: also places can be composed with isomorphisms. If P, Q are places of an arbitrary field L and there is an isomorphism $\sigma : LP \to LQ$ such that $\sigma(bP) = bQ$ for all $b \in \mathcal{O}_P$, then we call P and Q **equivalent places**. In fact, P and Q are equivalent if and only if $\mathcal{O}_P = \mathcal{O}_Q$. Nevertheless, it is often more convenient to work with places than with valuation rings, and we will just identify equivalent places wherever this causes no problems.

Two valuations v and w are called **equivalent valuations** if they only differ by an isomorphism of the value groups; this holds if and only if v and w have the same valuation ring. As for places, we will identify equivalent valuations wherever this causes no problems, and we will also identify the isomorphic value groups.

At this point, we shall introduce a useful notion. Given a function field $F|K$, we will call P a **place of** $F|K$ if it is a place of F whose restriction to K is the identity. We say that P is **trivial** on K if it induces an isomorphism on K. But then, composing P with the inverse of this isomorphism, we find that P is equivalent to a place of F whose restriction to K is the identity. Note that a place P of F is trivial on K if and only if v_P is **trivial** on K, i.e., $v_P K = \{0\}$. This is also equivalent to $K \subset \mathcal{O}_P$. A place P of $F|K$ is said to be a **rational place** if $FP = K$. The **dimension** of P, denoted by $\dim P$, is the transcendence degree of $FP|K$. Hence, P is **zero-dimensional** if and only if $FP|K$ is algebraic.

Let's get back to our problem. The first thing we learn from our example is the following. Clearly, we would like to extend our homomorphism of $K[V]$ to a place of $K(V)$ because then, it will induce a map on $K[V']$. Then we have the chance to say that the point we have to look at on the new variety (e.g., in order to see whether this one is simple) is the point given by this map on $K[V']$. But this only makes sense if this map is a homomorphism of $K[V']$. So we have to require that

$$y_1, \ldots, y_k \in \mathcal{O}_P$$

(since then, $K[y_1, \ldots, y_k] \subseteq \mathcal{O}_P$, which implies that P is a homomorphism on $K[y_1, \ldots, y_k]$).

This being granted, the next question coming to our mind is whether to every point there corresponds exactly one place (up to equivalence), as it was the case in Example 2.1. To destroy this hope, I give again a very simple example. It will also serve to introduce several types of places and their invariants.

Example 2.2 Consider the coordinate ring $K[x_1, x_2]$ of $V = \mathbb{A}^2_K$. That is, x_1 and x_2 are algebraically independent over K, and the function field $K(V) = K(x_1, x_2)$ is just the rational function field in two variables over K. A homomorphism of the polynomial ring $K[V] = K[x_1, x_2]$ is given by "evaluating" every polynomial $g(x_1, x_2)$ at $x_1 = a_1$, $x_2 = a_2$. For example, let us take $a_1 = a_2 = 0$ and try to extend the corresponding homomorphism of $K[x_1, x_2]$ to $K(V) = K(x_1, x_2)$. It is clear that $1/x_1$ and $1/x_2$ go to ∞. But what about x_1/x_2 or even x_1^m/x_2^n? Do they go to 0, ∞ or some non-zero element in K? The answer is: all that is possible, and there are infinitely many ways to extend our homomorphism to a place of $K(x_1, x_2)$.

There is one way, however, which seems to be the most well-behaved. It is to construct a **place of maximal rank**; we will explain this notion later in full generality. The idea is to learn from Example 2.1 where we replace K by $K(x_2)$ and x by x_1, and extend the homomorphism defined on $K(x_2)[x_1]$ by $x_1 \mapsto 0$ to a unique place Q of $K(x_1, x_2)$. Its residue field is $K(x_2)$ since $x_1 Q = 0 \in K(x_2)$, and its value group is $\mathbb{Z}$. Now we do the same for $K(x_2)$, extending the homomorphism given on $K[x_2]$ by $x_2 \mapsto 0$ to a unique place $\overline{Q}$ of $K(x_2)$ with residue field K and value group $\mathbb{Z}$. We compose the two places, in the following way. Take $b \in K(x_1, x_2)$. If $bQ = \infty$, then we set $bQ\overline{Q} = \infty$. If $bQ \neq \infty$, then $bQ \in K(x_2)$, and we know what $bQ\overline{Q} = (bQ)\overline{Q}$ is. In this way, we obtain a place $P = Q\overline{Q}$ on $K(x_1, x_2)$ with residue field K. We observe that for every $g \in K[x_1, x_2]$, we have that $g(x_1, x_2)Q\overline{Q} = g(0, x_2)\overline{Q} = g(0, 0)$, so our place P indeed extends the given homomorphism of $K[x_1, x_2]$. Now what happens to our critical fractions? Clearly, $(1/x_1)P = (1/x_1)Q\overline{Q} = (\infty)\overline{Q} = \infty$, and $(1/x_2)P = (1/x_2)Q\overline{Q} = (1/x_2)\overline{Q} = \infty$. But what interests us most is that for all $m > 0$ and $n \geq 0$, $(x_1^m/x_2^n)P = (x_1^m/x_2^n)Q\overline{Q} = 0\overline{Q} = 0$. We see that "$x_1$ goes more strongly to 0 than every x_2^n". We have achieved this by sending first x_1 to 0, and only afterwards x_2 to 0. We have arranged our action "lexicographically".

What is the associated value group? General valuation theory (cf. [V], §3 and §4, or [ZS]) tells us that for every composition $P = Q\overline{Q}$, the value group $v_{\overline{Q}}(FQ)$ of the place $\overline{Q}$ on FQ is a convex subgroup of the value group $v_P F$, and that the value group $v_Q F$ of P is isomorphic to $v_P F / v_{\overline{Q}}(FQ)$. If the subgroup $v_{\overline{Q}}(FQ)$ is a direct summand of $v_P F$ (as it is the case in our example), then $v_P F$ is the lexicographically ordered direct product $v_Q F \times v_{\overline{Q}}(FQ)$. Hence in our case, $v_P K(x_1, x_2) = \mathbb{Z} \times \mathbb{Z}$, ordered lexicographically. The **rank of an abelian ordered group** G is the number of proper convex subgroups of G (or rather the order type of the chain of convex subgroups, ordered by inclusion, if this is not finite). The **rank of**

(F, P) is defined to be the rank of $v_P F$. See under the name "hauteur" in [V]. In our case, the rank is 2. We will see in Section 7 that if P is a place of $F|K$, then the rank cannot exceed the transcendence degree of $F|K$. So our place $P = Q\overline{Q}$ has maximal possible rank.

There are other places of maximal rank which extend our given homomorphism, but there is also an abundance of places of smaller rank. In our case, "smaller rank" can only mean rank 1, i.e., there is only one proper convex subgroup of the value group, namely $\{0\}$. For an ordered abelian group G, having rank 1 is equivalent to being archimedean ordered and to being embeddable in the ordered additive group of $\mathbb{R}$. Which subgroups of $\mathbb{R}$ can we get as value groups? To determine them, we look at the **rational rank** of an ordered abelian group G. It is $\mathrm{rr}\, G := \dim_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Z}} G$ (note that $\mathbb{Q} \otimes_{\mathbb{Z}} G$ is the **divisible hull** of G). This is the maximal number of rationally independent elements in G. We will see in Section 7 that for every place P of $F|K$ we have that

$$\mathrm{rr}\, v_P F \;\leq\; \mathrm{trdeg}\, F|K\;. \tag{1}$$

Hence in our case, also the rational rank of P can be at most 2. The subgroups of $\mathbb{R}$ of rank 2 are well known: they are the groups of the form $r\mathbb{Z} + s\mathbb{Z}$ where $r > 0$ and $s > 0$ are rationally independent real numbers. Moreover, through multiplication by $1/r$, the group is order isomorphic to $\mathbb{Z} + \frac{s}{r}\mathbb{Z}$. As we identify equivalent valuations, we can assume all rational rank 2 value groups (of a rank 1 place) to be of the form $\mathbb{Z} + r\mathbb{Z}$ with $0 < r \in \mathbb{R} \setminus \mathbb{Q}$. To construct a place P with this value group on $K(x_1, x_2)$, we proceed as follows. We want that $v_P x_1 = 1$ and $v_P x_2 = r$; then it will follow that $v_P K(x_1, x_2) = \mathbb{Z} + r\mathbb{Z}$ (cf. Theorem 7.2 below). We observe that for such P, $v_P(x_1^m/x_2^n) = m - nr$, which is > 0 if $m/n > r$, and < 0 if $m/n < r$. Hence, $(x_1^m/x_2^n)P = 0$ if $m/n > r$, and $(x_1^m/x_2^n)P = \infty$ if $m/n < r$. I leave it to you as an exercise to verify that this defines a unique place P of $K(x_1, x_2)|K$ with the desired value group and extending our given homomorphism.

Observe that so far every value group was finitely generated, namely by two elements. Now we come to the groups of rational rank 1. If such a group is finitely generated, then it is simply isomorphic to $\mathbb{Z}$. How do we get places P on $K(x_1, x_2)$ with value group $\mathbb{Z}$? A place with value group $\mathbb{Z}$ is called a **discrete place**. The idea is to first construct a place on the subfield $K(x_1)$. We know from Example 2.1 that every place of $K(x_1)|K$ (if it is not trivial on $K(x_1)$) will have value group $\mathbb{Z}$ (cf. Theorem 7.2). Then we can try to extend this place from $K(x_1)$ to $K(x_1, x_2)$ in such a way that the value group doesn't change.

There are many different ways how this can be done. One possibility is to send the fraction x_1/x_2 to an element z which is transcendental over K. You may verify that there is a unique place which does this and extends the given homomorphism; it has value group $\mathbb{Z}$ and residue field $K(z)$. If, as in this case, a place P of $F|K$ has the property that $\mathrm{trdeg}\, FP|K = \mathrm{trdeg}\, F|K - 1$, then P is called a **prime divisor** and v_P is called a **divisorial valuation**. The places $Q, \overline{Q}$ were prime divisors, one of F, the other one of FQ.

But maybe we don't want a residue field which is transcendental over K? Maybe we even insist on having K as a residue field? Well, then we can employ another approach. Having already constructed our place P on $K(x_1)$ with residue field K, we can consider the completion of $(K(x_1), P)$. The **completion** of an arbitrary valued field (L, v) is the completion of L with respect to the topology induced by v. Both v and the associated place P extend canonically to this completion, whereby value group and residue field remain unchanged. Let us give a more concrete representation of this completion.

Let t be any transcendental element over K. We consider the unique place P of $F|K$ with $tP = 0$. The associated valuation is called the **t-adic valuation**, denoted by v_t. It is the unique valuation v on $K(t)$ (up to equivalence) which is trivial on K and satisfies that $vt > 0$. We want to write down the completion of $(K(t), v_t)$. We define the **field of formal Laurent series** (I prefer **power series field**) over K. It is denoted by $K((t))$ and consists of all formal sums of the form

$$\sum_{i=n}^{\infty} c_i t^i \quad \text{with } n \in \mathbb{Z} \text{ and } c_i \in K\;. \tag{2}$$

I suppose I don't have to tell you in which way the set $K((t))$ can be made into a field. But I tell you how v_t extends from $K(t)$ to $K((t))$: we set

$$v_t \sum_{i=n}^{\infty} c_i t^i \;=\; n \quad \text{if } c_n \neq 0\,. \tag{3}$$

One sees immediately that $v_t K((t)) = v_t K(t) = \mathbb{Z}$. For $b = \sum_{i=n}^{\infty} c_i t^i$ with $c_n \neq 0$, we have that $bv_t = \infty$ if $m < 0$, $bv_t = 0$ if $m > 0$, and $bv_t = c_0 \in K$ if $m = 0$. So we see that $K((t))v_t = K(t)v_t = K$. General valuation theory shows that $(K((t)), v_t)$ is indeed the completion of $(K(t), v_t)$.

It is also known that the transcendence degree of $K((t))|K(t)$ is uncountable. If K is countable, this follows directly from the fact that $K((t))$ then has the cardinality of the continuum. But it is quite easy to show that the transcendence degree is at least one, and already this suffices for our purposes here. The idea is to take any $y \in K((t))$, transcendental over $K(t)$; then $x_1 \mapsto t$, $x_2 \mapsto y$ induces an isomorphism $K(x_1, x_2) \to K(t, y)$. We take the restriction of v_t to $K(t, y)$ and pull it back to $K(x_1, x_2)$ through the isomorphism. What we obtain on $K(x_1, x_2)$ is a valuation v which extends our valuation v_P of $K(x_1)$. As is true for v_t, also this extension still has value group $\mathbb{Z} = v_P K(x_1)$ and residue field $K = K(x_1)P$. The desired place of $K(x_1, x_2)$ is simply the place associated with this valuation v.

We have now constructed essentially all places on $K(x_1, x_2)$ which extend the given homomorphism of $K[x_1, x_2]$ and have a finitely generated value group (up to certain variants, like exchanging the role of x_1 and x_2). The somewhat shocking experience to every "newcomer" is that on this rather simple rational function field, there are also places extending the given homomorphism and having a value group which is not finitely generated. For instance, the value group can be $\mathbb{Q}$. (In fact, it can be any subgroup of $\mathbb{Q}$.) We postpone the construction of such a place till Section 18. $\diamondsuit$

After we have become acquainted with places and how one obtains them from homomorphisms of coordinate rings, it is time to formulate our problem of local desingularization. Instead of looking for a desingularization "at a given point" of our variety V, we will look for a desingularization at a given place P of the function field $F|K$ (we forget about the variety from which F originates). Suppose we have any V such that $K(V) = F$, that is, we have generators $x_1, \ldots, x_\ell$ of $F|K$ and the coordinate ring $K[x_1, \ldots, x_\ell]$ of V. If we talk about the **center of P on V**, we always tacitly assume that $x_1, \ldots, x_\ell \in \mathcal{O}_P$, so that the restriction of P is a homomorphism on $K[x_1, \ldots, x_\ell]$. With this provision, the center of P on V is the point $(x_1 P, \ldots, x_\ell P)$ (or, if we so want, the induced homomorphism). We also say that P is **centered on V at** $(x_1 P, \ldots, x_\ell P)$. If V is a variety defined over K with function field F, then we call V a **model of $F|K$**. Our problem now reads:

(LU) *Take any function field $F|K$ and a place P of $F|K$. Does there exist a model of $F|K$ on which P is centered at a simple point?*

This was answered in the positive by Oscar Zariski in [Z] for the case of K having characteristic 0. Instead of "local desingularization", he called this principle **local uniformization**.

3 Local uniformization and the Implicit Function Theorem

Let's think about what we mean by "simple point". I don't really have to tell you, so let me pick the most valuation theoretic definition, which will show us our way on our excursion. It is the Jacobi criterion: Given our variety V defined by $f_1, \ldots, f_n \in K[X_1, \ldots, X_\ell]$ and having function field F, then a point $a = (a_1, \ldots, a_\ell)$ of V is called **simple** (or **smooth**) if $\operatorname{trdeg} F|K = \ell - r$, where r is the rank of the Jacobi matrix

$$\left(\frac{\partial f_i}{\partial X_j}(a) \right)_{\substack{1 \le i \le n \\ 1 \le j \le \ell}}$$

But wait — I have seen the Jacobi matrix long before I learned anything about algebraic geometry. Now I remember: I saw it in my first year calculus course in connection with the **Implicit Function**

Theorem. Let's have a closer look. First, let us assume that we don't have too many f_i's. Indeed, when looking for a local uniformization we will construct varieties V defined by $\ell - \operatorname{trdeg} F|K$ many polynomial relations, whence $n = \ell - \operatorname{trdeg} F|K$. In this situation, if a is a simple point, then n is equal to r and after a suitable renumbering we can assume that for $k := \ell - n = \operatorname{trdeg} F|K$, the submatrix

$$\left(\frac{\partial f_i}{\partial X_j}(a)\right)_{\substack{1\le i\le n\\ k+1\le j\le \ell}}$$

is invertible. Then, assuming that we are working over the reals, the Implicit Function Theorem tells us that for every $(a'_1,\ldots,a'_k)$ in a suitably small neighborhood of $(a_1,\ldots,a_k)$ there is a unique $(a'_{k+1},\ldots,a'_\ell)$ such that $(a'_1,\ldots,a'_\ell)$ is a point of V. Working in the reals, the existence is certainly interesting, but for us here, the main assertion is the uniqueness. Let's look at a very simple example.

Example 3.1 I leave it to you to draw the graph of the function $y^2 = x^3$ in $\mathbb{R}^2$. It only exists for $x \geq 0$. Starting from the origin, it has two branches, one positive, one negative. Now assume that we are sitting on one of these branches at a point (x, y), away from the origin. If somebody starts to manipulate x then we know exactly which way we have to run (depending on whether x increases or decreases). But if we are sitting at the origin and somebody increases x, then we have the freedom of choice into which of the two branches we want to run. So we see that everywhere but at the origin, y is an implicit function of x in a sufficiently small neighborhood. Indeed, with $f(x,y) = x^3 - y^2$, we have that $\frac{\partial f}{\partial x}(x,y) = 3x^2$. If $x \neq 0$, then this is non-zero, whence $r = 1$ while $\operatorname{trdeg} F|K = 1$ and $\ell = 2$, so for $x \neq 0$, (x,y) is a simple point. On the other hand, $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = 0$, so $(0,0)$ is singular. $\diamondsuit$

We have now seen the connection between simple points and the Implicit Function Theorem. "Wait!" you will interrupt me. "You have used the topology of $\mathbb{R}$. What if we don't have such a topology at hand? What then do you mean by 'neighborhood'?" Good question. So let's look for a topology. My luck, that the Implicit Function Theorem is also known in valuation theory. Indeed, we have already remarked in connection with the notion "completion of a valued field" that every valuation induces a topology. And since we have our place on F, we have the topology right at hand. That is why I said that the Jacobi criterion renders the most valuation theoretical definition of "simple".

But now this makes me think: haven't I seen the Jacobi matrix in connection with an even more famous valuation theoretical theorem, one of central importance in valuation theory? Indeed: it appears in the so-called "multidimensional version" of **Hensel's Lemma**. This brings us to our next sightseeing attraction on our excursion.

4 Hensel's Lemma

Hensel's Lemma is originally a lemma proved by Kurt Wilhelm Sebastian Hensel for the field of p-adic numbers $\mathbb{Q}_p$. It was then extended to all complete discrete valued fields and later to all maximal fields (see Corollary 5.17 below). A valued field (L, v) is called **maximal** (or **maximally complete**) if it has no proper extensions for which value group and residue field don't change. A complete field is not necessarily maximal, and if it is not of rank 1 (i.e., its value group is not archimedean), then it also does not necessarily satisfy Hensel's Lemma. However, complete discrete valued fields are maximal. In particular, $(K((t)), v_t)$ is maximal.

In modern valuation theory (and its model theory), Hensel's Lemma is rather understood to be a property of a valued field. The nice thing is that, in contrast to "complete" or "maximal", it is an elementary property (I will tell you in Section 12 what this means). We call a valued field **henselian** if it satisfies Hensel's Lemma. Here is one version of Hensel's Lemma for a valued field with valuation ring $\mathcal{O}_v$:

(Hensel's Lemma) *For every polynomial $f \in \mathcal{O}_v[X]$ the following holds: if $b \in \mathcal{O}_v$ satisfies*

$$vf(b) > 0 \quad \textit{and} \quad vf'(b) = 0\,, \tag{4}$$

then f admits a root $a \in \mathcal{O}_v$ such that $v(a-b) > 0$.

Here, f' denotes the derivative of f. Note that a more classical version of Hensel's Lemma talks only about monic polynomials.

For the multidimensional version, we introduce some notation. For polynomials $f_1, \ldots, f_n$ in variables $X_1, \ldots, X_n$, we write $f = (f_1, \ldots, f_n)$ and denote by J_f the Jacobian matrix $\left(\frac{\partial f_i}{\partial X_j}\right)_{i,j}$. For $a \in L^n$, $J_f(a) = \left(\frac{\partial f_i}{\partial X_j}(a)\right)_{i,j}$.

(Multidimensional Hensel's Lemma) *Let $f = (f_1, \ldots, f_n)$ be a system of polynomials in the variables $X = (X_1, \ldots, X_n)$ and with coefficients in $\mathcal{O}_v$. Assume that there exists $b = (b_1, \ldots, b_n) \in \mathcal{O}_v^n$ such that*

$$v f_i(b) \;>\; 0 \text{ for } 1 \le i \le n \quad \text{and} \quad v \det J_f(b) \;=\; 0\,. \tag{5}$$

Then there exists a unique $a = (a_1, \ldots, a_n) \in \mathcal{O}_v^n$ such that $f_i(a) = 0$ and $v(a_i - b_i) > 0$ for all i.

And here is the valuation theoretical Implicit Function Theorem:

(Implicit Function Theorem) *Take $f_1, \ldots, f_n \in L[X_1, \ldots, X_\ell]$ with $n < \ell$. Set*

$$\tilde{J} := \begin{pmatrix} \frac{\partial f_1}{\partial X_{\ell-n+1}} & \cdots & \frac{\partial f_1}{\partial X_\ell} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial X_{\ell-n+1}} & \cdots & \frac{\partial f_n}{\partial X_\ell} \end{pmatrix}. \tag{6}$$

Assume that $f_1, \ldots, f_n$ admit a common zero $a = (a_1, \ldots, a_\ell) \in L^\ell$ and that $\det \tilde{J}(a) \neq 0$. Then there is some $\alpha \in vL$ such that for all $(a'_1, \ldots, a'_{\ell-n}) \in L^{\ell-n}$ with $v(a_i - a'_i) > 2\alpha$, $1 \le i \le \ell - n$, there exists a unique $(a'_{\ell-n+1}, \ldots, a'_\ell) \in L^n$ such that $(a'_1, \ldots, a'_\ell)$ is a common zero of $f_1, \ldots, f_n$, and $v(a_i - a'_i) > \alpha$ for $\ell - n < i \le \ell$.

It is (not all too well) known that Hensel's Lemma holds in (L, v) if and only if the Multidimensional Hensel's Lemma holds in (L, v), and this in turn is true if and only if the Implicit Function Theorem holds in (L, v). For a proof, see [K2] or [PZ]. The latter paper is particularly interesting since it shows the connection between the Implicit Function Theorem in henselian fields and the "real" Implicit Function Theorem in $\mathbb{R}$.

There are many more versions of Hensel's Lemma which all are equivalent to the above (the classical Hensel's Lemma for monic polynomials, Krasner's Lemma, Newton's Lemma, Hensel–Rychlik,...). See [R2] or [K2] for a listing of them. It is indeed often very useful to have the different versions at hand. One particularly important is given in the following lemma:

Lemma 4.1 *A valued field (L, v) is henselian if and only if the extension of v to the algebraic closure $\tilde{L}$ of L is unique.*

Since any valuation of any field can always be extended to any extension field (cf. [V], §5), the following is an easy consequence of this lemma: *(L, v) is henselian if and only if v admits a unique extension to every algebraic extension field.* Also, we immediately obtain:

Corollary 4.2 *Every algebraic extension of a henselian field is again henselian.*

This is hard to prove if you use Hensel's Lemma instead of the unique extension property in the proof. On the other hand, the next lemma is hard to prove using the unique extension property, while it is immediate if you use Hensel's Lemma:

Lemma 4.3 *Take a henselian field (L, v) and a relatively algebraically closed subfield L' of L. Then also (L', v) is henselian.*

Let us take a short break to see how Hensel's Lemma can be applied. The following two examples will later have important applications.

Example 4.4 Assume that $\operatorname{char} L = p > 0$. A polynomial $f(X) = X^p - X - c$ with $c \in L$ is called an **Artin-Schreier polynomial** (over L). If ϑ is a root of f in some extension of L, then $\vartheta, \vartheta+1, \ldots, \vartheta+p-1$ are the distinct roots of f. Hence if f is irreducible over L, then $L(\vartheta)|L$ is a Galois extension of degree p. It is called an **Artin-Schreier extension**. Conversely, *every* Galois extension of degree p in characteristic p is generated by a root of a suitable Artin-Schreier polynomial, i.e., is an Artin-Schreier extension (see [K2] for a proof).

Let us prove our assertion about the roots of f. We note that in characteristic $p > 0$, the map $x \mapsto x^p$ is a ring homomorphism (the **Frobenius**). Therefore, the polynomial $\wp(X) := X^p - X$ is an **additive polynomial**. A polynomial g is called additive if $g(a+b) = g(a) + g(b)$ for all a, b (for details, cf. [L2], VIII, §11). Thus, if $i \in \mathbb{F}_p$, then $f(\vartheta + i) = \wp(\vartheta + i) - c = \wp(\vartheta) - c + \wp(i) = 0 + i^p - i = i - i = 0$ since $i^p = i$ for every $i \in \mathbb{F}_p$.

Now assume that (L, v) is henselian. Suppose first that $vc > 0$. Take $b = 0 \in \mathcal{O}_v$. Then $vf(b) = vc > 0$. On the other hand, $f'(X) = pX^{p-1} - 1 = -1$ since $p = 0$ in characteristic p. Hence, $vf'(b) = v(-1) = 0$. Therefore, Hensel's Lemma shows that f admits a root in L, which by our above observation about the roots of f means that f splits completely over L.

Suppose next that $vc = 0$. Then for $b \in \mathcal{O}_v$ we have that $v(b^p - b - c) > 0$ if and only if $0 = (b^p - b - c)v = (bv)^p - bv - cv$. Hence, $v(b^p - b - c) > 0$ if and only if bv is a root of the Artin-Schreier polynomial $X^p - X - cv \in Lv[X]$. If $cv = 0$, which is our previous case where $vc > 0$, then 0 is a root of $X^p - X - cv = X^p - X$ and we can choose $b = 0$. But in our present case, $cv \neq 0$, and everything depends on whether $X^p - X - cv$ has a root in Lv or not. If it has a root η in Lv, then we choose $b \in \mathcal{O}_v$ such that $bv = \eta$. We obtain that $(b^p - b - c)v = (bv)^p - bv - cv = \eta^p - \eta - cv = 0$, hence $vf(b) > 0$. Then by Hensel's Lemma, $X^p - X - c$ has a root $a \in \mathcal{O}_v$ with $v(a - b) > 0$, hence $av = bv = \eta$. Conversely, if $X^p - X - c$ has a root a in L, then one easily shows that $a \in \mathcal{O}_v$, and $0 = 0v = (a^p - a - c)v = (av)^p - av - cv$ yields that $X^p - X - cv$ has a root in Lv.

The only remaining case is that of $vc < 0$. In this case, $X^p - X - c \notin \mathcal{O}_v[X]$, so Hensel's Lemma doesn't give us any immediate information about whether f has a root in L or not. $\diamondsuit$

Example 4.5 Take a field K of characteristic $p > 0$. In the field $(K((t)), v_t)$ (which is henselian, cf. Corollary 5.17 below), the Artin-Schreier polynomial

$$X^p - X - t \tag{7}$$

has the root

$$a = \sum_{i=0}^{\infty} (-t)^{p^i} \tag{8}$$

since

$$a^p - a = \sum_{i=0}^{\infty} (-t)^{p^{i+1}} - \sum_{i=0}^{\infty} (-t)^{p^i} = \sum_{i=1}^{\infty} (-t)^{p^i} - \sum_{i=0}^{\infty} (-t)^{p^i} = t\,.$$

$\diamondsuit$

Take any polynomial $f \in \mathcal{O}_v[X]$. By fv we mean the **reduction of the polynomial f modulo v**, that is, the polynomial we obtain from f by replacing every coefficient c_i of f by its residue $c_i v$. As the residue map is a homomorphism on $\mathcal{O}_v$, we have that $f'v = (fv)'$. Suppose there is some $b \in L$ such that $vf(b) > 0$ and $vf'(b) = 0$. This is equivalent to $f(b)v = 0$ and $f'(b)v \neq 0$. But $f(b)v = fv(bv)$ and $f'(b)v = (fv)'(bv)$, so the latter is equivalent to bv being a simple root of fv. Conversely, if fv has a simple root ζ, find some b such that $bv = \zeta$, and you will have that $vf(b) > 0$ and $vf'(b) = 0$. Hence, Hensel's Lemma is also equivalent to the following version:

(Hensel's Lemma, Simple Root Version) *For every polynomial $f \in \mathcal{O}_v[X]$ the following holds: if fv has a simple root ζ in Lv, then f admits a root $a \in \mathcal{O}_v$ such that $av = \zeta$.*

Example 4.6 Take a henselian valued field (L,v) and a relatively algebraically closed subfield L' of L. Assume there is an element ζ of the residue field Lv which is algebraic over $L'v$, and denote its minimal polynomial over $L'v$ by $h\in L'v[X]$. Find a monic polynomial $f\in(\mathcal{O}_v\cap L')[X]$ such that $fv=h$.

If ζ is separable over $L'v$, then ζ is a simple root of h. As $\zeta\in Lv$ and (L,v) is henselian by assumption, the Simple Root Version of Hensel's Lemma tells us then that there is some $a\in L$ such that $h(a)=0$ and $av=\zeta$. But as a is algebraic over L' we have that $a\in L'$, so that $\zeta=av\in L'v$. If on the other hand ζ is not separable over $L'v$, then it is quite possible that $\eta\notin L'v$. But at least we have proved:

Lemma 4.7 *If (L,v) is henselian and L' is relatively algebraically closed in L, then $L'v$ is* **relatively separable-algebraically closed** *in Lv, i.e., every element of Lv already belongs to $L'v$ if it is separable-algebraic over $L'v$.*

Something similar can be shown for the value groups, provided that $Lv=L'v$. Pick an element $\delta\in vL$ such that for some $n>0$, $n\delta\in vL'$. Choose some $d\in L$ such that $vd=\delta$. Hence, $vd^n=nvd\in vL'$ and we can choose some $d'\in L'$ such that $vd'd^n=0$. Assuming that $Lv=L'v$, we can also pick some $d''\in L'$ such that $(d'd''d^n)v=1$.

An element u with $uv=1$ is called a **1-unit**. We consider the polynomial X^n-u. Its reduction modulo v is simply the polynomial X^n-1. Obviously, 1 is a root of that polynomial, but is it a simple root? The answer is: 1 is a simple root of X^n-1 if and only if the characteristic of Lv does not divide n. Hence in that case, Hensel's Lemma shows that there is a root $a\in L$ of the polynomial X^n-u such that $av=1$. This proves:

Lemma 4.8 *Take a 1-unit u in the henselian field (L,v) and $n\in\mathbb{N}$ such that the characteristic of Lv does not divide n. Then there is a unique 1-unit $a\in L$ such that $a^n=u$.*

In our present case, this provides an element $a\in L$ such that $a^n=d'd''d^n$. We find that $(a/d)^n=d'd''\in L'$. Since $a/d\in L$ and L' is relatively algebraically closed in L, this implies that $a/d\in L'$. On the other hand, $v(a/d)^n=vd'd''=vd'=n\alpha$ so that $v(a/d)=\alpha$. This proves that $\alpha\in vL'$. We have proved:

Lemma 4.9 *If (L,v) is henselian and L' is relatively algebraically closed in L and $Lv=L'v$, then the torsion subgroup of vL/vL' is trivial if* $\mathrm{char}\,Lv=0$*, and it is a p-group if* $\mathrm{char}\,Lv=p>0$.

It can be shown that the assertion is in general not true without the assumption that $Lv=L'v$. $\diamondsuit$

Let's return to our variety V which is defined by $f_1,\ldots,f_n\in K[X_1,\ldots,X_\ell]$ and has coordinate ring $K[x_1,\ldots,x_\ell]$. We have seen that a point $a=(a_1,\ldots,a_\ell)$ of V is simple if and only if after a suitable renumbering, the submatrix

$$\tilde{J} = \left(\frac{\partial f_i}{\partial X_j}(a)\right)_{\substack{1\leq i\leq n\\ k+1\leq j\leq \ell}} \tag{9}$$

of $J_f(a)$ is invertible, where $k:=\ell-n=\mathrm{trdeg}\,F|K$. That means that $f_1,\ldots,f_n$ and a satisfy the assumptions of the Implicit Function Theorem.

Since we are interested in the question whether the center of P on V is simple, we have to look at $a=(x_1P,\ldots,x_\ell P)$. As P is a homomorphism on $\mathcal{O}_P$ and leaves the coefficients of the f_i invariant, we see that

$$\left(\frac{\partial f_i}{\partial X_j}(x_1P,\ldots,x_\ell P)\right) = \left(\frac{\partial f_i}{\partial X_j}(x_1,\ldots,x_\ell)\right)P\,. \tag{10}$$

We have omitted the indices since this holds for *every* submatrix of J_f. Again because P is a homomorphism, it commutes with taking determinants (since this operation remains inside the ring $\mathcal{O}_P$). Hence,

$$\det\tilde{J}(x_1P,\ldots,x_\ell P) = (\det\tilde{J}(x_1,\ldots,x_\ell))P\,. \tag{11}$$

Therefore, $\det\tilde{J}(x_1P,\ldots,x_\ell P)\neq 0$ is equivalent to $v\det\tilde{J}(x_1,\ldots,x_\ell)=0$. This condition also appears in the Multidimensional Hensel's Lemma, but with J_f in the place of $\tilde{J}$. So we are led to the question: what is the connection? It is obvious that we have some variables too many for the case of the Multidimensional

Hensel's Lemma. But they are exactly trdeg $F|K$ too many, and on the other hand, at least the basic Hensel's Lemma obviously talks about algebraic elements (we will see that this is also true for the Multidimensional Hensel's Lemma). So why don't we just take $x_1,\ldots,x_k$ as a transcendence basis of $F|K$ and view $f_1,\ldots,f_n$ as polynomial relations defining the remaining $x_{k+1},\ldots,x_\ell$, which are algebraic over $K(x_1,\ldots,x_k)$? But then, we should write every f_i as a polynomial $\tilde{f}_i$ in the variables $X_{k+1},\ldots,X_\ell$ with coefficients in $K(x_1,\ldots,x_k)$, or actually, in $K[x_1,\ldots,x_k]$. Then we have that

$$f_i(x_1,\ldots,x_\ell)P \;=\; f_i(x_1P,\ldots,x_\ell P) \;=\; \tilde{f}_iP(x_{k+1}P,\ldots,x_\ell P) \;=\; \tilde{f}_i(x_{k+1},\ldots,x_\ell)P\;.$$

With $\tilde{f} := (\tilde{f}_1,\ldots,\tilde{f}_n)$ and $\tilde{f}P := (\tilde{f}_1P,\ldots,\tilde{f}_nP)$, it follows that

$$\det\tilde{J}(x_1P,\ldots,x_\ell P) \;=\; \det J_{\tilde{f}P}(x_{k+1}P,\ldots,x_\ell P) \;=\; (\det J_{\tilde{f}}(x_{k+1},\ldots,x_\ell))P\;. \tag{12}$$

Hence, $\det\tilde{J}(x_1P,\ldots,x_\ell P)\neq 0$ is equivalent to $v\det J_{\tilde{f}}(x_{k+1},\ldots,x_\ell)=0$, which means that the polynomials $\tilde{f}_1,\ldots,\tilde{f}_n$ and the elements $x_{k+1},\ldots,x_\ell$ satisfy the assumption (5) of the Multidimensional Hensel's Lemma. Indeed, as $\tilde{f}_i(x_{k+1},\ldots,x_\ell)=0$, we have that $v\tilde{f}_i(x_{k+1},\ldots,x_\ell)=\infty>0$. So we see:

To find a model of $F|K$ on which P is centered at a simple point means to find generators $x_1,\ldots,x_\ell\in\mathcal{O}_P$ such that $x_1,\ldots,x_k$ form a transcendence basis of $F|K$ and $x_{k+1},\ldots,x_\ell$ together with the polynomials which define them over $K[x_1,\ldots,x_k]$ satisfy the assumption of the Multidimensional Hensel's Lemma.

(Since f_i and $\tilde{f}_i$ are the same polynomial, just written in two different ways, we will later use "f_i" instead of "$\tilde{f}_i$"; cf. the definition of relative uniformization in Section 15. There, we will also prefer "$(\det J_{\tilde{f}}(x_{k+1},\ldots,x_\ell))P\neq 0$" over "$v\det J_{\tilde{f}}(x_{k+1},\ldots,x_\ell)=0$".)

What we have derived now is still quite vague, and before we can make more out of it, I'm sorry, you have to go to a course again.

5 A crash course in ramification theory

Throughout, we assume that $L|K$ is an algebraic extension, not necessarily finite, and that v is a *non-trivial* valuation on K. If w is a valuation on L which extends v, then there is a natural embedding of the value group vK of v in the value group wL of w. Similarly, there is a natural embedding of the residue field Kv of v in the residue field Lw of w. If both embeddings are onto (which we just express by writing $vK=wL$ and $Kv=Lw$), then the extension $(L,w)|(K,v)$ is called **immediate**. WARNING: It may happen that $vK\simeq wL$ or $Kv\simeq Lw$ although the corresponding embedding is not onto and therefore, the extension is not immediate. For example, every finite extension of the p-adics $(\mathbb{Q}_p,v_p)$ will again have a value group isomorphic to $\mathbb{Z}$, but v_pp may not be anymore the smallest positive element in this value group.

We choose an arbitrary extension of v to the algebraic closure $\tilde{K}$ of K. Then for every $\sigma\in\mathrm{Aut}\,(\tilde{K}|K)$, the map

$$\tilde{v}\sigma=\tilde{v}\circ\sigma:\; L\ni a\;\mapsto\;\tilde{v}(\sigma a)\in\tilde{v}\tilde{K} \tag{13}$$

is a valuation of L which extends v.

Theorem 5.1 *The set of all extensions of v from K to L is*

$$\{\tilde{v}\sigma\mid\sigma\text{ an embedding of }L\text{ in }\tilde{K}\text{ over }K\}\,.$$

(We say that "all extensions of v from K to L are **conjugate**".)

Corollary 5.2 *If $L|K$ is finite, then the number g of distinct extensions of v from K to L is smaller or equal to the extension degree $[L:K]$. More precisely, g is smaller or equal to the degree of the maximal separable subextension of $L|K$. In particular, if $L|K$ is purely inseparable, then v has a unique extension from K to L.*

Theorem 5.3 *Assume that* $n := [L : K]$ *is finite, and denote the extensions of* v *from* K *to* L *by* $v_1, \ldots, v_{\mathrm{g}}$. *Then for every* $i \in \{1, \ldots, \mathrm{g}\}$, *the* **ramification index** $\mathrm{e}_i = (v_i L : vK)$ *and the* **inertia degree** $\mathrm{f}_i = [Lv_i : Kv]$ *are finite, and we have the* **fundamental inequality**

$$n \geq \sum_{i=1}^{\mathrm{g}} \mathrm{e}_i \mathrm{f}_i \ . \tag{14}$$

From now on, let us assume that $L|K$ is normal. Hence, the set of all extensions of v from K to L is given by $\{\tilde{v}\sigma \mid \sigma \in \mathrm{Aut}\,(L|K)\}$. For simplicity, we denote the restriction of $\tilde{v}$ to L again by v. The valuation ring of v on L will be denoted by $\mathcal{O}_L$. We define distinguished subgroups of $G := \mathrm{Aut}\,(L|K)$. The subgroup

$$G^d := G^d(L|K, v) := \{\sigma \in G \mid \forall x \in \mathcal{O}_L : \ v\sigma x \geq 0\} \tag{15}$$

is called the **decomposition group of** $(L|K, v)$. It is easy to show that σ sends $\mathcal{O}_L$ into itself if and only if the valuations v and $v\sigma$ agree on L. Thus,

$$G^d = \{\sigma \in G \mid v\sigma = v \text{ on } L\} \ . \tag{16}$$

Further, the **inertia group** is defined to be

$$G^i := G^i(L|K, v) := \{\sigma \in \mathrm{Aut}\,(L|K) \mid \forall x \in \mathcal{O}_L : v(\sigma x - x) > 0\} \ , \tag{17}$$

and the **ramification group** is

$$G^r := G^r(L|K, v) := \{\sigma \in \mathrm{Aut}\,(L|K) \mid \forall x \in \mathcal{O}_L : v(\sigma x - x) > vx\} \ . \tag{18}$$

Let S denote the maximal separable extension of K in L (we call it the **separable closure of** K **in** L). The fixed fields of G^d, G^i and G^r in S are called **decomposition field**, **inertia field** and **ramification field of** $(L|K, v)$. For simplicity, let us abbreviate them by Z, T and V. (These letters refer to the german words "Zerlegungskörper", "Trägheitskörper" and "Verzweigungskörper".)

Remark 5.4 WARNING: In contrast to the classical definition used by other authors, we take decomposition field, inertia field and ramification field to be the fixed fields of the respective groups *in the maximal separable subextension.* The reason for this will become clear in Section 8.

By our definition, V, T and Z are separable-algebraic extensions of K, and S|V, S|T, S|Z are (not necessarily finite) Galois extensions. Further,

$$1 \subset G^r \subset G^i \subset G^d \subset G \ \text{ and thus, } \ \mathrm{S} \supset \mathrm{V} \supset \mathrm{T} \supset \mathrm{Z} \supset K \ . \tag{19}$$

(For the inclusion $G^i \subset G^d$ note that $vx \geq 0$ and $v(\sigma x - x) > 0$ implies that $v\sigma x \geq 0$.)

Theorem 5.5 G^i *and* G^r *are normal subgroups of* G^d, *and* G^r *is a normal subgroup of* G^i. *Therefore,* T|Z, V|Z *and* V|T *are (not necessarily finite) Galois extensions.*

First, we consider the decomposition field Z. In some sense, it represents all extensions of v from K to L.

Theorem 5.6 *a)* $v\sigma = v\tau$ *on* L *if and only if* $\sigma\tau^{-1}$ *is trivial on* Z.
b) $v\sigma = v$ *on* Z *if and only if* σ *is trivial on* Z.
c) The extension of v *from* Z *to* L *is unique.*
d) The extension $(\mathrm{Z}|K, v)$ *is immediate.*

WARNING: It is in general not true that $v\sigma \neq v\tau$ holds already on Z if it holds on L.
a) and b) are easy consequences of the definition of G^d. c) follows from b) by Theorem 5.1. For d), there is a simple proof using a trick which is mentioned in the paper [AX] by James Ax.

Now we turn to the inertia field T. Let $\mathcal{M}_L$ denote the valuation ideal of v on L (the unique maximal ideal of $\mathcal{O}_L$). For every $\sigma \in G^d(L|K,v)$ we have that $\sigma\mathcal{O}_L = \mathcal{O}_L$, and it follows that $\sigma\mathcal{M}_L = \mathcal{M}_L$. Hence, every such σ induces an automorphism $\overline{\sigma}$ of $\mathcal{O}_L/\mathcal{M}_L = Lv$ which satisfies $\overline{\sigma}\,\overline{a} = \overline{\sigma a}$. Since σ fixes K, it follows that $\overline{\sigma}$ fixes Kv.

Lemma 5.7 *Since $L|K$ is normal, the same is true for $Lv|Kv$. Moreover, the map*

$$G^d(L|K,v) \ni \sigma \;\mapsto\; \overline{\sigma} \in \mathrm{Aut}\,(Lv|Kv) \tag{20}$$

is a group homomorphism.

Theorem 5.8 *a) The homomorphism (20) is onto and induces an isomorphism*

$$\mathrm{Aut}\,(\mathrm{T}|\mathrm{Z}) \;=\; G^d/G^i \;\simeq\; \mathrm{Aut}\,(\mathrm{T}v|\mathrm{Z}v)\;. \tag{21}$$

b) For every finite subextension $F|\mathrm{Z}$ of $\mathrm{T}|\mathrm{Z}$,

$$[F:\mathrm{Z}] \;=\; [Fv:\mathrm{Z}v]\;. \tag{22}$$

c) We have that $v\mathrm{T} = v\mathrm{Z} = vK$. Further, $\mathrm{T}v$ is the separable closure of Kv in Lv, and therefore,

$$\mathrm{Aut}\,(\mathrm{T}v|\mathrm{Z}v) \;=\; \mathrm{Aut}\,(Lv|Kv)\;. \tag{23}$$

If $F|\mathrm{Z}$ is normal, then b) is an easy consequence of a). From this, the general assertion of b) follows by passing from F to the normal hull of the extension $F|\mathrm{Z}$ and then using the multiplicativity of the extension degree. c) follows from b) by use of the fundamental inequality.

We set $p := \mathrm{char}\,Kv$ if this is positive, and $p := 1$ if $\mathrm{char}\,Kv = 0$. Given any extension $\Delta \subset \Delta'$ of abelian groups, the **p'-divisible closure of Δ in Δ'** is defined to be the subgroup $\{\alpha \in \Delta' \mid \exists n \in \mathbb{N} : (p,n) = 1 \wedge n\alpha \in \Delta\}$ of all elements in Δ' whose order modulo Δ is prime to p.

Theorem 5.9 *a) There is an isomorphism*

$$\mathrm{Aut}\,(\mathrm{V}|\mathrm{T}) \;=\; G^i/G^r \;\simeq\; \mathrm{Hom}\left(v\mathrm{V}/v\mathrm{T}\,,\,(\mathrm{T}v)^\times\right)\;, \tag{24}$$

where the character group on the right hand side is the full character group of the abelian group $v\mathrm{V}/v\mathrm{T}$. Since this group is abelian, $\mathrm{V}|\mathrm{T}$ is an abelian Galois extension.
b) For every finite subextension $F|\mathrm{T}$ of $\mathrm{V}|\mathrm{T}$,

$$[F:\mathrm{T}] \;=\; (vF:v\mathrm{T})\;. \tag{25}$$

c) $\mathrm{V}v = \mathrm{T}v$, and $v\mathrm{V}$ is the p'-divisible closure of vK in vL.

b) follows from a) since for a finite extension $F|\mathrm{T}$, the group $vF/v\mathrm{T}$ is finite and therefore, there exists an isomorphism of $vF/v\mathrm{T}$ onto its full character group. The equality $\mathrm{V}v = \mathrm{T}v$ follows from b) by the fundamental inequality. The second assertion of c) follows from the next theorem and the fact that the order of all elements in $(\mathrm{T}v)^\times$ and thus also of all elements in $\mathrm{Hom}\,(v\mathrm{V}/v\mathrm{T}\,,\,(\mathrm{T}v)^\times)$ is prime to p.

Theorem 5.10 *The ramification group G^r is a p-group and therefore, $\mathrm{S}|\mathrm{V}$ is a p-extension. Further, $vL/v\mathrm{V}$ is a p-group, and the residue field extension $Lv|\mathrm{V}v$ is purely inseparable. If $\mathrm{char}\,Kv = 0$, then $\mathrm{V} = \mathrm{S} = L$.*

Remark 5.11 Every p-extension is a tower of Galois extensions of degree p. In characteristic p, all of them are Artin–Schreier–extensions, as we have mentioned in Example 4.4.

From the last theorem it follows that there is a canonical isomorphism

$$\mathrm{Hom}\left(v\mathrm{V}/v\mathrm{T}\,,\,(\mathrm{T}v)^\times\right) \;\simeq\; \mathrm{Hom}\left(vL/vK\,,\,(Lv)^\times\right)\;. \tag{26}$$

We summarize our main results in the following table:

Galois group	**field**		**value group**	**residue field**
	L		vL	Lv
	purely inseparable			
1	S	maximal separable subextension	division by p	purely inseparable
	Galois p-extension			
$G^r(L\mid K,v)$	V	ramification field	$(vL\mid vK)^{p'}$	$(Lv\mid Kv)^{\mathrm{sep}}$
Char	abelian Galois p'-extension		division prime to p	$\Vert$
$G^i(L\mid K,v)$	T	inertia field	vK	$(Lv\mid Kv)^{\mathrm{sep}}$
Aut $(Lv\mid Kv)$	Galois		$\Vert$	Galois
$G^d(L\mid K,v)$	Z	decomposition field	vK	Kv
	immediate		$\Vert$	$\Vert$
Aut $(L\mid K)$	K		vK	Kv

where $(vL|vK)^{p'}$ denotes the p'-divisible closure of vK in vL, $(Lv|Kv)^{\mathrm{sep}}$ denotes the separable closure of Kv in Lv, and Char denotes the character group (26).

We state two more useful theorems from ramification theory. If we have two subfields K, L of a field M (in our case, we will have the situation that $L \subset \tilde{K}$) then $K.L$ will denote the smallest subfield of M which contains both K and L; it is called the **field compositum of** K **and** L.

Theorem 5.12 *If $K \subseteq K' \subseteq L$, then the decomposition field of the normal extension $(L|K',v)$ is $\mathrm{Z}.K'$, its inertia field is $\mathrm{T}.K'$, and its ramification field is $\mathrm{V}.K'$.*

Theorem 5.13 *If $E|K$ is a normal subextension of $L|K$, then the decomposition field of $(E|K,v)$ is $\mathrm{Z}\cap E$, its inertia field is $\mathrm{T}\cap E$, and its ramification field is $\mathrm{V}\cap E$.*

If we take for $L|K$ the normal extension $\tilde{K}|K$, then we speak of **absolute ramification theory**. The fixed fields K^d, K^i and K^r of $G^d(\tilde{K}|K,v)$, $G^i(\tilde{K}|K,v)$ and $G^r(\tilde{K}|K,v)$ in the separable-algebraic closure K^{sep} of K are called **absolute decomposition field**, **absolute inertia field** and **absolute ramification field of** (K,v) (with respect to the given extension of v from K to its algebraic closure $\tilde{K}$). If $\mathrm{char}\, Kv = 0$, then by Theorem 5.10, $K^r = K^{\mathrm{sep}} = \tilde{K}$.

Lemma 5.14 *Fix an extension of v from K to $\tilde{K}$. Then the absolute inertia field of (K,v) is the unique maximal extension of (K,v) within the absolute ramification field having the same value group as K.*

Proof: Let $(L|K,v)$ be any extension within the absolute ramification field s.t. $vL = vK$. Then $vL^i = vL = vK = vK^i$. By Theorem 5.12, $L^i = L.K^i$. Further, $L \subseteq K^r$ yields that $L.K^i \subseteq K^r$. If the subextension $L^i|K^i$ of $K^r|K^i$ were proper, it contained a proper finite subextension $L_1|K^i$, and by part b) of Theorem 5.9 we had that $vK^i \subsetneqq vL_1 \subseteq vL^i$. As this contradicts the fact that $vL^i = vK^i$, we find that $L^i = K^i$, that is, $L \subseteq K^i$. □

From part c) of Theorem 5.6 we infer that the extension of v from K^d to $\tilde{K}$ is unique. On the other hand, if L is any extension field of K within K^d, then by Theorem 5.12, $K^d = L^d$. Thus, if $L \neq K^d$, then it follows from part b) of Theorem 5.6 that there are at least two distinct extensions of v from L to K^d and thus also to $\tilde{K} = \tilde{L}$. This proves that the absolute decomposition field K^d is a minimal algebraic extension of K admitting a unique extension of v to its algebraic closure. So it is the minimal algebraic extension of K which is henselian (cf. Lemma 4.1). We call it the **henselization of** (K,v) **in** $(\tilde{K},v)$. Instead of K^d, we also write K^h. A valued field is henselian if and only if it is equal to its henselization. Henselizations have the following universal property:

Theorem 5.15 *Let (K,v) be an arbitrary valued field and (L,v) any henselian extension field of (K,v). Then there is a unique embedding of (K^h,v) in (L,v) over K.*

From the definition of the henselization as a decomposition field, together with part d) of Theorem 5.6, we obtain another very important property of the henselization:

Theorem 5.16 *The henselization (K^h,v) is an immediate extension of (K,v).*

Corollary 5.17 *Every maximal valued field is henselian. In particular, $(K((t)),v_t)$ is henselian.*

Finally, we employ Theorem 5.12 to obtain:

Theorem 5.18 *If $K'|K$ is an algebraic extension, then the henselization of K' is $K'.K^h$.*

6 A valuation theoretical interpretation of local uniformization

We return to where we stopped before entering the crash course in ramification theory. The first question is: what does it mean that $x_{k+1},\ldots,x_\ell$ together with the polynomials which define them over $K[x_1,\ldots,x_k]$ satisfy the assumption of the Multidimensional Hensel's Lemma? First of all, general valuation theory tells us that a rational function field $K(x_1,\ldots,x_k)$ is much too small to be henselian (unless the valuation is trivial). But we could pass to the henselization of $(K(x_1,\ldots,x_k),P)$. So does it mean that $x_{k+1},\ldots,x_\ell$ lie in this henselization? If we look closely, there is something fishy in the way we have satisfied the assumption of the Multidimensional Hensel's Lemma. Instead of talking about a so-called "approximative root" $b = (b_1,\ldots,b_n)$ which lies in the henselian field we wish to work in, we have talked already about the actual root, and we do not know where it lies. Let us modify our Example 3.1 a bit to see that it does not always lie in the henselization of $(K(x_1,\ldots,x_k),v)$.

Example 6.1 Let us consider the function field $\mathbb{Q}(x,y)$ where $y^2 = x^3$. Take the place given by $xP = 2$, $yP = 2\sqrt{2}$. The minimal polynomial of y over $\mathbb{Q}(x)$ is $f(Y) = Y^2 - x^3$. As $f(y) = 0$, we have that $v_P f(y) = \infty > 0$. As $f'(Y) = 2Y$, we have that $v_P f'(y) = v_P 2y = 0$ (since $2yP = 4\sqrt{2} \neq 0$). Hence, f

and y satisfy the assumption (4) of Hensel's Lemma. But y does not lie in the henselization of $(\mathbb{Q}(x), P)$. Indeed, P on $\mathbb{Q}(x)$ is just the place coming from the evaluation homomorphism given by $x \mapsto 2$; hence, $\mathbb{Q}(x)^h P = \mathbb{Q}(x)P = \mathbb{Q}$. But $\mathbb{Q}(x,y)P \neq \mathbb{Q}$ since $yP = 2\sqrt{2} \notin \mathbb{Q}$. $\diamondsuit$

So we see that extensions of the residue field can play a role. We could try to suppress them by requiring that K be algebraically closed. This works for those P for which $FP|K$ is algebraic, but if this is not the case, then we have no chance to avoid them. At least, we can show that they are the only reason why $x_{k+1}, \ldots, x_\ell$ may not lie in the henselization of $(K(x_1, \ldots, x_k), P)$.

Theorem 6.2 *If $x_{k+1}, \ldots, x_\ell$ together with the polynomials f_i which define them over $K[x_1, \ldots, x_k]$ satisfy the assumption (5) of the Multidimensional Hensel's Lemma, then the elements $x_{k+1}, \ldots, x_\ell$ lie in the absolute inertia field of $(K(x_1, \ldots, x_k), P)$, and the extension $FP|K(x_1P, \ldots, x_kP)$ is separable-algebraic. If in addition P is a rational place, then $x_{k+1}, \ldots, x_\ell$ lie in the henselization of $(K(x_1, \ldots, x_k), P)$.*

Proof: Denote by (L, P) the absolute inertia field of $(K(x_1, \ldots, x_k), P)$. First,

$$\det J_{\tilde{f}P}(x_{k+1}P, \ldots, x_\ell P) \;=\; \det J_{\tilde{f}}(x_{k+1}, \ldots, x_\ell)P \;\neq\; 0 \tag{27}$$

and the fact that the f_iP are polynomials over $K(x_1P, \ldots, x_kP)$ imply that $x_{k+1}P, \ldots, x_\ell P$ are separable algebraic over $K(x_1P, \ldots, x_kP)$ (cf. [L2], Chapter X, §7, Proposition 8). On the other hand, LP is the separable-algebraic closure of $K(x_1, \ldots, x_k)P$. Therefore, there are elements $b_1, \ldots, b_n$ in L such that $b_iP = x_{k+i}P$. Since (L, P) is henselian, the Multidimensional Hensel's Lemma now shows the existence of a common root $(b'_1, \ldots, b'_n) \in L^n$ of the f_i such that $b'_iP = b_iP = x_{k+i}P$. But the uniqueness assertion of the Multidimensional Hensel's Lemma also holds in the algebraic closure $\tilde{L}$ of L (which is also henselian). So we find that $(b'_1, \ldots, b'_n) = (x_{k+1}, \ldots, x_\ell)$. Hence, $x_{k+1}, \ldots, x_\ell$ are elements of L.

If we have in addition that P is a rational place, then $x_{k+1}P, \ldots, x_\ell P \in K$. In this case, we can choose $b_1, \ldots, b_n$ and $b'_1, \ldots, b'_n$ already in the henselization of $(K(x_1, \ldots, x_k), P)$, which implies that also $x_{k+1}, \ldots, x_\ell$ lie in this henselization. $\square$

Since the absolute inertia field is a separable-algebraic extension and every rational function field is separable, we obtain:

Corollary 6.3 *If the place P of $F|K$ admits local uniformization, then $F|K$ is separable.*

We see that we are slowly entering the **structure theory of valued function fields**, that is, the algebraic theory of function fields $F|K$ equipped with a valuation (which may or may not be trivial on K). Later, we will see some main results from this theory (Theorems 14.1 and 17.5).

Given a place P of F, not necessarily trivial on K, we will say that $(F|K, P)$ is **inertially generated** if there is a transcendence basis T of $F|K$ such that (F, P) lies in the absolute inertia field of $(K(T), P)$. Similarly, $(F|K, P)$ is **henselian generated** if there is a transcendence basis T of $F|K$ such that (F, P) lies in henselization of $(K(T), P)$. Now we see a valuation theoretical interpretation of local uniformization:

Theorem 6.4 *If the place P of $F|K$ admits local uniformization, then $(F|K, P)$ is inertially generated. If in addition $FP = K$, then $(F|K, P)$ is henselian generated.*

So if local uniformization holds in arbitrary characteristic for every $F|K$ with perfect K, then for every place P of $F|K$, the valued function field $(F|K, P)$ is inertially generated. In the context of valuation theory, at least to me, this is a quite surprising assertion. Here is our first open problem:

Open Problem 1: Is the converse also true, i.e., if $(F|K, P)$ is inertially generated, does it then admit local uniformization?

I will discuss this question in Section 15. A partial answer to this question is given in the papers [K5] and [K6]. What we see is that in order to get local uniformization, one has to avoid ramification. Indeed, ramification is the valuation theoretical symptom of branching, the violation of the Implicit Function Theorem at a point of the variety. Let us look again at our simple Example 3.1:

Example 6.5 Consider the function field $\mathbb{R}(x,y)$ where $y^2 = x^3$. Take the place given by $xP = 0 = yP$. As P on $K(x)$ originates from the evaluation homomorphism given by $x \mapsto 0$, we have that $v_P K(x) = \mathbb{Z}$, with $v_P x = 1$ the smallest positive element in the value group. Now compute $v_P y$. We have that $y^2 = x^3$, whence $2vy = vy^2 = vx^3 = 3$. It follows that $vy = 3/2 \notin \mathbb{Z}$, that is, the extension $(K(x,y)|K(x),P)$ is **ramified**, or in other words, $(K(x,y),P)$ does not lie in the absolute inertia field of $(K(x),P)$. We see that we have ramification at the singular point $(0,0)$. As an exercise, you may check that $(K(x,y),Q)$ lies in the absolute inertia field of $(K(x),Q)$ whenever $xQ \neq 0$. $\diamondsuit$

7 Inertial generation and Abhyankar places

We may now ask ourselves: How could we show that for a given place P of $F|K$, the valued function field $(F|K,P)$ is inertially generated?

Example 7.1 Let us start with the most simple case, where $\mathrm{trdeg}\, F|K = 1$. Assuming that P is not trivial on F (if it is trivial, then local uniformization is trivial if $F|K$ is separable), we pick some $z \in F$ such that $zP = 0$. As we have seen in Example 2.1, $v_P K(z) = \mathbb{Z}$. Since $z \notin K$ and $\mathrm{trdeg}\, F|K = 1$, we know that $F|K(z)$ is algebraic; since $F|K$ is finitely generated, it follows that $F|K$ is finite. From Theorem 5.3 we infer that the ramification index $(v_P F : v_P K(z))$ is finite. Therefore, $v_P F$ is again isomorphic to $\mathbb{Z}$ and we can pick some $x \in F$ such that $x \in \mathcal{O}_P$ and $v_P F = \mathbb{Z} v_P x$.

We have achieved that $v_P F = v_P K(x)$. If the residue characteristic $\mathrm{char}\, FP = \mathrm{char}\, K$ is 0, then we know from Lemma 5.14 that the absolute inertia field $K(x)^i$ is the unique maximal extension still having the same value group as $K(x)$. In this case, we find that F must lie in this absolute inertia field, and we have proved that $(F|K,P)$ is inertially generated. But we are lost, it seems, if the characteristic is $p > 0$, since in this case, the absolute inertia field is not necessarily the maximal algebraic extension of $K(x)$ having the same value group. To solve this case, we yet have to learn some additional tools. $\diamondsuit$

In this example, the fact that $v_P F$ was finitely generated played a crucial role. As we have shown, this is always the case if $\mathrm{trdeg}\, F|K = 1$. But in general, we can't expect this to hold. We will give counterexamples in Section 18. But prior to the negative, we want to start with the positive, i.e., criteria for the value group to be finitely generated.

The following theorem has turned out in the last years to be amazingly universal in many different applications of valuation theory. It plays an important role in algebraic geometry as well as in the model theory of valued fields, in real algebraic geometry, or in the structure theory of exponential Hardy fields (= nonarchimedean ordered fields which encode the asymptotic behaviour of real-valued functions including exp and log, cf. [KK]). For more details and the easy proof of the theorem, see [V], Theorem 5.5, or [B], Chapter VI, §10.3, Theorem 1, or [K2].

Theorem 7.2 *Let $(L|K,P)$ be an extension of valued fields. Take elements $x_i, y_j \in L$, $i \in I$, $j \in J$, such that the values $v_P x_i$, $i \in I$, are rationally independent over $v_P K$, and the residues $y_j P$, $i \in J$, are algebraically independent over KP. Then the elements x_i, y_j, $i \in I$, $j \in J$, are algebraically independent over K, the value of each polynomial in $K[x_i, y_j \mid i \in I, j \in J]$ is equal to the least of the values of its monomials, and*

$$v_P K(x_i, y_j \mid i \in I, j \in J) = v_P K \oplus \bigoplus_{i \in I} \mathbb{Z} v_P x_i \tag{28}$$

$$K(x_i, y_j \mid i \in I, j \in J)P = KP\,(y_j P \mid j \in J)\,. \tag{29}$$

Moreover, the valuation v_P on $K(x_i, y_j \mid i \in I, j \in J)$ is uniquely determined by its restriction to K, the values $v_P x_i$ and the residues $y_j P$.

For the proof of the following corollary, see [V] or [K2].

Corollary 7.3 *Let $(L|K,P)$ be an extension of valued fields of finite transcendence degree. Then*

$$\mathrm{trdeg}\, L|K \;\geq\; \mathrm{trdeg}\, LP|KP \,+\, \mathrm{rr}\,(v_P L / v_P K)\,. \tag{30}$$

If in addition $L|K$ is a function field, and if equality holds in (30), then the extensions $v_P L|v_P K$ and $LP|KP$ are finitely generated. In particular, if P is trivial on K, then $v_P L$ is a product of finitely many (namely, $\mathrm{rr}\, v_P L$) copies of $\mathbb{Z}$, and LP is again a function field over K.

If P is a place of $F|K$, then (30) reads as follows:

$$\mathrm{trdeg}\, F|K \;\geq\; \mathrm{trdeg}\, FP|K \;+\; \mathrm{rr}\, v_P F\;. \tag{31}$$

The famous **Abhyankar inequality** is a generalization of this inequality to the case of noetherian local rings (see [V]). We call P an **Abhyankar place** if equality holds in (31).

The rank of an ordered abelian group is always smaller or equal to its rational rank. This is seen as follows. If G_1 is a subgroup of G, then its divisible hull $\mathbb{Q}\otimes G_1$ lies in the convex hull of G_1 in $\mathbb{Q}\otimes G$. Hence if G_1 is a proper convex subgroup of G, then $\mathbb{Q}\otimes G_1$ is a proper convex subgroup of $\mathbb{Q}\otimes G$ and thus, $\dim_{\mathbb{Q}}\mathbb{Q}\otimes G_1 < \dim_{\mathbb{Q}}\mathbb{Q}\otimes G$. It follows that if $\{0\} = G_0 \subsetneqq G_1 \subsetneqq \ldots \subsetneqq G_n = G$ is a chain of convex subgroups of G, then $\mathrm{rr}\, G = \dim_{\mathbb{Q}}\mathbb{Q}\otimes G \geq n$. In view of (31), this proves that the rank of a place P of a function field $F|K$ cannot exceed $\mathrm{trdeg}\, F|K$ and thus is finite. We say that P is of **maximal rank** if the rank is equal to $\mathrm{trdeg}\, F|K$.

If $\mathrm{trdeg}\, F|K = 1$, then every place P of $F|K$ is an Abhyankar place. It is of maximal rank if and only if it is non-trivial. Indeed, if $v_P F$ is not trivial, then $\mathrm{rr}\, v_P F \geq 1$, and it follows from (31) that $\mathrm{trdeg}\, F|K = 1 = \mathrm{rr}\, v_P F$. Then also the rank is 1 since a group of rational rank 1 is a non-trivial subgroup of $\mathbb{Q}$. If on the other hand $v_P F$ is trivial, then P is an isomorphism on F so that $\mathrm{trdeg}\, F|K = 1 = \mathrm{trdeg}\, FP|K$.

Using Corollary 7.3, we can now generalize our construction given in Example 7.1. Let P be an arbitrary place of $F|K$. We set $\rho = \mathrm{rr}\, v_P F$ and $\tau = \mathrm{trdeg}\, FP|K$. We take elements $x_1,\ldots,x_\rho \in F$ such that $v_P x_1,\ldots,v_P x_\rho$ are rationally independent elements in $v_P F$. Further, we take elements $y_1,\ldots,y_\tau \in F$ such that $y_1 P,\ldots,y_\tau P$ are algebraically independent over K. Then by Theorem 7.2, $x_1,\ldots,x_\rho,y_1,\ldots,y_\tau$ are algebraically independent over K. The restriction of P to $K(x_1,\ldots,x_\rho,y_1,\ldots,y_\tau)$ is an Abhyankar place. We fix this situation for later use. We call

$$\left.\begin{array}{l} F_0 \;:=\; K(x_1,\ldots,x_\rho,y_1,\ldots,y_\tau) \text{ with}\\ \rho = \mathrm{rr}\, v_P F \;\;\text{ and }\;\; \tau = \mathrm{trdeg}\, FP|K,\\ v_P x_1,\ldots,v_P x_\rho \text{ rationally independent in } v_P F, \text{ and}\\ y_1 P,\ldots,y_\tau P \text{ algebraically independent over } K \end{array}\right\} \tag{32}$$

an **Abhyankar subfunction field** of $(F|K,P)$.

Now let us assume in addition that P is an Abhyankar place of $F|K$. That is, $\rho+\tau = \mathrm{trdeg}\, F|K$. It follows that $x_1,\ldots,x_\rho,y_1,\ldots,y_\tau$ is a transcendence basis of $F|K$. We refine our choice of these elements as follows. From Corollary 7.3 we know that $v_P F$ is product of ρ copies of $\mathbb{Z}$. So we can choose $x_1,\ldots,x_\rho \in \mathcal{O}_P$ in such a way that $v_P F = \mathbb{Z} v_P x_1 \oplus\ldots\oplus \mathbb{Z} v_P x_\rho$, which implies that $v_P F = v_P K(x_1,\ldots,x_\rho)$. From Corollary 7.3 we also know that $FP|K$ is finitely generated. We shall also assume that $FP|K$ is separable. Then it follows that there is a separating transcendence basis for $FP|K$. We choose $y_1,\ldots,y_\tau \in \mathcal{O}_P$ in such a way that $y_1 P,\ldots,y_\tau P$ is such a separating transcendence basis. Now we can choose some $a\in FP$ such that $FP = K(y_1 P,\ldots,y_\tau P,a)$. We take a monic polynomial f with coefficients in the valuation ring of (F_0,P) such that its reduction fv_P is the minimal polynomial of a over $F_0 P = K(y_1 P,\ldots,y_\tau P)$. Since $a\in FP$ is separable-algebraic over $K(y_1 P,\ldots,y_\tau P)$, by Hensel's Lemma (Simple Root Version) there exists a root η of f in the henselization of (F,P) such that $\eta P = a$. Take $\sigma \in \mathrm{Aut}\,(\tilde{F}_0|F_0)$ such that $v(\sigma x - x) > 0$ for all x in the valuation ring of P on $\tilde{F}$. Then in particular, $v(\sigma\eta-\eta) > 0$. But if $\sigma\eta \neq \eta$, then it follows from $\deg(f) = \deg(fv_P)$ that $(\sigma\eta)P \neq \eta P$, i.e., $v(\sigma\eta - \eta) = 0$. Hence, $\sigma\eta = \eta$, which shows that η lies in the absolute inertia field of F_0.

Now the field $F_0(\eta)$ has the same value group and residue field as F, and it is contained in the henselization of F. Hence by Theorem 5.16,

$$(F^h|F_0(\eta)^h, P) \tag{33}$$

is an immediate algebraic extension. As η lies in the absolute inertia field of F_0 and this field is henselian, we have that $F_0(\eta)^h$ is a subfield of this absolute inertia field. If we could show that $F^h = F_0(\eta)^h$, then F itself would lie in this absolute inertia field, which would prove that $(F|K,P)$ is inertially generated. If the residue characteristic char $FP =$ char K is 0, then again Lemma 5.14 tells us that the absolute inertia field of (F_0,P) is the unique maximal extension having the same value group as F_0; so F^h must be a subfield of it. Hence in characteristic 0 we have now shown that $(F|K,P)$ is inertially generated. But what happens in positive characteristic? Can the extension (33) be non-trivial? To answer this question, we have to take a closer look at the main problem of valuation theory in positive characteristic.

8 The defect

Assume that (K,v) is henselian and $(L|K,v)$ is a finite extension of degree n. Then we have to deal only with a single ramification index e and a single inertia degree f. Hence, the fundamental inequality now reads as

$$n \geq \mathrm{e}\,\mathrm{f}\,. \tag{34}$$

If L is contained in K^i then by Theorem 5.8, $n = \mathrm{f}$. If $K = K^i$ and L is contained in K^r, then by Theorem 5.9, $n = \mathrm{e}$. Putting these observations together (using Theorems 5.12 and 5.13 and the fact that extension degree, ramification index and inertia degree are multiplicative), one finds:

Lemma 8.1 *If (K,v) is henselian and $L|K$ is a finite subextension of $K^r|K$, then it satisfies the* **fundamental equality**

$$n = \mathrm{e}\,\mathrm{f}\,. \tag{35}$$

Hence, an inequality can only result from some part of the extension which lies beyond the absolute ramification field. So Theorem 5.10 shows that the missing factor can only be a power of p. In this way, one proves the important **Lemma of Ostrowski**:

Theorem 8.2 *Set $p :=$ char Kv if this is positive, and $p := 1$ if char $Kv = 0$. If (K,v) is henselian and $L|K$ is of degree n, then*

$$n \;=\; \mathrm{d}\,\mathrm{e}\,\mathrm{f}\;, \tag{36}$$

where d *is a power of p. In particular, if* char $Kv = 0$*, then we always have the fundamental equality $n = \mathrm{e}\,\mathrm{f}$.*

The integer $\mathrm{d} \geq 1$ is called the **defect** of the extension $(L|K,v)$. This can also be taken as a definition for the defect if (K,v) is not henselian, but the extension of v from K to L is unique. We note:

Corollary 8.3 *If (K,v) is henselian and $(L|K,v)$ is a finite immediate extension, then the defect of $(L|K,v)$ is equal to $[L:K]$.*

A henselian field (K,v) is called **defectless** if every finite extension has trivial defect $\mathrm{d} = 1$. In rigid analysis, this is also called **stable**. A not necessarily henselian field is called defectless if for every finite extension of it, equality holds in the fundamental inequality (14) (if the field is henselian, this coincides with our first definition). A proof of the next theorem can be found in [K2] and, partially, also in [E].

Theorem 8.4 *A valued field is a defectless field if and only if its henselization is.*

We also note the following fact, which is easy to prove:

Lemma 8.5 *Every finite extension field of a defectless field is again a defectless field.*

The following are examples of defectless fields:

(DF1) All valued fields with residue characteristic 0. This is a direct consequence of the Lemma of Ostrowski.

(DF2) Every discretely valued field of characteristic 0. An easy argument shows that every finite extension with non-trivial defect of a discretely valued field must be inseparable. In particular, the field $(\mathbb{Q}_p, v_p)$ of p-adic numbers with its p-adic valuation, and all of its subfields, are defectless fields.

(DF3) All maximal fields (and hence also all power series fields, see Section 9) are defectless fields. For the proof, see [R1] or [K2].

We have seen that the extensions beyond the absolute ramification field are responsible for non-trivial defects. To get this picture, we have chosen a modified approach to ramification theory (see Remark 5.4). We have shifted the purely inseparable extensions to the top (cf. our table). In fact, that is where the purely inseparable extensions belong, because from the ramification theoretical point of view, they can be nasty, and in this respect, they have much in common with the extension S|V.

The defect, appearing only for positive residue characteristic, is essentially the cause of the problems that we have in algebraic geometry as well as in the model theory of valued fields, in positive characteristic. Therefore, it is very important that you get a feeling for what the defect is. Let us look at three main examples. The first one is the most basic and was probably already known to most of the early valuation theorists. But it seems reasonable to attribute it to F. K. Schmidt.

Example 8.6 We consider the power series field $K((t))$ with its t-adic valuation $v = v_t$. We have already remarked in Example 2.2 that $K((t))|K(t)$ is transcendental. So we can choose an element $s \in K((t))$ which is transcendental over $K(t)$. Since $vK((t)) = \mathbb{Z} = vK(t)$ and $K((t))v = K = K(t)v$, the extension $(K((t))|K(t), v)$ is immediate. The same must be true for the subextension $(K(t,s)|K(t), v)$ and thus also for $(K(t,s)|K(t,s^p), v)$. The latter extension is purely inseparable of degree p (since s, t are algebraically independent over K, the extension $K(s)|K(s^p)$ is linearly disjoint from $K(t,s^p)|K(s^p)$). Hence, Corollary 5.2 shows that there is only one extension of the valuation v from $K(t,s^p)$ to $K(t,s)$. Consequently, its defect is p. ♢

To give an example of a henselian field which is not defectless, we build on the foregoing example.

Example 8.7 By Theorem 5.15, there is a henselization $(K(t,s), v)^h$ of $(K(t,s), v)$ in the henselian field $K((t))$ and a henselization $(K(t,s^p), v)^h$ of $(K(t,s^p), v)$ in $(K(t,s), v)^h$. We find that the extension $(K(t,s), v)^h|(K(t,s^p), v)^h$ is again purely inseparable of degree p. Indeed, $K(t,s)|K(t,s^p)$ is linearly disjoint from the separable extension $K(t,s^p)^h|K(t,s^p)$, and by virtue of Corollary 5.18, $K(t,s)^h = K(t,s).K(t,s^p)^h$. Also for this extension, we have that $\mathrm{e} = \mathrm{f} = 1$ and again, the defect is p. Note that by what we have said earlier, an extension of degree p with non-trivial defect over a discretely valued field like $(K(t,s^p), v)^h$ can only be purely inseparable. ♢

Now we will give an example of a finite *separable* extension with non-trivial defect. It seems to be the generic example for our purposes since its importance is also known in algebraic geometry.

Example 8.8 Take an arbitrary field K of characteristic $p > 0$, and t to be transcendental over K. On $K(t)$ we take the t-adic valuation $v = v_t$. We set $L := K(t^{1/p^i} \mid i \in \mathbb{N})$. This is a purely inseparable extension of $K(t)$; if K is perfect, then it is the perfect hull of $K(t)$. By Corollary 5.2, v has a unique extension to L. We set $L_k := K(t^{1/p^k})$ for every $k \in \mathbb{N}$; so $L = \bigcup_{k\in\mathbb{N}} L_k$. We observe that $1/p^k = vt^{1/p^k} \in vL_k$, so $(vL_k : vK(t)) \geq p^k$. Now the fundamental inequality shows that $(vL_k : vK(t)) = p^k$ and that $L_k v = K(t)v = K$. The former shows that $vL_k = \frac{1}{p^k}\mathbb{Z}$. We obtain that $vL = \bigcup_{k\in\mathbb{N}} vL_k = \frac{1}{p^\infty}\mathbb{Z}$ and that $Lv = \bigcup_{k\in\mathbb{N}} L_k v = K$.

We consider the extension $L(\vartheta)|L$ generated by a root ϑ of the Artin–Schreier–polynomial $X^p - X - \frac{1}{t}$. We set

$$\vartheta_k := \sum_{i=1}^{k} t^{-1/p^i} \tag{37}$$

and compute

$$\vartheta_k^p - \vartheta_k - \frac{1}{t} = \sum_{i=1}^{k} t^{-1/p^{i-1}} - \sum_{i=1}^{k} t^{-1/p^i} - t^{-1} = \sum_{i=0}^{k-1} t^{-1/p^i} - \sum_{i=1}^{k} t^{-1/p^i} - t^{-1} = -t^{-1/p^k} . \quad (38)$$

Therefore,

$$(\vartheta - \vartheta_k)^p - (\vartheta - \vartheta_k) = \vartheta^p - \vartheta - \frac{1}{t} - \left(\vartheta_k^p - \vartheta_k - \frac{1}{t}\right) = 0 + t^{-1/p^k} = t^{-1/p^k} . \quad (39)$$

If we have the equation $b^p - b = c$ and know that $vc < 0$, then we can conclude that $vb < 0$ since otherwise, $v(b^p - b) \geq 0 > vc$, a contradiction. But since $vb < 0$, we have that $vb^p = pvb < vb$, which implies that $vc = v(b^p - b) = pvb$. Consequently, $vb = \frac{vc}{p}$. In our case, we obtain that

$$v(\vartheta - \vartheta_k) = \frac{vt^{-1/p^k}}{p} = -\frac{1}{p^{k+1}} . \quad (40)$$

We see that $1/p^{k+1} \in vL_k(\vartheta)$, so that $(vL_k(\vartheta) : vL_k) \geq p$. Since $[L_k(\vartheta) : L_k] \leq p$, the fundamental inequality shows that $[L_k(\vartheta) : L_k] = p$, $vL_k(\vartheta) = 1/p^{k+1}\mathbb{Z}$, $L_k(\vartheta)v = L_kv = K$ and that the extension of the valuation v from L_k to $L_k(\vartheta)$ is unique. As $L(\vartheta) = \bigcup_{k\in\mathbb{N}} L_k(\vartheta)$, we obtain that $vL(\vartheta) = \bigcup_{k\in\mathbb{N}} 1/p^{k+1}\mathbb{Z} = \frac{1}{p^\infty}\mathbb{Z} = vL$ and that $L(\vartheta)v = \bigcup_{k\in\mathbb{N}} L_kv = K = Lv$. We have thus proved that the extension $(L(\vartheta)|L, v)$ is immediate. Since ϑ has degree p over every L_k, it must also have degree p over their union L. Further, as the extension of v from L_k to $L_k(\vartheta)$ is unique for every k, also the extension of v from L to $L(\vartheta)$ is unique. So we have $n = p$, $\mathrm{e} = \mathrm{f} = \mathrm{g} = 1$, and we find that the defect of $(L(\vartheta)|L, v)$ is p.

To obtain a defect extension of a henselian field, we show that L can be replaced by its henselization. By Theorem 5.18, $L^h(\vartheta) = L^h.L(\vartheta) = L(\vartheta)^h$. By Theorem 5.16, $vL(\vartheta)^h = vL(\vartheta) = vL = vL^h$ and $L(\vartheta)^hv = L(\vartheta)v = Lv = L^hv$. Hence, also the extension $(L^h(\vartheta)|L^h, v)$ is immediate. We only have to show that it is of degree p. This follows from the general valuation theoretical fact that if an extension $L'|L$ admits a unique extension of the valuation v from L to L', then $L'|L$ is linearly disjoint from $L^h|L$. But we can also give a direct proof. Again by Theorem 5.18, $L_k^h(\vartheta) = L_k(\vartheta)^h$, and by Theorem 5.16, $vL_k(\vartheta)^h = vL_k(\vartheta)$ and $vL_k^h = vL_k$. Therefore, $(vL_k^h(\vartheta) : vL_k^h) = (vL_k(\vartheta) : vL_k) = p$, showing that $[L_k^h(\vartheta) : L_k^h] = p$ for every k. Again by Theorem 5.18, $L^h = L.L_1^h = (\bigcup_{k\in\mathbb{N}} L_k).L_1^h = \bigcup_{k\in\mathbb{N}} L_k.L_1^h = \bigcup_{k\in\mathbb{N}} L_k^h$. By the same argument as before, it follows that $[L^h(\vartheta) : L^h] = p$.

Hence, we have found an immediate Artin–Schreier extension of degree p and defect p of a henselian field which is only of transcendence degree 1 over K. $\diamondsuit$

A valued field (K, v) is called **algebraically maximal** if it admits no proper immediate algebraic extension, and it is called **separable-algebraically maximal** if it admits no proper immediate separable-algebraic extension. Since the henselization is an immediate separable-algebraic extension by Theorem 5.16, every separable-algebraically maximal field is henselian. The converse is not true, since the field (L^h, v) of our foregoing example is henselian but not separable-algebraically maximal. Corollary 8.3 shows that every henselian defectless field is algebraically maximal. The converse is not true, as was shown by Françoise Delon [DEL] (cf. also [K2]).

Example 8.9 In the foregoing example, we may replace $K(t)$ by $K((t))$, taking L to be the field $K((t))(t^{1/p^k} \mid k \in \mathbb{N})$. It is not hard to show (by splitting up the power series in a suitable way) that $K((t^{1/p^k})) = K((t))[t^{1/p^k}]$, which is algebraic over $K((t))$. Hence, $L = \bigcup_{k\in\mathbb{N}} K((t^{1/p^k}))$, a union of an ascending chain of power series fields. By Lemma 12.2 below, (L, v) is henselian, and $(L(\vartheta)|L, v)$ gives an instant example of an immediate extension of a henselian field. But this L is "very large": it is of infinite transcendence degree over K. On the other hand, this version of our example shows that an infinite algebraic extension of a maximal field (or a union over an ascending chain of maximal fields) is in general not even defectless (and hence also not maximal). The example can also be transformed into the p-adic situation, showing that there are infinite extensions of $(\mathbb{Q}_p, v_p)$ which are not defectless fields (cf. [K2]). $\diamondsuit$

9 Maximal immediate extensions

Based on our examples, we can observe another obstruction in positive characteristic. In many applications of valuation theory, one is interested in the embedding of a given valued field in a power series field which, if possible, should have the same value group and residue field. (We give an example relevant for algebraic geometry in the next section.) Then this power series field would be an immediate extension of our field, and since every power series field is maximal, it would be a **maximal immediate extension** of our field. So we see that we are led to the problem of determining maximal immediate extensions, in particular, whether maximal immediate extensions of a given valued field are unique up to valuation preserving isomorphism. It was shown by Wolfgang Krull [KR] that maximal immediate extensions exist for every valued field. The proof uses Zorn's Lemma in combination with an upper bound for the cardinality of valued fields with prescribed value group and residue field. Krull's deduction of this upper bound is hard to read; later, Kenneth A. H. Gravett [GRA] gave a nice and simple proof.

The uniqueness problem for maximal immediate extensions was considered by Irving Kaplansky in his important paper [KA1]. He showed that if the so-called **hypothesis A** holds, then the field has a unique maximal immediate extension (up to valuation preserving isomorphism). For a Galois theoretic interpretation of hypothesis A and more information about the uniqueness problem, see [KPR]. Let us mention a problem which was only partially solved in [KPR] and in [WA1]:

Open Problem 2: If a valued field does not satisfy Kaplansky's hypothesis A, does it then admit two non-isomorphic maximal immediate extensions?

We can give a quick example of a valued field with two non-isomorphic maximal immediate extensions.

Example 9.1 In the setting of Example 8.9, assume that K is not **Artin–Schreier closed**, that is, there is an element $c \in K$ such that $X^p - X - c$ is irreducible over K. Take ϑ_c to be a root of $X^p - X - (\frac{1}{t} + c)$; note that $v(\frac{1}{t} + c) = v\frac{1}{t} < 0$ since $vc = 0 > v\frac{1}{t}$. Then in exactly the same way as for ϑ, one shows that the extension $(L(\vartheta_c)|L, v)$ is immediate of degree p and defect p. So we have two distinct immediate extensions of L. We take (M_1, v) to be a maximal immediate extension of $L(\vartheta)$, and (M_2, v) to be a maximal immediate extension of $L(\vartheta_c)$. Then (M_1, v) and (M_2, v) are also maximal immediate extensions of (L, v). If they were isomorphic over L, then M_1 would also contain a root of $X^p - X - (\frac{1}{t} + c)$; w.l.o.g., we can assume that it is the one called ϑ_c. Now we compute:

$$(\vartheta_c - \vartheta)^p - (\vartheta_c - \vartheta) \;=\; \vartheta_c^p - \vartheta_c - (\vartheta^p - \vartheta) \;=\; \frac{1}{t} + c - \frac{1}{t} \;=\; c\,.$$

Since $vc = 0$, we also have that $v(\vartheta_c - \vartheta) = 0$ (you may prove this along the lines of an argument given earlier). Applying the residue map to $\vartheta_c - \vartheta$, we thus obtain a root of $X^p - X - c$. But by our assumption, this root is not contained in $K = Lv$. Consequently, $M_1 v \neq Lv$, contradicting the fact that (M_1, v) was an immediate extension of (L, v). This proves that (M_1, v) and (M_2, v) cannot be isomorphic over (L, v).

We have used that K is not Artin–Schreier closed. And in fact, one of the consequences of hypothesis A for a valued field (L, v) is that its residue field be Artin–Schreier closed (see Section 11). $\diamondsuit$

We will need a generalization of the field of formal Laurent series, called **(generalized) power series field**. Take any field K and any ordered abelian group G. We take $K((G))$ to be the set of all maps μ from G to K with well-ordered **support** $\{g \in G \mid \mu(g) \neq 0\}$. You can visualize the elements of $K((G))$ as formal power series $\sum_{g \in G} c_g t^g$ for which the support $\{g \in G \mid c_g \neq 0\}$ is well-ordered. Using this condition one shows that $K((G))$ is a field in a similar way as it is done for $K((t))$. Also, one uses it to introduce the valuation:

$$v \sum_{g \in G} c_g t^g \;=\; \min\{g \in G \mid c_g \neq 0\} \tag{41}$$

(the minimum exists because the support is well-ordered). This valuation is often called the **canonical valuation of** $K((G))$, and sometimes called the **minimum support valuation**. With this valuation, $K((G))$ is a maximal field.

The fields L constructed in Examples 8.8 and 8.9 are subfields of $K((\mathbb{Q}))$ in a canonical way. It is interesting to note that the element ϑ is an element of $K((\mathbb{Q}))$:

$$\vartheta \;=\; \sum_{i\in\mathbb{N}} t^{-1/p^i} \;=\; t^{-1/p} + t^{-1/p^2} + \ldots + t^{-1/p^i} + \ldots\,. \tag{42}$$

Indeed,

$$\vartheta^p - \vartheta - \frac{1}{t} \;=\; \sum_{i\in\mathbb{N}} t^{-1/p^{i-1}} \;-\; \sum_{i\in\mathbb{N}} t^{-1/p^i} \;-\; t^{-1} \;=\; \sum_{i=0}^{\infty} t^{-1/p^i} \;-\; \sum_{i=1}^{\infty} t^{-1/p^i} \;-\; t^{-1} \;=\; 0\,.$$

Note that the values $vt^{-1/p^n} = -1/p^n$ converge from below to 0. Therefore, ϑ does not even lie in the completion of L. In fact, there cannot be a root of $X^p - X - 1/t$ in the completion; if a would be such a root, then there would be some $b \in L$ such that $v(a-b) > 0$. We would have that

$$(a-b)^p - (a-b) \;=\; a^p - a - (b^p - b) \;=\; \frac{1}{t} - (b^p - b)\,. \tag{43}$$

Because of $v(a-b) > 0$, the left hand side and consequently also the right hand side has value > 0. But as we have seen in Example 4.4, the polynomial $X^p - X - c$ splits over every henselian field containing c if $vc > 0$. Hence, in the cases where L is henselian, there exists a root $a' \in L$ of $X^p - X - 1/t + b^p - b$. It follows that $(a'+b)^p - (a'+b) - 1/t = 1/t - (b^p - b) + b^p - b - 1/t = 0$. As $a' + b \in L$, this would imply that $X^p - X - 1/t$ splits over L, a contradiction.

Let us illustrate the influence of the defect by considering an object which is well-known in algebraic geometry.

10 A quick look at Puiseux series fields

Recall that $K((\mathbb{Q}))$ is the field of all formal sums $\sum_{q\in\mathbb{Q}} c_q t^q$ with $c_q \in K$ and well-ordered support. The subset

$$\mathrm{P}(K) \;:=\; \left\{ \sum_{i=n}^{\infty} c_i t^{i/k} \mid c_i \in K,\, n \in \mathbb{Z},\, k \in \mathbb{N} \right\} \;=\; \bigcup_{k\in\mathbb{N}} K((t^{1/k})) \;\subset\; K((\mathbb{Q})) \tag{44}$$

is itself a field, called the **Puiseux series field over** K. Here, the valuation v on $K((t^{1/k}))$ is again the minimum support valuation, in particular, we have that $vt^{1/k} = 1/k$. In this way, the valuation v on every $K((t^{1/k}))$ is an extension of the t-adic valuation v_t of $K((t))$ and of the valuation of every subfield $K((t^{1/m}))$ where m divides k.

$\mathrm{P}(K)$ can also be written as a union of an ascending chain of power series fields in the following way. We take p_i to be the i-th prime number and set $m_k := \prod_{i=1}^{k} p_i^k$. Then $m_k | m_{k+1}$ and thus $K((t^{1/m_k})) \subset K((t^{1/m_{k+1}}))$ for every $k \in \mathbb{N}$, and every natural number will divide m_k for large enough k. Therefore,

$$\mathrm{P}(K) \;=\; \bigcup_{k\in\mathbb{N}} K((t^{1/m_k}))\,. \tag{45}$$

If one does not want to work in the power series field $K((\mathbb{Q}))$, then one simply has to choose a compatible system of k-th roots $t^{1/k}$ of t (that is, for $k = \ell m$ we must have $(t^{1/k})^\ell = t^{1/m}$; this is automatic for the elements $t^{1/k} \in K((\mathbb{Q}))$ by definition of the multiplication in this field). Then (44) can serve as a definition for the Puiseux series field over K.

Lemma 10.1 *The Puiseux series field $P(K)$ is an algebraic extension of $K((t))$, and it is henselian with respect to its canonical valuation v. Its residue field is K and its value group is $\mathbb{Q}$.*

Proof: For every $k \in \mathbb{N}$, the element $t^{1/k}$ is algebraic over $K((t))$. Similarly as in Example 8.9, we have that $K((t^{1/k})) = K((t))[t^{1/k}]$, which is algebraic over $K((t))$. Consequently, also the union $\mathrm{P}(K)$ of the $K((t^{1/k}))$ is algebraic over $K((t))$. By Corollary 5.17, $K((t))$ is henselian w.r.t. its canonical valuation v_t. As the canonical valuation v of $\mathrm{P}(K)$ is an extension of v_t, Corollary 4.2 yields that $(\mathrm{P}(K), v)$ is henselian.

The value group of every $(K((t^{1/k})), v)$ is $\mathbb{Z}vt^{1/k} = \mathbb{Z}\frac{vt}{k} = \frac{1}{k}\mathbb{Z}$, so the union over all $K((t^{1/k}))$ has value group $\bigcup_{k\in\mathbb{N}} \frac{1}{k}\mathbb{Z} = \mathbb{Q}$. The residue field of every $(K((t^{1/k})), v)$ is K, hence also the residue field of their union is K. □

Theorem 10.2 *Let K be a field of characteristic 0. Then $(\mathrm{P}(K), v)$ is a defectless field. Further, $P(K)$ is the algebraic closure of $K((t))$ if and only if K is algebraically closed.*

Proof: The residue field of $(\mathrm{P}(K), v)$ is K, hence if $\operatorname{char} K = 0$, then $(\mathrm{P}(K), v)$ is a defectless field by **(DF1)** in Section 8.

For the second assertion, we use the following valuation theoretical fact (try to prove it, it is not hard):

Let (L, v) be a valued field and choose any extension of v to the algebraic closure $\tilde{L}$. Then $v\tilde{L}$ is the divisible hull of vL, and $\tilde{L}v$ is the algebraic closure of Lv.

Hence, $v\widetilde{K((t))} = \mathbb{Q} = v\mathrm{P}(K)$ and $\widetilde{K((t))}P = \tilde{K}$. Thus if $\widetilde{K((t))} = \mathrm{P}(K)$, then $\tilde{K} = \mathrm{P}(K) = K$, which shows that K must be algebraically closed. For the converse, note that by the foregoing lemma, $\mathrm{P}(K) \subseteq \widetilde{K((t))}$. Assume that $\tilde{K} = K$. Then the extension $(\widetilde{K((t))}|\mathrm{P}(K), v)$ is immediate. But since $(\mathrm{P}(K), v)$ is henselian (by the foregoing lemma) and defectless, every finite subextension must be trivial by Theorem 8.2. This proves that $\widetilde{K((t))} = \mathrm{P}(K)$, i.e., $\mathrm{P}(K)$ is algebraically closed. □

The assertion of this theorem does not hold if K has positive characteristic:

Example 10.3 In Example 8.8, we can replace L_k by $K((t^{1/k}))$ for every $k \in \mathbb{N}$ (as opposed to $K((t^{1/p^k}))$, which we used in Example 8.9). Still, everything works the same, producing the henselian Puiseux series field $L = \mathrm{P}(K)$ with an immediate Artin–Schreier extension $(L(\vartheta)|L, v)$ of degree p and defect p.

By construction, $\mathrm{P}(K)$ is a subfield of $K((\mathbb{Q}))$. Hence, the arguments at the end of the last section show that there is no root of $X^p - X - 1/t$ in the completion of $\mathrm{P}(K)$. The arguments of Example 9.1 show that $\mathrm{P}(K)$ has non-isomorphic maximal immediate extensions if K is not Artin–Schreier closed. ♢

Our example proves:

Theorem 10.4 *Let K be a field of characteristic $p > 0$. Then $(\mathrm{P}(K), v)$ is not defectless. In particular, $\mathrm{P}(K)$ is not algebraically closed, even if K is algebraically closed. Not even the completion of $\mathrm{P}(K)$ is algebraically closed.*

There is always a henselian defectless field extending $K((t))$ and having residue field K and divisible value group, even if K has positive characteristic. We just have to take the power series field $K((\mathbb{Q}))$. But in contrast to the Puiseux series field, this field is "very large": it has uncountable transcendence degree over $K((t))$. Nevertheless, having serious problems with the Puiseux series field in positive characteristic, we tend to replace it by $K((\mathbb{Q}))$. But this seems problematic since it might not be the unique maximal immediate extension of the Puiseux series field. However, if K is perfect and does not admit afinite extension whose degree is divisible by p (and in particular if K is algebraically closed), then Kaplansky's uniqueness result shows that the maximal immediate extension is unique. On the other hand, our example shows that the assumption "K is perfect" alone is not sufficient, since there are perfect fields which are not Artin–Schreier closed.

11 The tame and the wild valuation theory

Before we carry on, let us describe some advanced ramification theory based on the material of Sections 5 and 8. Throughout, we let (K,v) be a henselian non-trivially valued field. We set $p = \mathrm{char}\, Kv$ if this is positive, and $p = 1$ otherwise. If $(L|K,v)$ is an algebraic extension, then we call $(L|K,v)$ a **tame extension** if for every finite subextension $L'|K$ of $L|K$,

1) $(vL' : vK)$ is not divisible by the residue characteristic $\mathrm{char}\, Kv$,
2) $L'v|Kv$ is separable,
3) $[L' : K] = (vL' : vK)[L'v : Kv]$, i.e., $(L'|K,v)$ has trivial defect.

From the ramification theoretical facts presented in Section 5, one derives:

Theorem 11.1 *If (K,v) is henselian, then its absolute ramification field (K^r,v) is the unique maximal tame extension of (K,v), and its absolute inertia field (K^i,v) is the unique maximal tame extension of (K,v) having the same value group as K.*

An extension $(L|K,v)$ is called **purely wild** if $L|K$ is linearly disjoint from $K^r|K$. An ordered group G is called p-divisible if for every $\alpha \in G$ and $n \in \mathbb{N}$ there is $\beta \in G$ such that $p^n\beta = \alpha$. The **p-divisible hull** of G, denoted by $\frac{1}{p^\infty}G$, is the smallest subgroup of the divisible hull $\mathbb{Q} \otimes G$ which contains G and is p-divisible; it can be written as $\{\alpha/p^n \mid \alpha \in G\,,\, n \in \mathbb{N}\}$. The following was proved by Matthias Pank (cf. [KPR]):

Theorem 11.2 *If (K,v) is henselian, then there exists a field complement W to K^r over K, that is, $W.K^r = \tilde{K}$ and $W \cap K^r = K$. The degree of every finite subextension of $W|K$ is a power of p. Further, vW is the p-divisible hull $\frac{1}{p^\infty}vK$ of vK, and Wv is the perfect hull of Kv.*

So (W,v) is a maximal purely wild extension of (K,v). It was shown by Pank and is shown in [KPR] via Galois theory that W is unique up to isomorphism over K if Kv does not admit finite separable extensions whose degree are divisible by p. On the other hand, if vK is p-divisible and Kv is perfect, then $(W|K,v)$ is an immediate extension, and since every subextension of $K^r|K$ has trivial defect, it follows that the field complements W of K^r over K are precisely the maximal immediate algebraic extensions of (K,v).

It was shown by George Whaples [WH2] and by Francoise Delon [D] that Kaplansky's original hypothesis A consists of the following three conditions:

1) Kv does not admit finite separable extensions whose degree are divisible by p,
2) vK is p-divisible,
3) Kv is perfect.

So if (K,v) satisfies Kaplansky's hypothesis A, then it follows from what we said above that the maximal immediate algebraic extensions of (K,v) are unique up to isomorphism over K. But this is the kernel of the uniqueness problem for the maximal immediate extensions: using Theorem 2 of [KA1], one can easily show that the maximal immediate extensions are unique as soon as the maximal immediate algebraic extensions are.

Since all finite tame extensions have trivial defect, the defect is located in the purely wild extensions $(W|K,v)$. So we are interested in their structure. Here is one amazing result, due to Florian Pop (for the proof, see [K2], and for the notion of "additive polynomial", see Example 4.4):

Theorem 11.3 *Let $(L|K,v)$ be a minimal purely wild extension, i.e., there is no subextension $L'|K$ of $L|K$ such that $L \neq L' \neq K$. Then there is an additive polynomial $\mathcal{A} \in K[X]$ and some $c \in K$ such that $L|K$ is generated by a root of $\mathcal{A}(X) + c$.*

The degree of $\mathcal{A}$ is a power of p (since it is additive), and in general it may be larger than p.

Now we shall quickly develop the theory of tame fields. The henselian field (K,v) is said to be a **tame field** if all of its algebraic extensions are tame extensions. By Theorem 11.2, this holds if and only if K^r is algebraically closed. Similarly, (K,v) is said to be a **separably tame field** if all of its separable-algebraic extensions are tame extensions. This holds if and only if K^r is separable-algebraically closed.

By Theorem 5.12, $\tilde{K} = K^r.W$ is the absolute ramification field of W. If $W'|K$ is a proper subextension, then $\tilde{K} \neq K^r.W'$. This proves:

Lemma 11.4 *Every maximal purely wild extension W is a tame field. No proper subextension of $W|K$ is a tame field. The maximal separable subextension is a separably tame field.*

By Theorem 5.10, $K^{\mathrm{sep}}|K^r$ is a p-extension. Hence if $\operatorname{char} Kv = 0$, then this extension is trivial. Since then also $\operatorname{char} K = 0$, it follows that $K^r = K^{\mathrm{sep}} = \tilde{K}$. Therefore,

Lemma 11.5 *Every henselian field of residue characteristic 0 is a tame field.*

Suppose that $K_1|K$ is an algebraic extension. Then by Theorem 5.12, $K^r \subseteq K_1^r$. Hence if K^r is algebraically closed, then so is K_1^r, and if K^r is separable-algebraically closed, then so is K_1^r. This proves:

Lemma 11.6 *Every algebraic extension of a tame field is again a tame field. Every algebraic extension of a separably tame field is again a separably tame field.*

If $K^r = \tilde{K}$, then every finite extension of (K, v) is a tame extension and thus has trivial defect, which shows that (K, v) is a defectless field. If $K^r = K^{\mathrm{sep}}$, then every finite separable extension has trivial defect. So we note:

Lemma 11.7 *Every tame field is henselian defectless and perfect. Every separably tame field is henselian and all of its finite separable extensions have trivial defect.*

We give a characterization of tame and separably tame fields (for the proof, see [K2]):

Lemma 11.8 *A valued field (K, v) is tame if and only if it is algebraically maximal, vK is p-divisible and Kv is perfect. If* $\operatorname{char} K = \operatorname{char} Kv$ *then (K, v) is tame if and only if it is algebraically maximal and perfect.*

A non-trivially valued field (K, v) is separably tame if and only if it is separable-algebraically maximal, vK is p-divisible and Kv is perfect.

This lemma together with Lemma 11.7 shows that for perfect valued fields (K, v) with $\operatorname{char} K = \operatorname{char} Kv$, the properties "algebraically maximal" and "henselian and defectless" are equivalent.

Corollary 11.9 *Assume that* $\operatorname{char} K = \operatorname{char} Kv$. *Then every maximal immediate algebraic extension of the perfect hull of (K, v) is a tame field (and no proper subextension of it is a tame field). If* $\operatorname{char} Kv = 0$ *then already the henselization (K^h, v) is a tame field.*

The following is a crucial lemma in the theory of tame fields. For its proof, see [K1] or [K2].

Lemma 11.10 *Let (L, v) be a tame field and $K \subset L$ a relatively algebraically closed subfield. If in addition $Lv|Kv$ is an algebraic extension, then (K, v) is also a tame field and moreover, vK is pure in vL and $Kv = Lv$. The same holds for "separably tame" in the place of "tame".*

The break we took for the development of the theory of tame fields is at the same time a good occasion to do some model theoretic preparation for later sections.

12 Some notions and tools from model theoretic algebra

The basic idea of model theoretic algebra is to analyze the assertions that an algebraist wants to prove, and to apply principles that are valid for certain types of assertions. Such principles prove once and for all facts that otherwise are proved over and over again in different settings (as a little example, see Lemma 12.1 below). To state and apply such principles, it is necessary to make it precise what it is that we are talking about, and in which mathematical language we are talking. The reader may interpose that mathematicians are talking about mathematical structures, which are fixed by definitions, that is, by axioms. If for instance we are talking about a group, then we talk about a set of elements G and a binary function $G\times G \to G$ which associates with every two elements a third one. So besides the underlying set, we are using a **function symbol** for this function, which is $+$ if we write the group additively, and $\cdot$ if we write it multiplicatively. But a group also has a unit element, for which we use the **constant symbol** 0 in the additive and 1 in the multiplicative case. If we talk about ordered groups, then for expressing the ordering we need a further symbol, which might be $<$ or $\leq$. Although it is also binary like $+$ or $\cdot$, it is not a function symbol. Since the ordering is a relation between the elements of the group, it is called a **relation symbol**. The description of a mathematical object may need more than one constant, function or relation symbol. For a field, we need two binary functions, $+$ and $\cdot$, and two constants, 0 and 1. Further, we may need function symbols or relation symbols of any (fixed) number of entries.

A **language** is defined to be

$$\mathcal{L} = \mathcal{F} \cup \mathcal{C} \cup \mathcal{R}$$

where

- $\mathcal{F}$ is a set of function symbols,
- $\mathcal{C}$ is a set of constant symbols,
- $\mathcal{R}$ is a set of relation symbols.

For example,

$$\mathcal{L}_{\mathrm{G}} := \{+,-,0\}$$

is the **language of groups** (where $-$ is a function symbol with one entry), $\mathcal{L}_{\mathrm{OG}} := \{+,-,0,<\}$ is the **language of ordered groups**,

$$\mathcal{L}_{\mathrm{F}} := \{+,\cdot\,,-,{}^{-1},0,1\}$$

is the **language of fields**, and

$$\mathcal{L}_{\mathrm{VF}} := \{+,\cdot\,,-,{}^{-1},0,1,\mathcal{O}\}$$

is the **language of valued fields**, where $\mathcal{O}$ is a relation symbol with one entry.

For a given language $\mathcal{L}$, an **$\mathcal{L}$-structure** is a quadruple

$$\mathfrak{A} = (A, \mathcal{R}_{\mathfrak{A}}, \mathcal{F}_{\mathfrak{A}}, \mathcal{C}_{\mathfrak{A}})$$

where

- A is a set, called the **universe of** $\mathfrak{A}$,
- $\mathcal{F}_{\mathfrak{A}} = \{f_{\mathfrak{A}} \mid f \in \mathcal{F}\}$ such that every $f_{\mathfrak{A}}$ is a function on A of the same arity as the function symbol f,
- $\mathcal{C}_{\mathfrak{A}} = \{c_{\mathfrak{A}} \mid c \in \mathcal{C}\}$ such that every $c_{\mathfrak{A}}$ is an element of A (called a **constant**),
- $\mathcal{R}_{\mathfrak{A}} = \{R_{\mathfrak{A}} \mid R \in \mathcal{R}\}$ such that every $R_{\mathfrak{A}}$ is a relation on A of the same arity as the relation symbol R.

We call $R_{\mathfrak{A}}$ the **interpretation of R on** A, and similarly for the functions $f_{\mathfrak{A}}$ and the constants $c_{\mathfrak{A}}$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two $\mathcal{L}$-structures. Then we will call $\mathfrak{A}$ a **substructure** of $\mathfrak{B}$ if the universe A of $\mathfrak{A}$ is a subset of the universe B of $\mathfrak{B}$ and the restrictions to A of the interpretations of the relation and function symbols and the constant symbols on B coincide with the interpretations of the same relation, function and constant symbols on A.

Any set A with a distinguished element 0, a binary function $+ :\ A\times A \to A$ and a unary function $- :\ A \to A$ will be a structure for the language $\mathcal{L}_{\mathrm{G}} = \{+,-,0\}$ of groups. This does not say anything about the behaviour of the functions $+$ and $-$ and the element 0. For instance, 0 may not at all behave as a neutral element for $+$. Such properties of structures cannot be fixed by the language. They have to be described by axioms.

By an **elementary $\mathcal{L}$-formula** we mean a syntactically correct string built up using the symbols of the language $\mathcal{L}$, variables, $=$, and the logical symbols $\forall$, $\exists$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$. An elementary $\mathcal{L}$-formula is called an **elementary $\mathcal{L}$-sentence** if every variable is bound by some quantifier. An elementary $\mathcal{L}$-sentence is called **existential** if it is of the form $\exists X_1 \ldots \exists X_n \varphi(X_1, \ldots, X_n)$ where $\varphi(X_1, \ldots, X_n)$ is a quantifier free $\mathcal{L}$-formula and $X_1, \ldots, X_n$ are the only variables appearing in φ. Hence an existential sentence is a sentence which only talks about the existence of certain elements. An elementary $\mathcal{L}$-sentence is called **universal existential** if it is of the form $\forall X_1 \ldots \forall X_k \exists X_{k+1} \ldots \exists X_n \varphi(X_1, \ldots, X_n)$ where φ is as above ($k = 0$ or $k = n$ are admissible, so existential is also universal existential).

For example, the usual sentences expressing associativity, commutativity, the fact that 0 is a neutral element, and the existence of inverses are universal existential elementary $\mathcal{L}_{\mathrm{G}}$-sentences. They form an elementary axiom system for the class of abelian groups. Similarly, we have elementary axiom systems for the classes of ordered abelian groups, fields, valued fields. It is not necessary that an axiom system consists of only finitely many axioms. For instance, properties like "algebraically closed" or "real closed" can be axiomatized by an infinite scheme of universal existential elementary axioms. One can quantify over all possible polynomials of fixed degree n by quantifying over their $n + 1$ coefficients. But in order to express that *all* polynomials have a root in a field K we need countably many axioms talking about polynomials of increasing degree. In a similar way, the property "henselian" is axiomatized by an infinite scheme of universal existential elementary axioms in the language of valued fields. In contrast to this, properties like "complete" or "maximal" have no elementary axiomatization in $\mathcal{L}_{\mathrm{VF}}$; we would have to quantify over subsets of the universe, which is impossible in elementary sentences. This shows that it makes sense to replace "complete" or "maximal" by "henselian", wherever possible.

The following lemma expresses (once and for all) a fact that is (intuitively) known to every good mathematician.

Lemma 12.1 *The union over an ascending chain of $\mathcal{L}$-structures $\mathfrak{A}_i$, $i \in \mathbb{N}$, satisfies all universal existential elementary sentences which are satisfied in all of the $\mathfrak{A}_i$.*

This proves, for instance:

Lemma 12.2 *The union of an ascending chain of henselian valued fields (L_i, v), $i \in \mathbb{N}$, is again a henselian valued field.*

Any set $\mathcal{T}$ of elementary $\mathcal{L}$-sentences is called an **elementary axiom system** (or an **$\mathcal{L}$-theory**). If an $\mathcal{L}$-structure satisfies all axioms in $\mathcal{T}$, then we call it a **model of $\mathcal{T}$**. So if we have an $\mathcal{L}_{\mathrm{G}}$-structure which is a model of the axiom system of groups, then we know that $+, -, 0$ behave as we expect them to do. The axiom system for valued fields expresses that $\{x \mid \mathcal{O}(x) \text{ holds}\}$ is a valuation ring. Since v is not a function from the field into itself, we cannot simply take a function symbol for v into the language. However, we can express "$vx \leq vy$" by the elementary sentence "$\mathcal{O}(yx^{-1}) \vee x = y = 0$".

We say that two $\mathcal{L}$-structures $\mathfrak{A}, \mathfrak{B}$ are **elementarily equivalent**, denoted by $\mathfrak{A} \equiv \mathfrak{B}$, if $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same elementary $\mathcal{L}$-sentences. An axiom system is **complete** if and only if all of its models are elementarily equivalent. Syntactically, that means that every elementary $\mathcal{L}$-sentence or its negation can be deduced from that axiom system. For example, the axiom system of divisible ordered abelian groups and the axiom system of algebraically closed fields of fixed characteristic are complete. It was shown by Abraham Robinson [RO] that the axiom system of algebraically closed non-trivially valued fields of fixed characteristic and fixed residue characteristic is complete.

The completeness of an axiom system yields a **Transfer Principle** for the class axiomatized by it. For example, the completeness of the axiom system for algebraically closed fields of fixed characteristic tells us that every elementary $\mathcal{L}_{\mathrm{F}}$-sentence which holds in one algebraically closed field will also hold in all other algebraically closed fields of the same characteristic. This reminds of the **Lefschetz Principle** which was stated and partially proved by Weil and Lefschetz and says (roughly speaking) that algebraic geometry over all algebraically closed fields of a fixed characteristic is the same ("there is but one algebraic geometry in characteristic p"). As for the elementary $\mathcal{L}_{\mathrm{F}}$-sentences of algebraic geometry, this indeed follows from the completeness. However, Lefschetz and Weil had in mind more than just the elementary sentences. That is why Weil worked with so-called "universal domains" which are algebraically closed and of infinite transcendence degree over their prime field. So the assertion was that there is but one algebraic geometry over universal domains of characteristic p. A satisfactory formalization and model theoretic proof by use of an **infinitary language** is due to Paul Eklof [EK]. Infinitary languages admit the conjunction of infinitely many elementary sentences. With such infinitary sentences, one can also express the fact that a field has infinite transcendence degree over its prime field. This cannot be done by elementary sentences. Indeed, algebraically closed fields are elementarily equivalent to the algebraic closure of their prime field, even if they have infinite transcendence degree.

In the theory of valued fields, we consider two important invariants: value groups and residue fields. If two valued fields are equivalent (in the language of valued fields), then so are their value groups (in the language of ordered groups) and their residue fields (in the language of fields). We are interested in a converse of this implication: under which additional assumptions do we have the so-called **Ax–Kochen–Ershov principle**:

$$vK \equiv vL \text{ and } Kv \equiv Lv \;\Rightarrow\; (K,v) \equiv (L,v) \tag{46}$$

It follows from the fact that the henselization is an immediate extension that this principle can only hold for henselian fields. And indeed, it was shown by James Ax and Simon Kochen [AK1] and independently by Yuri Ershov [ER2] that this principle holds for all henselian fields of residue characteristic 0. This is the famous **Ax–Kochen–Ershov Theorem**. Ax and Kochen proved it in order to deduce a correct version of **Artin's conjecture** about the fields $\mathbb{Q}_p$ of p-adic numbers ([AK1]; cf. [CK] or [K2]). From the Ax–Kochen–Ershov Theorem, one obtains an equivalence of two ultraproducts:

$$\prod_{p \text{ prime}} \mathbb{Q}_p \,/\mathcal{D} \equiv \prod_{p \text{ prime}} \mathbb{F}_p((t))\,/\mathcal{D} \tag{47}$$

because both fields carry a canonical henselian valuation of residue characteristic 0 and have equal value groups and residue fields. Since a product like $\prod_p \mathbb{Q}_p$ would not even be a field, one has to take the product modulo a non-principal ultrafilter $\mathcal{D}$ on the set of all primes. Let us quickly give the main facts about ultraproducts.

A filter $\mathcal{D}$ on a set I is called an **ultrafilter** if

$$J \subset I \,\wedge\, J \notin \mathcal{D} \;\Longrightarrow\; I \setminus J \in \mathcal{D}\,, \tag{48}$$

and it is called **non-principal** if it is not of the form $\{J \subset I \mid i \in J\}$, for no $i \in I$. If $\mathfrak{A}_i$, $i \in I$ are $\mathcal{L}$-structures, then the **ultraproduct** $\prod_{i\in I} \mathfrak{A}_i \,/\mathcal{D}$ is defined by setting, for all $(a_i),(b_i) \in \prod_{i\in I} \mathfrak{A}_i$,

$$(a_i) \equiv (b_i) \text{ modulo } \mathcal{D} \iff \{i \in I \mid a_i = b_i\} \in \mathcal{D}\,. \tag{49}$$

The following theorem is due to J. Łos. For a proof, see [K2] or [CK].

Theorem 12.3 (Fundamental Theorem of Ultraproducts)
For every $\mathcal{L}$-sentence φ, $\prod_{i\in I} \mathfrak{A}_i \,/\mathcal{D}$ satisfies φ if and only if

$$\{i \in I \mid \mathfrak{A}_i \text{ satisfies } \varphi\} \in \mathcal{D}\,. \tag{50}$$

Thus, elementary sentences which are true for all $\mathbb{F}_p((t))$ can be transferred to $\prod_p \mathbb{F}_p((t))/\mathcal{D}$, from there via (47) to $\prod_p \mathbb{Q}_p/\mathcal{D}$, and from there, by varying over all possible ultrafilters on the set of primes, to almost all $\mathbb{Q}_p$. The elementary sentences we are interested in are deduced from the analogue of Artin's conjecture which holds for all $\mathbb{F}_p((t))$, as proved by Serge Lang [L1].

Since the proof of the Ax–Kochen–Ershov Theorem, the Ax–Kochen–Ershov principle (46) has also been proved for other classes of valued fields, like p-adically closed fields, or algebraically maximal fields satisfying Kaplansky's hypothesis A ("algebraically maximal" is stronger than "henselian" if the residue characteristic is positive, as we have seen in Example 8.7). The proofs used, more or less explicitly, that those fields have unique maximal immediate extensions. But this is not necessary for the validity of the Ax–Kochen–Ershov principle (46). Using instead the Generalized Stability Theorem (Theorem 14.1 below) and the Henselian Rationality of Immediate Function Fields (Theorem 17.5 below), I proved that Ax–Kochen–Ershov principle (46) also holds for all tame fields ([K1], [K2]).

There is another version of (46) which will bring us closer to applications in algebraic geometry. There is a notion which in the past years has turned out to be more basic and flexible than that of elementary equivalence. Through general tools of model theory (like Theorem 13.6 below), notions like elementary equivalence can often be reduced to it. Take $\mathfrak{B}$ to be an $\mathfrak{L}$-structure, and $\mathfrak{A}$ a substructure of $\mathfrak{B}$. We form a language $\mathcal{L}(A)$ by adjoining the universe A of $\mathfrak{A}$ to the language $\mathcal{L}$. That is, in the language $\mathcal{L}(A)$ we have a constant symbol for every element of the structure $\mathfrak{A}$, so we can talk about every single element. We say that $\mathfrak{A}$ **is existentially closed in** $\mathfrak{B}$, denoted by $\mathfrak{A} \prec_\exists \mathfrak{B}$, if every existential $\mathcal{L}(A)$-sentence holds already in $\mathfrak{A}$ if it holds in $\mathfrak{B}$. (The other direction is trivial: if something exists in $\mathfrak{A}$, then it also exists in $\mathfrak{B}$.) Let us illustrate the use of this notion by three important examples.

Example 12.4 Take a field extension $L|K$. If $K \prec_\exists L$ in the language of fields, then K is relatively algebraically closed in L. To see this, take $a \in L$ algebraic over K. Take $f = X^n + c_{n-1}X^{n-1} + \ldots + c_0 \in K[X]$ to be the minimal polynomial of a over K. Since $f(a) = 0$, we know that the existential $\mathcal{L}_{\mathrm{F}}(K)$-sentence "$\exists X\, X^n + c_{n-1}X^{n-1} + \ldots + c_0 = 0$" holds in L. ("X^n" is an abbreviation for "$X \cdot \ldots \cdot X$" where X appears n times. Observe that we need the constants from K since we use the coefficients c_i in our sentence.) Since $K \prec_\exists L$, it must also hold in K. That means, that f also has a root in K. But as a minimal polynomial, f is irreducible over K. This shows that f must be linear, i.e., $a \in K$. One can also show that $L|K$ must be separable.

Similarly, let $G \subset H$ be an extension of abelian groups such that $G \prec_\exists H$ in the language of groups. Take $\alpha \in H$ such that $n\alpha \in G$ for some integer $n > 0$. Set $\beta = n\alpha$. Then the existential $\mathcal{L}_{\mathrm{G}}(G)$-sentence "$\exists X\, nX = \beta$" holds in H. (Here, "nX" is just an abbreviation for "$X + \ldots + X$" where X appears n times.) Hence, it must also hold in G. That is, $\alpha = \beta/n \in G$. Hence, we have:

Lemma 12.5 *If $L|K$ is an extension of fields such that $K \prec_\exists L$ in the language of fields, then K is relatively algebraically closed in L and $L|K$ is separable. If $G \subset H$ is an extension of abelian groups such that $G \prec_\exists H$ in the language of groups, then H/G is torsion free.*

$\diamondsuit$

Example 12.6 Take a function field $F|K$ such that $K \prec_\exists F$ in the language of fields. Since $F|K$ is separable by Lemma 12.5, we can choose a separating transcendence basis $t_1, \ldots, t_k$ of $F|K$ and an element $z \in F$ such that $F = K(t_1, \ldots, t_k, z)$ with z separable-algebraic over $K(t_1, \ldots, t_k)$. Take $f \in K[X_1, \ldots, X_k, Z]$ to be the irreducible polynomial of z over $K[t_1, \ldots, t_k]$ (obtained from the minimal polynomial by multiplication with the common denominator of the coefficients from $K(t_1, \ldots, t_k)$). We have that $f(t_1, \ldots, t_k, z) = 0$. Since z is a simple root of f, we also have that $\frac{\partial f}{\partial Z}(t_1, \ldots, t_k, z) \neq 0$. Further, we take n arbitrary non-zero elements $z_1, \ldots, z_n \in F$ which we write as $g_1/h_1, \ldots, g_n/h_n$ with non-zero elements $g_i, h_i \in K[t_1, \ldots, t_k, z]$. Now the existential $\mathcal{L}_{\mathrm{F}}(K)$-sentence

$$\begin{aligned} \text{“}\, \exists Y_1 \ldots \exists Y_k \exists Y\ f(Y_1, \ldots, Y_k, Y) = 0 \ \wedge\ \frac{\partial f}{\partial Z}(Y_1, \ldots, Y_k, Y) \neq 0 \ \wedge\ g_1(Y_1, \ldots, Y_k, Y) \neq 0 \ \wedge \ldots \\ \ldots \wedge\ g_n(Y_1, \ldots, Y_k, Y) \neq 0 \ \wedge\ h_1(Y_1, \ldots, Y_k, Y) \neq 0 \ \wedge \ldots \wedge\ h_n(Y_1, \ldots, Y_k, Y) \neq 0\,\text{”} \end{aligned}$$

holds in F. Hence it must also hold in K, that is, there are elements $c_1, \ldots, c_k, d \in K$ such that $f(c_1, \ldots, c_k, b) = 0$, $\frac{\partial f}{\partial Z}(c_1, \ldots, c_k, d) \neq 0$ and $g_i(c_1, \ldots, c_k, b)/h_i(c_1, \ldots, c_k, d) \neq 0, \infty$.

On $K[t_1, \ldots, t_k]$, we have the homomorphism given by $t_i \mapsto c_i$, or equivalently, by $t_i - c_i \mapsto 0$. As in Example 2.2, we can construct a place P of maximal rank of $K(t_1, \ldots, t_k) = K(t_1 - c_1, \ldots, t_k - c_k)$ which extends this homomorphism. Its residue field is K. Now we consider the polynomial $g(Z) = f(t_1, \ldots, t_k, Z)$. Its reduction modulo P is the polynomial $f(c_1, \ldots, c_k, Z)$ which admits d as a simple root. Hence by Hensel's Lemma, $g(Z)$ has a root z' in the henselization $(K(t_1, \ldots, t_k)^h, P)$ of $(K(t_1, \ldots, t_k), P)$. Thus, the assignment $z \mapsto z'$ defines an embedding of F over $K(t_1, \ldots, t_k)$ in $K(t_1, \ldots, t_k)^h$, and pulling the place P from the image of this embedding back to F, we obtain on F a place P with residue field K and having maximal rank. In addition, it satisfies $z_i P \neq 0, \infty$. We have proved:

Lemma 12.7 *Take a function field $F|K$ such that $K \prec_\exists F$ in the language of fields. Take non-zero elements $z_1, \ldots, z_n \in F$. Then there exists a rational place P of $F|K$ of maximal rank and such that $z_i P \neq 0, \infty$ for $1 \leq i \leq n$, and a model of $F|K$ on which P is centered at a smooth point.*

Note that it was crucial for our proof that $F|K$ is finitely generated (because elementary sentences can only talk about finitely many elements). If a field extension $L|K$ is not finitely generated, then there may not exist a place P of L such that $LP = K$, even if $K \prec_\exists L$. ♢

Example 12.8 A field K is called a **large field** (cf. [POP]) if every smooth curve over K has infinitely many K-rational points, provided it has at least one K-rational point. For the proof of the following theorem, see [K2] or [K3].

Theorem 12.9 *A field K is large if and only if $K \prec_\exists K((t))$ (in the language of fields).*

♢

The use of "$\prec_\exists$" gives us another version of the Ax–Kochen–Ershov principle:

$$vK \prec_\exists vL \text{ and } Kv \prec_\exists Lv \;\Rightarrow\; (K,v) \prec_\exists (L,v)\;. \tag{51}$$

This principle also holds for the classes of valued fields that we mentioned above: henselian fields of residue characteristic 0, p-adically closed fields, algebraically maximal fields satisfying Kaplansky's hypothesis A, tame fields. A short proof for the case of henselian fields of residue characteristic 0 is given in the appendix of [KP]. This form of the Ax–Kochen–Ershov principle is applied in the proof of Theorem 20.1 below.

13 Saturation and embedding lemmas

How can a principle like (51) be proved? In fact, nice model theoretic results often just represent a good algebraic structure theory. Indeed, using a very useful model theoretic tool, we can easily transfer "$\prec_\exists$" to an algebraic fact. The tool is that of a **κ-saturated model** (where κ is a cardinal number). Saturation is a property which is not elementary, quite similar to "complete" or "maximal", but still different (in fact, "maximal" and "saturated" are to some extent mutually exclusive). Before defining "κ-saturated", I want to illustrate its meaning by an example which plays a remarkable role in the theory of ordered structures. We take fields, but the same can be done for ordered abelian groups and other ordered structures.

Example 13.1 Take any ordinal number α and an ordered field $(K,<)$. For $A, B \subset K$ we will write $A < B$ if every element of A is smaller than every element of B. Now $(K,<)$ is said to be an **η_α-field** if for every two subsets $A, B \subset K$ of cardinality less than $\aleph_\alpha$ (= the α-th cardinal number) such that $A < B$, there is some $c \in K$ such that $A < \{c\} < B$, i.e., c lies between A and B. Note that because of the restriction of the cardinality of A and B, this does not mean that $(K,<)$ is cut-complete (in fact, the only cut-complete field is $\mathbb{R}$, while there is an abundance of η_α-fields).

Given $A, B \subset K$ such that $A < B$, we consider the collection of elementary sentences (in the language of ordered fields with constants from K) "$a < X$", $a \in A$, and "$X < b$", $b \in B$. It is clear that if we take any finite subset of these, then there is some element in K that we can insert for X so that all of these finitely many sentences hold. That is, our collection of sentences is **finitely realizable** in $(K,<)$. Now if $(K,<)$ is κ-saturated, then this tells us the following: if the cardinality of $A \cup B$ is smaller than κ, then there is an element $c \in K$ which simultaneously satisfies *all* of our above sentences (we say that c **realizes** the above set of elementary sentences). But this means that $A < \{c\} < B$. So we see that every $\aleph_\alpha$-saturated ordered field is an η_α-field. ♢

Let us extract a definition from our example. An $\mathcal{L}$-structure $\mathfrak{A}$ will be called **κ-saturated** if for every subset S of its universe A of cardinality less than κ, every set of elementary $\mathcal{L}(S)$-sentences is realizable in $\mathfrak{A}$, provided that it is finitely realizable in $\mathfrak{A}$. To express the fact that there are enough κ-saturated $\mathcal{L}$-structures, we need one further notion. Given an $\mathcal{L}$-structure $\mathfrak{B}$ with substructure $\mathfrak{A}$, we say that $\mathfrak{B}$ **is an elementary extension of** $\mathfrak{A}$ and write $\mathfrak{A} \prec \mathfrak{B}$ if *every* $\mathcal{L}(A)$-sentence holds in $\mathfrak{A}$ if and only if it holds in $\mathfrak{B}$. (So in contrast to "existentially closed", here we do not restrict the scope to existential sentences.) For example, an algebraically closed field K is existentially closed in every extension field, and every algebraically closed extension field of K is an elementary extension of K. If $K \prec L$, then K is existentially closed in every intermediate field.

We are going to state the theorem which provides us with sufficiently many κ-saturated structures. It is a consequence of one of the basic theorems of model theory:

Theorem 13.2 (Compactness Theorem) *A set of elementary $\mathcal{L}$-sentences has a model if and only if each of its finite subsets has a model.*

For the proof of the next theorems, see [CK] or [K2].

Theorem 13.3 *For every infinite $\mathcal{L}$-structure $\mathfrak{A}$ and every large enough κ there exists a κ-saturated elementary extension of $\mathfrak{A}$.*

Here "large enough" means: larger than the cardinality of the language (which, if infinite, will in most cases be the cardinality of the set of constants appearing in the language), and larger than the cardinality of the universe of $\mathfrak{A}$. Now the reduction of "$\prec_\exists$" to an algebraic statement is done as follows:

Theorem 13.4 *Take an $\mathcal{L}$-structure $\mathfrak{B}$ with substructure $\mathfrak{A}$. Take κ larger than the cardinality of $\mathcal{L}$ and the cardinality of the universe of $\mathfrak{B}$. Further, choose a κ-saturated elementary extension $\mathfrak{A}^*$ of $\mathfrak{A}$. Then $\mathfrak{A} \prec_\exists \mathfrak{B}$ holds if and only if there is an embedding of $\mathfrak{B}$ over $\mathfrak{A}$ in $\mathfrak{A}^*$.*

If there is an embedding of $\mathfrak{B}$ over $\mathfrak{A}$ in $\mathfrak{A}^*$, then every existential sentence holding in $\mathfrak{B}$ will carry over to the image of $\mathfrak{B}$ in $\mathfrak{A}^*$, from where it goes up to $\mathfrak{A}^*$. Since $\mathfrak{A} \prec \mathfrak{A}^*$, it then also holds in $\mathfrak{A}$.

A nice additional feature of saturation is the following:

Theorem 13.5 *There is an embedding of $\mathfrak{B}$ over $\mathfrak{A}$ in $\mathfrak{A}^*$ already if for every finitely generated subextension $\mathfrak{A} \subset \mathfrak{B}'$ of $\mathfrak{A} \subset \mathfrak{B}$ there is an embedding of $\mathfrak{B}'$ over $\mathfrak{A}$ in $\mathfrak{A}^*$.*

So if we have a field extension $L|K$ and want to prove that $K \prec_\exists L$, we take a κ-saturated elementary extension K^* of K, for κ larger than the cardinality of L, and seek to embed L over K in K^*. By the last theorem, we only have to show that every finitely generated subextension $F|K$ of $L|K$ embeds in K^*. But a finitely generated extension is a function field (in view of Lemma 12.5 we can exclude the case where $F|K$ is algebraic).

If K is an algebraically closed field, then so is K^* because it is an elementary extension of K. The assumption that K^* is κ-saturated with κ larger than the cardinality of L implies that the transcendence degree of $K^*|K$ is at least as large as that of $L|K$. So we see that L embeds over K in K^* (even without employing the last theorem). This proves that every algebraically closed field is existentially closed in every extension field.

Let's see how we can prove a principle like (51) with the above tools. We take a κ-saturated elementary extension $(K,v)^* = (K^*,v^*)$ of (K,v) (with respect to the language of valued fields). Then it is easy to prove that v^*K^* is a κ-saturated elementary extension of vK (with respect to the language of ordered groups) and that K^*v^* is a κ-saturated elementary extension of Kv (with respect to the language of fields). Thus, we see from Theorem 13.4 that $vK \prec_\exists vL$ implies that vL embeds over vK in v^*K^*, and that $Kv \prec_\exists Lv$ implies that Lv embeds over Kv in K^*v^*. So Theorem 13.4 shows that we can prove (51) by an **embedding lemma** of the form: *If vL embeds over vK in v^*K^* and Lv embeds over Kv in K^*v^* and* (additional assumptions) *then (L,v) embeds over K in $(K,v)^*$ (as a valued field).* See Example 14.3 and Example 17.6 below for two different cases and a sketch of the proof of (51) for tame fields.

To conclude this section, let us come back to elementary extensions. An $\mathcal{L}$-theory $\mathcal{T}$ is called **model complete** if for every two models $\mathfrak{A}$ and $\mathfrak{B}$ of $\mathcal{T}$ such that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ we have that $\mathfrak{A} \prec \mathfrak{B}$. This is closely connected to the relation $\mathfrak{A} \prec_\exists \mathfrak{B}$ through the following important criterion (cf. [K2] or [CK]):

Theorem 13.6 (Robinson's Test)
Assume that for every two models $\mathfrak{A}$ and $\mathfrak{B}$ of $\mathcal{T}$ such that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ we have that $\mathfrak{A} \prec_\exists \mathfrak{B}$. Then $\mathcal{T}$ is model complete.

For example, this theorem together with the fact that every algebraically closed field is existentially closed in every extension field shows that the axiom system of algebraically closed fields is model complete. Furthermore, with this theorem together with Theorem 13.4, Theorem 7.2, Theorem 2 of [KA1] and Theorem 5.1, it is not hard to prove the following theorem of Abraham Robinson:

Theorem 13.7 *The elementary axiom system of non-trivially valued algebraically closed fields is model complete.*

Observe that we do not need "side conditions" about the value groups and the residue fields here (because they are divisible and algebraically closed, respectively). But there is also an Ax–Kochen–Ershov principle with $\prec$ in the place of $\prec_\exists$ that again holds for the classes of valued fields which I mentioned above.

Example 13.8 Another simple but useful example for a fact proved by an embedding lemma is the following:

Theorem 13.9 *If (K,v) is henselian and (L,v) is a separable extension of (K,v) within its completion, then $(K,v) \prec_\exists (L,v)$.*

This fact can be seen as the (much simpler) "field version of Artin Approximation". It was observed in the 1960s by Yuri Ershov; for a proof, see [K2]. Together with Theorem 12.9, a modification of Theorem 14.4 and the transitivity of $\prec_\exists$, this theorem (applied to $(K(t),v)^h$) can be used to prove (cf. [K2], [K3]):

Theorem 13.10 *If the field K admits a henselian valuation, then $K \prec_\exists K((t))$, i.e., K is a large field.*
$\diamondsuit$

14 The Generalized Grauert–Remmert Stability Theorem

Let us return to our problem of inertial generation as considered at the end of Section 7. Our problem was to show that the finite immediate extension (33) of henselian fields is trivial. If it is not, then by Corollary 8.3 it has non-trivial defect (which then is equal to its degree). So we would like to show that the field $F_0(\eta)^h$ is a defectless field. The reason for this would have to lie in the special way we have constructed this field.

At this point, let us invoke a deep and important theorem from the theory of valued function fields ([K1], [K2]). For historical reasons, I call it the **Generalized Grauert–Remmert Stability Theorem** although I do not like the notion "stable". It is one of those words in mathematics which is very often used in different contexts, but in most cases does not reflect its meaning. I replace it by "defectless".

If $(F|K,v)$ is an extension of valued fields of finite transcendence degree, then by inequality (30) of Corollary 7.3, $\operatorname{trdeg} F|K - \operatorname{trdeg} Fv|Kv - \operatorname{rr}(vF/vK)$ is a non-negative integer. It will be called the **transcendence defect** of $(F|K,v)$. We say that $(F|K,v)$ is **without transcendence defect** if the transcendence defect is 0.

Theorem 14.1 *Let $(F|K,v)$ be a valued function field without transcendence defect. If (K,v) is a defectless field, then also (F,v) is a defectless field.*

This theorem has a long and interesting history. Hans Grauert and Reinhold Remmert [GR] first proved it in a very restricted case, where (K,v) is a complete discrete valued field and (F,v) is discrete too. There are generalizations by Laurent Gruson [GRU], Michel Matignon, and Jack Ohm [OH]. All of these generalizations are restricted to the case $\mathrm{trdeg}\, F|K = \mathrm{trdeg}\, Fv|Kv$, the case of **constant reduction**. The classical origin of it is the study of curves over number fields and the idea to reduce them modulo a p-adic valuation. Certainly, the reduction should again render a curve, this time over a finite field. This is guaranteed by the condition $\mathrm{trdeg}\, F|K = \mathrm{trdeg}\, Fv|Kv$, where F is the function field of the curve and Fv will be the function field of its reduction. Naturally, one seeks to relate the genus of $F|K$ to that of $Fv|Kv$. Several authors proved **genus inequalities**. To illustrate the use of the defect, we cite an inequality proved by Barry Green, Michel Matignon and Florian Pop in [GMP1]. Let $F|K$ be a function field of transcendence degree 1 and assume that K coincides with the constant field of $F|K$ (the relative algebraic closure of K in F). Let $v_1,\ldots,v_s$ be distinct constant reductions of $F|K$ which have a common restriction to K. Then:

$$1 - g_F \leq 1 - s + \sum_{i=1}^{s} \delta_i e_i r_i (1 - g_i) \tag{52}$$

where g_F is the genus of $F|K$ and g_i the genus of $Fv_i|Kv_i$, r_i is the degree of the constant field of $Fv_i|Kv_i$ over Kv_i, δ_i is the defect of $(F^{h(v_i)}|K^{h(v_i)},v_i)$ where "$.^{h(v_i)}$" denotes the henselization with respect to v_i, and $e_i = (v_iF : v_iK)$ (which is always finite in the constant reduction case by virtue of Corollary 7.3). It follows that constant reductions v_1, v_2 with common restriction to K and $g_1 = g_2 = g_F \geq 1$ must be equal. In other words, for a fixed valuation on K there is at most one extension v to F which is a **good reduction**, that is, (i) $g_F = g_{Fv}$, (ii) there exists $f \in F$ such that $vf = 0$ and $[F : K(f)] = [Fv : Kv(fv)]$, (iii) Kv is the constant field of $Fv|Kv$. An element f as in (ii) is called a **regular function**.

More generally, f is said to have the **uniqueness property** if fv is transcendental over Kv and the restriction of v to $K(f)$ has a unique extension to F. In this case, $[F : K(f)] = \delta\, e\, [Fv : Kv(fv)]$ where δ is the defect of $(F^h|K^h,v)$ and $e = (vF : vK(f)) = (vF : vK)$. If K is algebraically closed, then $e = 1$, and it follows from the Stability Theorem that $\delta = 1$; hence in this case, every element with the uniqueness property is regular.

It was proved in [GMP2] that F has an element with the uniqueness property already if the restriction of v to K is henselian. The proof uses Theorem 13.7 and ultraproducts of function fields. Elements with the uniqueness property also exist if vF is a subgroup of $\mathbb{Q}$ and Kv is algebraic over a finite field. This follows from work in [GMP4] where the uniqueness property is related to the **local Skolem property** which gives a criterion for the existence of algebraic v-adic integral solutions on geometrically integral varieties.

As an application to rigid analytic spaces, the Stability Theorem is used to prove that the quotient field of the free Tate algebra $T_n(K)$ is a defectless field, provided that K is. This in turn is used to deduce the **Grauert–Remmert Finiteness Theorem**, in a generalized version due to Gruson; see [BGR].

Surprisingly, it was not before the model theory of valued fields developed in positive characteristic that an interest in a generalized version of the Stability Theorem arose. But a criterion like Robinson's Test (Theorem 13.6) forces us to deal with arbitrary extensions of arbitrarily large valued fields. For instance, it is virtually impossible to restrict oneself to rank 1 in order to prove model completeness or completeness of a class of valued fields. And the extensions $(L|K,v)$ in question won't obey a restriction like "vL/vK is a torsion group". Therefore, I had to prove the above Generalized Stability Theorem. At that time, I had not heard of the Grauert–Remmert Theorem, so I gave a purely valuation theoretic proof ([K1], [K2]), not based on the original proofs of Grauert–Remmert or Gruson like the other cited generalizations.

Later, I was amazed to see that the Generalized Stability Theorem is also the suitable version for an application to the problem of local uniformization. (If your valuation v is trivial on the base field K and you ask that $\mathrm{trdeg}\, F|K = \mathrm{trdeg}\, Fv|Kv$, then $vL/\{0\}$ is torsion, so $vL = \{0\}$ and v is also trivial on F; this is not quite the case we are interested in.) So let's now describe this application. By our assumption at the end of Section 7, P is an Abhyankar place on F and hence also on $F_0(\eta)$. That is, $(F_0(\eta)|K,P)$ is a function field without transcendence defect. As P is trivial on K, also v_P is trivial on

K. But a trivially valued field (K,v) is always a defectless field since for every finite extension $L|K$ we have that $[L:K]=[Lv:Kv]$. Hence by the Generalized Stability Theorem, $(F_0(\eta),v_P)$ is a defectless field. By Theorem 8.4, also $(F_0(\eta)^h,v_P)$ is a defectless field. Therefore, since $(F^h|F_0(\eta)^h,v_P)$ is an immediate extension, Corollary 8.3 shows that it must be trivial. We have proved that $F^h=F_0(\eta)^h$. By construction, $F_0(\eta)^h$ was a subfield of the absolute inertia field of (F_0,P). Hence also F is a subfield of that absolute inertia field, showing that (F,P) is inertially generated. We have thus proved the first part of the following theorem (I leave the rest of the proof to you as an exercise; cf. [K6]):

Theorem 14.2 *Assume that P is an Abhyankar place of $F|K$ and that $FP|K$ is a separable extension. Then $(F|K,P)$ is inertially generated. If in addition $FP=K$ or $FP|K$ is a rational function field, then $(F|K,P)$ is henselian generated. In all cases, if $v_PF=\mathbb{Z}v_Px_1\oplus\ldots\oplus\mathbb{Z}v_Px_\rho$ and $y_1P,\ldots,y_\tau P$ is a separating transcendence basis of $FP|K$, then $\{x_1,\ldots,x_\rho,y_1,\ldots,y_\tau\}$ is a generating transcendence basis.*

Example 14.3 Let's now describe the model theoretic use of the Generalized Stability Theorem. Take $(L|K,v)$ of finite transcendence degree and without transcendence defect. Note that then for every function field $F|K$ contained in $L|K$, also the extension $(F|K,v)$ is without transcendence defect. Further, we assume that (K,v) is a non-trivially valued henselian defectless field and that $vK\prec_\exists vL$ and $Kv\prec_\exists Lv$. Making again essential use of the Generalized Stability, one proves a generalization of Theorem 14.2 to the case of non-trivially valued ground fields. Using also Theorem 7.2, Lemma 12.5 and Theorem 5.15, it is easy to show that the embeddings of vL and Lv (which induce embeddings of vF and Fv) lift to an embedding of (F,v) in $(K,v)^*$. This proves:

Theorem 14.4 *Take a non-trivially valued henselian defectless field (K,v) and an extension $(L|K,v)$ of finite transcendence degree, without transcendence defect. If $vK\prec_\exists vL$ and $Kv\prec_\exists Lv$, then $(K,v)\prec_\exists(L,v)$.*

This theorem shows the advantage of the notion "$\prec_\exists$" since there is no analogue for "$\prec$". $\diamondsuit$

To conclude this section, we give a short sketch of a main part of the proof of the Generalized Stability Theorem. This is certainly interesting because very similar methods have been used by Shreeram Abhyankar for the proof of his results in positive characteristic (see, e.g., [A1], [A6], [A7]). We have to prove that a certain henselian field (L,v) is a defectless field. We take an arbitrary finite extension $(L'|L,v)$ and have to show that it has trivial defect. We may assume that this extension is separable since the case of purely inseparable extensions can be considered separately and is much easier. Looking at $(L'|L,v)$, we are completely lost since we have not the slightest chance to develop a good structure theory. But we only have to deal with the defect, and we remember that a defect only appears if extensions beyond the absolute ramification field L^r are involved. So instead of $(L'|L,v)$ we consider the extension $(L'.L^r|L^r,v)$ which has the same defect as $(L'|L,v)$, although it will in general not have the same degree (the use of this fact reminds of Abhyankar's "Going Up" and "Coming Down"; cf. [A1]). Now we use the fact that by Theorem 5.10 the separable-algebraic closure of L^r is a p-extension. It follows that its subextension $L'.L^r|L^r$ is a tower of Artin–Schreier extensions (cf. Remark 5.11). Since the defect is multiplicative, to prove that $(L'.L^r|L^r,v)$ has trivial defect it suffices to show that each of these Artin–Schreier extensions has trivial defect. So we take such an extension, generated by a root ϑ of an irreducible polynomial X^p-X-c over some field L'' in the tower. By what we learned in Example 4.4, $vc\leq 0$. If $vc=0$, the extension (if it is not trivial) would correspond to a proper separable extension of the residue field; but as we are working beyond the absolute ramification field, our residue field is already separable-algebraically closed. So we see that $vc<0$. If $b\in L''$, then also the element $\vartheta-b$ generates the same extension. By the additivity of the polynomial X^p-X, $\vartheta-b$ is a root of the Artin–Schreier polynomial $X^p-X-(c-b^p+b)$. The idea now is to use this principle to deduce a "normal form" for c from which we can read off that the extension has trivial defect. Still, we are quite lost if we do not make some reductions beforehand. First, it is clear that one can proceed by induction on the transcendence degree; so we can reduce to the case of $\mathrm{trdeg}\,L|K=1$. Second, as v may not be trivial on K, it may have a very large rank. By general valuation theory, one has to reduce first to finite rank and then to rank 1. This being done, one can show that c can be taken to be a polynomial $g\in K[x]$, where $x\in L''$ is transcendental over K. Now the idea is the following: if $k = p \cdot \ell$ and g contains a non-zero summand $c_k x^k$, then we replace it by $c_k^{1/p} x^\ell$. This is done by setting $b = c_k^{1/p} x^\ell$ in the above computation. In this way one eliminates all p-th powers in g, and the thus obtained normal form for c will show that the extension has trivial defect. This method (which I call "Artin–Schreier surgery") seems to have several applications; I used it again to prove a quite different result (Theorem 17.5 below). It can also be found in the paper [EPP].

Let us note that the Artin–Schreier polynomials appear in Abhyankar's work in a somewhat disguised form. This is because the coefficients have to lie in the local ring he is working in. For example, if $vc < 0$, we would rather prefer to have a polynomial having coefficients in the valuation ring, defining the same extension as $X^p - X - c$. Setting $X = cY$, we find that if ϑ is a root of $X^p - X - c$, then ϑ/c is a root of $Y^p - c^{1-p}Y - c^{1-p}$ with $vc^{1-p} = (1-p)vc > 0$. Therefore, Abhyankar considers polynomials of the form $Z^p - c_1 Z - c_2$ (cf. e.g., [A1], page 515, [A6], Theorem (2.2), or [A7], page 34). In an extension obtained from L by adjoining a $(p-1)$th root of c_1 (if (L,v) is henselian, then such an extension is tame), this polynomial can be transformed back into an Artin–Schreier polynomial.

Having shown inertial generation for function fields with Abhyankar places, let us return to our problem of local uniformization.

15 Relative local uniformization

Throughout, I have stressed the function field aspect of local uniformization. I have shown that function fields with Abhyankar places can be generated in a nice way. I have talked about the algebraic elements satisfying the assumption of the Multidimensional Hensel's Lemma. The logical consequence of all this is to try to build up our function field step by step: first, choose a nice transcendence basis, according to Theorem 14.2, then find algebraic elements, one after the other, each of them satisfying the assumptions of Hensel's Lemma over the previously generated field. This is the origin of the following definition.

Take a finitely generated extension $F|K$, not necessarily transcendental, and a place P on F, not necessarily trivial on K. We write $\mathcal{O}_F$ for the valuation ring of P on F, and $\mathcal{O}_K$ for the valuation ring of its restriction to K. Further, take elements $\zeta_1, \ldots, \zeta_m \in \mathcal{O}_F$. We will say that $(F|K, P)$ **is uniformizable with respect to** $\zeta_1, \ldots, \zeta_m$ if there are

- a transcendence basis $T = \{t_1, \ldots, t_s\} \subset \mathcal{O}_F$ of $F|K$ (which may be empty),
- elements $\eta_1, \ldots, \eta_n \in \mathcal{O}_F$, with $\zeta_1, \ldots, \zeta_m$ among them,
- polynomials $f_i(X_1, \ldots, X_n) \in \mathcal{O}_K[t_1, \ldots, t_s, X_1, \ldots, X_n]$, $1 \leq i \leq n$,

such that $F = K(t_1, \ldots, t_s, \eta_1, \ldots, \eta_n)$, and

(U1) for $i < j$, X_j does not occur in f_i,
(U2) $f_i(\eta_1, \ldots, \eta_n) = 0$ for $1 \leq i \leq n$,
(U3) $(\det J_f(\eta_1, \ldots, \eta_n))P \neq 0$.

Assertion **(U1)** implies that J_f is triangular. Assertion **(U3)** says that $f_1, \ldots, f_n$ and $\eta_1, \ldots, \eta_n$ satisfy the assumption (5) of the Multidimensional Hensel's Lemma. By the triangularity, this implies that for each i, f_i and η_i satisfy the hypothesis of Hensel's Lemma over the ground field $K(t_1, \ldots, t_s, \eta_1, \ldots, \eta_{i-1})$.

We say that $(F|K, P)$ **is uniformizable** if it is uniformizable with respect to *every* choice of finitely many elements in $\mathcal{O}_F$.

Now assume in addition that P is trivial on K. Then $\mathcal{O}_K = K$, and the P-residues of the coefficients are obtained by just replacing t_j by $t_j P$, for $1 \leq j \leq n$. Hence if we view the polynomials f_i as polynomials in the variables $Z_1, \ldots, Z_s, X_1, \ldots, X_n$, then assertion **(U3)** means that at the point $(t_1 P, \ldots, t_s P, \eta_1 P, \ldots, \eta_n P)$ the Jacobian matrix has maximal rank. This assertion says that on the variety defined over K by the f_i (and having generic point $(t_1, \ldots, t_s, \eta_1, \ldots, \eta_n)$ and function field F), the place P is centered at the smooth point $(t_1 P, \ldots, t_s P, \eta_1 P, \ldots, \eta_n P)$.

By uniformizing with respect to the ζ's, we obtain the following important information: if we have already a model V of $F|K$ with generic point $(z_1, \ldots, z_k)$, where $z_1, \ldots, z_k \in \mathcal{O}_F$, then we can choose our new model $\mathcal{V}$ of $F|K$ in such a way that the local ring of the center of P on $\mathcal{V}$ contains the local ring of the center $(z_1P,\dots,z_kP)$ of P on V. For this, we only have to let $z_1,\dots,z_k$ appear among the ζ's. In fact, Zariski proved the following **Local Uniformization Theorem** (cf. [Z]):

Theorem 15.1 *Suppose that $F|K$ is a function field of characteristic 0, P is a place of $F|K$, and $\zeta_1,\dots,\zeta_m$ are elements of $\mathcal{O}_F$. Then $(F|K,P)$ is uniformizable with respect to $\zeta_1,\dots,\zeta_m$.*

But the ζ's also play another role. Through their presence, the above property becomes transitive (see [K6]):

Theorem 15.2 (Transitivity of Relative Uniformization) *If $E|F$ is uniformizable with respect to $\zeta_1,\dots,\zeta_m$ and $F|K$ is uniformizable with respect to certain finitely many elements derived from $E|F$ and $\zeta_1,\dots,\zeta_m$, then $E|K$ is uniformizable with respect to $\zeta_1,\dots,\zeta_m$. In particular, if $E|F$ and $F|K$ are uniformizable, then so is $E|K$.*

As we do not require that P is trivial on K, we call the property defined above **relative (local) uniformization**. Its transitivity enables us to build up our function field step by step by extensions which admit relative uniformization. I give examples of uniformizable extensions; for the proofs, see [K5], [K6], [K7].

I) We consider a function field $F|K$ and a place P of F such that v_PK is a convex subgroup of v_PF. The latter always holds if P is trivial on K since then, $v_PK=\{0\}$. We take elements $x_1,\dots,x_\rho$ in F such that $v_Px_1,\dots,v_Px_\rho$ form a maximal set of rationally independent elements in v_PF modulo v_PK. Further, we take elements $y_1,\dots,y_\tau$ in F such that $y_1P,\dots,y_\tau P$ form a transcendence basis of FP over KP.

Proposition 15.3 *In the situation described above, $(K(x_1,\dots,x_\rho,y_1,\dots,y_\tau)|K,P)$ is uniformizable. More precisely, the transcendence basis T can be chosen of the form $\{x'_1,\dots,x'_\rho,y_1,\dots,y_\tau\}$, where $x'_1,\dots,x'_\rho\in\mathcal{O}_{K(x_1,\dots,x_\rho)}$ and for some $c\in\mathcal{O}_K\setminus\{0\}$ (with $c=1$ if P is trivial on K), the elements $cx'_1,\dots,cx'_\rho$ generate the same multiplicative subgroup of $K(x_1,\dots,x_\rho)^\times$ as $x_1,\dots,x_\rho$.*

The proof uses Theorem 7.2. It also uses the following lemma, which was proved (but not explicitly stated) by Zariski in [Z] for subgroups of $\mathbb{R}$, using the **Algorithm of Perron**. I leave it as an easy exercise to you to prove the general case by induction on the (finite!) rank of the ordered abelian group. However, an instant proof of the lemma can also be found in [EL] (Theorem 2.2), and I am sure there are several more authors who reproved the lemma, not knowing about Zariski's application.

Lemma 15.4 *Let G be a finitely generated ordered abelian group. Take any positive elements $\alpha_1,\dots,\alpha_\ell$ in G. Then there exist positive elements $\gamma_1,\dots,\gamma_\rho\in G$ such that $G=\mathbb{Z}\gamma_1\oplus\ldots\oplus\mathbb{Z}\gamma_\rho$ and every α_i can be written as a sum $\sum_j n_{ij}\gamma_j$ with non-negative integers n_{ij}.*

II) The next result is based on Kaplansky's work [KA1]. First, we need:

Lemma 15.5 *Let $(K(z)|K,P)$ be an immediate transcendental extension. Assume further that (K,P) is separable-algebraically maximal or that $(K(z),P)$ lies in the completion of (K,P). Then for every polynomial $f\in K[X]$, the value $v_Pf(a)$ is fixed for all $a\in K$ sufficiently close to z. That is,*

$$\begin{gathered}\forall f\in K[X]\;\exists\alpha\in v_PK\;\exists\beta\in\{v_P(z-b)\mid b\in K\}\;\forall a\in K:\\ v_P(z-a)\geq\beta\Rightarrow v_Pf(a)=\alpha\,.\end{gathered}\tag{53}$$

Kaplansky proves that if (53) does not hold, then there is a proper immediate algebraic extension of (K,P). If $(K(z),P)$ does not lie in the completion of (K,P), then this can be transformed into a proper immediate separable-algebraic extension ([K1], [K2]; the proof uses a variant of the Theorem on the Continuity of Roots). The existence of such an extension is excluded if (K,v) is separable-algebraically maximal. If on the other hand we assume that $(K(z),P)$ lies in the completion of (K,P), then one can show that if f does not satisfy (53), then $v_Pf(z)=\infty$. But this means that $f(z)=0$, contradicting the assumption that $K(z)|K$ is transcendental. Note that by Lemma 11.8, every separably tame and hence also every tame field (K,P) satisfies the assumption of Lemma 15.5.

The following proposition is somewhat complementary to Proposition 15.3.

Proposition 15.6 *Let $(K(z)|K,P)$ be an immediate transcendental extension. If z satisfies (53), then $(K(z)|K,P)$ is uniformizable.*

III) The proof of the following proposition is quite easy; for the transcendental part, it uses Lemma 15.5 and Proposition 15.6.

Proposition 15.7 *Every separable extension of a valued field within its completion is uniformizable.*

The henselization of a valued field (K,P) is always a separable-algebraic extension. If (K,P) has rank 1, then moreover, the henselization lies in the completion of (K,P) (since in this case the completion is henselian). Therefore, Proposition 15.7 yields:

Corollary 15.8 *Assume that (K,P) has rank 1. If $(L|K,P)$ is a finite subextension of the henselization of (K,P), then it is uniformizable.*

IV) The following proposition is of interest in view of the inertial generation of function fields with Abhyankar places. Its proof is again quite easy.

Proposition 15.9 *Let (K,P) be a henselian field and $(L|K,P)$ a finite extension within the absolute inertia field of (K,P). Then $(L|K,P)$ is uniformizable.*

After these positive results, I have to talk about a serious problem. Throughout my work in the model theory of valued fields, my experience was that henselizations are very nice extensions and do not harm at all. Because of their universal property (Theorem 5.15), they behave well in embedding lemmas. Unfortunately, for the problem of local uniformization, this seems to be entirely different. While we have relative uniformization if the henselization lies in the completion, we get problems if this is not the case. And for a rank greater than 1, we cannot expect in general that the henselization lies in the completion (since the completion will in general not be henselian). This leads to the following important open problem:

Open Problem 3: Prove or disprove: every finite subextension in the henselization of a valued field is uniformizable.

This is a special case of a slightly more general problem, however. Again in view of inertial generation, we would like to know the following.

Open Problem 4: Assume that (K,v) is a field which is *not* henselian. Prove or disprove: every finite subextension in the absolute ramification field of (K,v) is uniformizable.

The obstruction is the following. Assume that $(L|K,v)$ is a finite subextension in the absolute ramification field of (K,v). Suppose there is an intermediate field L' such that $Lv = L'v$ and $[L' : K] = [L'v : Kv]$. The former yields that L lies in the henselization of L'. The latter yields that $(L'|K,v)$ admits relative local uniformization, which can be proved in exactly the same way as Proposition 15.9, although (K,v) need not be henselian. So by the transitivity of relative uniformization, the problem would be reduced to that of subextensions within the henselization. But such intermediate fields L' may not exist!

A closer look reveals that this problem is also the kernel of our problem about subextensions within the henselization. Indeed, if we have rank > 1, then P is the composition $P = Q\overline{Q}$ of two non-trivial places. Advanced ramification theory shows that if $(L|K,P)$ is a subextension of the henselization of (K,P), then $(L|K,Q)$ is a subextension of the absolute inertia field of (K,Q). If we could split everything up by intermediate fields, then we could reduce to extensions like the above $(L'|K,v)$, and extensions within completions; this would solve our problem. But the necessary intermediate fields may only exist after enlarging the extension $L|K$. Nevertheless, in [K6] I prove a weak form of relative uniformization for finite extensions within the absolute ramification field.

Let us see what we get and what we do not get from the above positive results.

16 Local uniformization for Abhyankar places

Using the transitivity of relative uniformization, we can combine Proposition 15.3 with Corollary 15.8 to obtain:

Theorem 16.1 *Take an Abhyankar place of $F|K$ such that (F,P) has rank 1. If $FP = K$ or $FP|K$ is a rational function field, then $(F|K,P)$ is uniformizable.*

This works since $(F|K,P)$ is henselian generated. But as soon as $FP|K$ is not a rational function field, $(F|K,P)$ will not be henselian generated. Then we may run into the problems described in the last section. For the time being, our only chance is to accept to extend the function field F. Then we can prove ([K6], [K7]):

Theorem 16.2 *Assume that P is an Abhyankar place of $F|K$ and take $\zeta_1,\ldots,\zeta_m \in \mathcal{O}_F$. Then there is a finite purely inseparable extension $\mathcal{K}|K$, a finite separable extension $\mathcal{F}|F.\mathcal{K}$, and an extension of P from F to $\mathcal{F}$ such that $(\mathcal{F}|\mathcal{K},P)$ is uniformizable with respect to $\zeta_1,\ldots,\zeta_m$. In addition, we have:*
1) If (F,P) has rank 1, then $\mathcal{F}$ can be obtained from $F.\mathcal{K}$ by a Galois extension.
2) If (F,P) has rank $r > 1$, then $\mathcal{F}$ can be obtained from $F.\mathcal{K}$ by a sequence of at most $r-1$ Galois extensions if $FP|K$ is algebraic, or at most r Galois extensions otherwise.
3) Alternatively, $\mathcal{F}$ can always be chosen to lie in the henselization of $(F.\mathcal{K},P)$.
4) If $FP|K$ is separable, then in all cases we can choose $\mathcal{K} = K$.

Unfortunately, "Galois extension" and "lie in the henselization" are mutually incompatible. Indeed, the normal hull of a subextension of the henselization will in general not again lie in the henselization.

We could prove local uniformization for all Abhyankar places of $F|K$ with separable $FP|K$, if we would know a positive answer to the following problem:

Open Problem 5: Take any Abhyankar place of $F|K$. Is it possible to choose the generating transcendence basis T for the inertial generation of $(F|K,P)$ in such a way that the extension $(F|K(T),P)$ is uniformizable?

Here is an even more daring idea. Since we have seen that we have problems with the henselization, why don't we try to avoid it?

Open Problem 6: Take any Abhyankar place of $F|K$. Is it possible to choose the generating transcendence basis T for the inertial generation of $(F|K,P)$ in such a way that the extension of P from $K(T)$ to F is unique?

In that case, general valuation theory shows that the extension $F|K(T)$ is linearly disjoint from the extension $K(T)^h|K(T)$. If T has that property, we would say that T **has the uniqueness property for** $F|K$ (this is a generalization of the definition given in Section 14). But I warn you: this problem seems to be very hard. It is already non-trivial to prove the existence of elements with the uniqueness property in case of transcendence degree 1. For arbitrary transcendence degree, the necessary algebraic geometry is not in sight. But perhaps there is some connection with local uniformization or resolution of singularities?

A weak form of local uniformization, without extending the function field, can be proved for all Abhyankar places in the case of perfect ground fields ([K6]). Also T. Urabe [U], building on Abhyankar's original approach, has a comparable result for the special case of places of maximal rank.

17 Non-Abhyankar places and the Henselian Rationality of immediate function fields

What can we do if the place P of $F|K$ is *not* an Abhyankar place? Still, the place may be nice. Assume for instance that $v_P F$ is finitely generated and $FP = K$. Then we can choose $x_1,\ldots,x_\rho$ such that

$v_P = \mathbb{Z}v_P x_1 \oplus \ldots \oplus \mathbb{Z}v_P x_\rho$, and set $F_0 := K(x_1,\ldots,x_\rho)$. Consequently, $(F|F_0, v_P)$ is an immediate extension. If P is not an Abhyankar place, then this extension is not algebraic. But we have already stated two tools to treat transcendental immediate extensions, namely Proposition 15.6 and Proposition 15.7.

Let us first apply Proposition 15.7. If (F, v_P) is a separable extension within the completion of (F_0, v_P), then by this proposition, the transitivity and Proposition 15.3, we find that $(F|K,P)$ is uniformizable. Do we need the assumption that the extension be separable? The answer is: this is automatically true. Indeed, by the Generalized Stability Theorem 14.1, (F_0, v_P) is a defectless field; thus, our assertion follows from the following lemma:

Lemma 17.1 *If (L,v) is a defectless field and $(L'|L,v)$ is an immediate extension, then $L'|L$ is separable.*

Proof: We have to show that $L'|L$ is linearly disjoint from every purely inseparable finite extension $E|L$. As the extension of the valuation from L to E is unique by Corollary 5.2, this implies that $[E:L] = (vE:vL)[Ev:Lv]$. Now we consider the compositum $L'.E$ (with the unique extension of the valuation from L' to the purely inseparable extension $L'.E$). Since $vE \subseteq v(L'.E)$, $vL' = vL$, $Ev \subseteq (L'.E)v$ and $L'v = Lv$, we have that $(v(L'.E):vL') \geq (vE:vL)$ and $[(L'.E)v:L'v] \geq [Ev:Lv]$. Hence,

$$[L'.E:L'] \;\geq\; (v(L'.E):vL')[(L'.E)v:L'v] \;\geq\; (vE:vL)[Ev:Lv] \;=\; [E:L] \;\geq\; [L'.E:L']\,.$$

Thus, equality holds everywhere, showing that $L'|L$ is linearly disjoint from $E|L$. □

In view of this result, Proposition 15.7 and the transitivity prove:

Theorem 17.2 *The assertions of Theorem 16.1 and Theorem 16.2 remain true if (F,P) is replaced by a finitely generated extension within its completion.*

If P is a discrete rational place of $F|K$, then we only have to choose $t \in F$ such that $v_P t$ is the smallest positive element in $v_P F \simeq \mathbb{Z}$; this will imply that $v_P F = \mathbb{Z}v_P t$. The immediate extension (F,P) of $(K(t),P)$ will automatically lie in the completion of $(K(t),P)$ (since the completion is of the form $K((t))$ and hence maximal, and one can easily show that the completion is unique up to isomorphism). So we obtain:

Theorem 17.3 *Every discrete rational place is uniformizable.*

Now let us turn to the general case. Not every immediate extension lies in the completion, not even in rank 1.

Example 17.4 Take the valuation v on the rational function field $K(x_1,x_2)$ such that $vK(x_1,x_2) = \mathbb{Z} + r\mathbb{Z}$ with $r \in \mathbb{R}\setminus\mathbb{Q}$. We can view $K(x_1,x_2)$ as a subfield of the power series field $K((\mathbb{Z}+r\mathbb{Z}))$, with $x_1 = t$ and $x_2 = t^r$. In $\mathbb{Z}+r\mathbb{Z}$, we can choose a monotonically increasing sequence r_i converging to 0 from below. Then we take the element $z = \sum_{i=1}^{\infty} t^{r_i} \in K((\mathbb{Z}+r\mathbb{Z}))$. It does not lie in the completion of $(K(x_1,x_2),v)$. Nevertheless, the extension $(K(x_1,x_2,z)|K(x_1,x_2),v)$ is immediate. I leave it to you to show that z is transcendental over $K(x_1,x_2)$. ◇

So let us look at the case of an immediate extension $(F_0(z)|F_0,P)$ which does not lie in the completion of (F_0,P). Then to apply Proposition 15.6, we need to know that z satisfies (53). By Lemma 15.5, this would hold if (F_0,P) were separable-algebraically maximal. But as the rational function field F_0 is not henselian unless P is trivial, it will admit its henselization as a proper immediate separable-algebraic extension. The only way out at this point is to pass to the immediate extension $(F_0^h(z)|F_0^h,P)$. There, we can apply Proposition 15.6 since by the Generalized Stability Theorem, (F_0^h,P) is a defectless field and thus is algebraically maximal. But now we have extended our function field F. (In the end, we will only need a finite subextension since the statement of local uniformization only talks about finitely many elements.) We will return to this aspect below.

Beforehand, let us think about two further problems. First, if our extension $F|F_0$ is of transcendence degree > 1, how do we carry on by induction? As the Generalized Stability Theorem does not apply any more in this situation, we do not know whether $(F_0(z), P)$ is a defectless field (and in fact, in general it will not be). Then we have to enlarge $F_0(z)$ even more to achieve the next induction step; in positive characteristic, the henselization will be too small for this purpose.

Second, if for instance $F|F_0(z)$ is a proper algebraic extension, then does it have relative uniformization? The only answers we have apply to the case where (F, P) lies in the henselization of $(F_0(z), P)$ (and even in this case our discussion has shown a bunch of problems). If (F, P) does not lie in the henselization, then we know nothing. Observe that since the extension $(F|F_0(z), P)$ is finite and immediate, (F, P) does not lie in the henselization if and only if $(F^h|F_0(z)^h, P)$ has non-trivial defect (by Theorem 5.16 and Corollary 8.3).

So the question arises: how can we avoid the defect in the case of immediate extensions? The answer is a theorem that I proved in [K1] (cf. [K2]). As for the Generalized Stability Theorem, the proof uses ramification theory and the deduction of normal forms for Artin-Schreier extensions. It also uses significantly a theory of immediate extensions which builds on Kaplansky's paper [KA1].

Theorem 17.5 (Henselian Rationality of Immediate Function Fields) *Let (K, P) be a tame field and $(F|K, P)$ an immediate function field of transcendence degree 1. Then*

$$\text{there is } x \in F \text{ such that } (F^h, P) = (K(x)^h, P)\,, \tag{54}$$

that is, $(F|K, P)$ is henselian generated. The same holds over a separably tame field (K, P) if in addition $F|K$ is separable.

Since the assertion says that F^h is equal to the henselization of a rational function field, we also call F **henselian rational**. For valued fields of residue characteristic 0, the assertion is a direct consequence of the fact that every such field is defectless. Indeed, take any $x \in F \setminus K$. Then $K(x)|K$ cannot be algebraic since otherwise, $(K(x)|K, P)$ would be a proper immediate algebraic extension of the tame field (K, P), a contradiction to Lemma 11.8. Hence, $F|K(x)$ is algebraic and immediate. Therefore, $(F^h|K(x)^h, P)$ is algebraic and immediate too. But since it cannot have a non-trivial defect, it must be trivial. This proves that $(F, P) \subset (K(x)^h, P)$. In contrast to this, in the case of positive residue characteristic only a very carefully chosen $x \in F \setminus K$ will do the job.

To illustrate the use of Theorem 17.5 in the model theory of valued fields, we give an example which is "complementary" to Example 14.3, treating the case of immediate extensions:

Example 17.6 Suppose that (K, v) is a tame field and that $(L|K, v)$ is an immediate extension. Then the conditions $vK \prec_\exists vL$ and $Kv \prec_\exists Lv$ are trivially satisfied. So do we have that $(K, v) \prec_\exists (L, v)$? Using the theory of tame fields, in particular the crucial Lemma 11.10, one reduces the proof to the case of transcendence degree 1. So we have to prove an embedding lemma for immediate function fields $(F|K, v)$ of transcendence degree 1 over tame fields. If we take any $x \in F \setminus K$ then it will satisfy (53) because (K, v) is tame and thus also algebraically maximal. Then with the help of Theorem 2 of [KA1] and the saturation of $(K, v)^*$, we can find an embedding of $(K(x), v)$ in $(K, v)^*$. But how do we carry on? We know that $(F|K(x), v)$ is immediate, but this does not mean that (F, v) lies in the henselization of $(K(x), v)$. But if it does, we can just use the universal property of the henselization (Theorem 5.15). Indeed, being a tame field, (K, v) is henselian. Since $(K, v)^*$ is an elementary extension of (K, v) (in the language of valued fields), it is also henselian. Hence if $(K(x), v)$ embeds in $(K, v)^*$, then this embedding can be extended to an embedding of $(K(x)^h, v)$ in $(K, v)^*$. This induces the desired embedding of F.

If F does not lie in the henselization of $(K(x), v)$, then $(F^h|K(x)^h, v)$ has non-trivial defect, and we have no clue how the embedding of $K(x)^h$ could be extended to an embedding of F^h. So as usual, our enemy is the defect, and we have to avoid it. Now this can be done by Theorem 17.5. It tells us that there is some $x \in F$ such that (F, v) lies in the henselization of $(K(x), v)$. So we have proved:

Theorem 17.7 *Suppose that (K, v) is a tame field and that $(L|K, v)$ is an immediate extension. Then $(K, v) \prec_\exists (L, v)$.*

Given an extension of tame fields of finite transcendence degree, then by use of Lemma 11.10, one can separate it into an extension without transcendence defect and an immediate extension. Both can be treated separately by Theorem 14.4 and Theorem 17.7. As $\prec_\exists$ is transitive, this proves (cf. [K1], [K2]):

Theorem 17.8 *The Ax–Kochen–Ershov principle (51) holds for every extension $(L|K,v)$ of tame fields.* $\diamondsuit$

Let us return to our problem of local uniformization. So far, we have worked with the assumption that we can find a subfunction field F_0 in F such that the restriction of P to F_0 is an Abhyankar place and $(F|F_0,P)$ is immediate. But it is not always possible to achieve the latter. For example, take F to be the rational function field $K(x_1,x_2)$ and P such that $FP=K$ and v_PF is a not finitely generated subgroup of $\mathbb{Q}$; we will construct such a place P in the next section. But for any F_0 on which P is an Abhyankar place, v_PF_0 is finitely generated, so we will always have that $v_PF\neq v_PF_0$.

In this situation, passing to henselizations may help again. Given an arbitrary place P of $F|K$, we choose an Abhyankar subfunction field as in (32). We have that v_PF/v_PF_0 is a torsion group and that $FP|F_0P$ is algebraic. Take F_1 to be the relative algebraic closure of F_0 in F^h. If $\operatorname{char} FP=\operatorname{char} K=0$, then by Lemma 4.7 and Lemma 4.9, $(F^h|F_1,P)$ is an immediate extension; so we succeeded again in reducing to the case of immediate extensions. But if $\operatorname{char} FP=\operatorname{char} K=p>0$, then we only know that v_PF^h/v_PF_1 is a p-group and that $F^hP|F_1P$ is purely inseparable. So in this case, passing to henselizations is not enough. But we obtain an immediate extension if we replace F^h and F_1 by their perfect hulls. In fact, to make all of our tools work, we have to take even bigger extensions. Namely, we have to pass to smallest algebraic extensions which are tame or at least separably tame fields. But these extensions still have nice properties; we will talk about them in Section 23.

With this approach, one can deduce the following theorems from the Generalized Stability Theorem, the Henselian Rationality of Immediate Function Fields, the results described in Section 15, and the transitivity of relative uniformization:

Theorem 17.9 *Let $F|K$ be a function field of arbitrary characteristic and P a place of $F|K$. Take any elements $\zeta_1,\ldots,\zeta_m\in\mathcal{O}_P$. Then there exist a finite extension $\mathcal{F}$ of F, an extension of P to $\mathcal{F}$, and a finite purely inseparable extension $\mathcal{K}$ of K within $\mathcal{F}$ such that $(\mathcal{F}|\mathcal{K},P)$ is uniformizable with respect to $\zeta_1,\ldots,\zeta_m$.*

Theorem 17.10 *The extension $\mathcal{F}|F$ can always be chosen to be normal.*

See [K6] for the proof. These theorems also follow from the results of Johan de Jong ([dJ]; cf. also [AO], [OO]). So this should be an interesting question:

Open Problem 7: Is it possible to recognize counterparts of the Generalized Stability Theorem and the Henselian Rationality of Immediate Function Fields in the theory of semi-stable reduction, or in any other part of de Jong's proof of desingularization by alteration?

The advantage of proving the above theorems by the described valuation theoretical approach is that we get additional information about the extension $\mathcal{F}|F$. We have already seen in this and the previous section that in certain cases we do not need an extension, i.e., we have local uniformization already for $(F|K,P)$. In other cases, we can obtain $\mathcal{F}$ from F by one or a tower of Galois extensions. We will see in Section 23 that in general, we can choose $\mathcal{F}|F.\mathcal{K}$ to be a separable (but not Galois) extension with additional information about the related extensions of value group and residue field.

A little bit of horror makes an excursion even more interesting. So let's watch out for bad places.

18 Bad places

In this section we will show that there are places of function fields $F|K$ whose value group or residue field are not finitely generated. By combining the methods you can construct examples where both is the case. The following two examples can already be found in [ZS], Chapter VI, §15. But our approach (using Hensel's Lemma) is somewhat easier and more conceptual.

Example 18.1 We construct a place on the rational function field $K(x_1,x_2)|K$ whose value group $G\subset\mathbb{Q}$ is not finitely generated, assuming that the order of every element in $G/\mathbb{Z}$ is prime to char K. To this end, we just find a suitable embedding of $K(x_1,x_2)$ in $K((G))$. We do this by setting $S:=\{n\in\mathbb{N}\mid 1/n\in G\}$ and

$$x_1 := t \quad\text{and}\quad x_2 := \sum_{n\in S} t^{-1/n} . \tag{55}$$

Further, take the valuation v on $K(x_1,x_2)$ to be the restriction of the canonical valuation v of $K((G))$. We wish to show that $1/S\subseteq vK(x_1,x_2)$, so that $G\subset vK(x_1,x_2)$. Since $(K(x_1,x_2),v)\subset(K((G)),v)$, it follows that $G=vK(x_1,x_2)$. If x_2 were algebraic over $K(x_1)$, we would know by Corollary 7.3 that $vK(x_1,x_2)$ is finitely generated. Hence if it is not, then x_2 must be transcendental over $K(x_1)$, so that $K(x_1,x_2)$ is indeed the rational function field over K in two variables.

Suppose that char $K=0$; then we can get $G=\mathbb{Q}$. Also in positive characteristic one can define the valuation in such a way that the value group becomes $\mathbb{Q}$; since then we have to deal with inseparability, our construction has to be refined slightly, which we will not do here.

Now let us prove our assertion. We take (L,v) to be the henselization of $(K(x_1,x_2),v)$. We are going to show that $t^{1/n}\in L$ for all $n\in S$. Suppose we have shown this for all $n<k$, where $k\in S$ (we can assume that $k>1$). Then also $s_k:=\sum\limits_{n\in S,n<k} t^{-1/n}\in L$. We write

$$x_2-s_k \;=\; \sum_{n\in S,n\geq k} t^{-1/n} \;=\; t^{-1/k}(1+c) \tag{56}$$

where $c\in K((G))$ with $vc>0$. Hence, $1+c$ is a 1-unit. We have that $(1+c)^k=t(x_2-s_k)^k\in L$. On the other hand, $(1+c)^k v=((1+c)v)^k=1^k=1$, which shows that $(1+c)^k$ is again a 1-unit. Since $k\in S$ we know that char $LP=$ char K does not divide k. Hence by Lemma 4.8, $1+c\in L$. This proves that $t^{1/k}=(1+c)(x_2-s_k)^{-1}\in L$.

We have now proved that $t^{1/k}\in L$ for all $k\in S$. Hence, $1/k=vt^{1/k}\in vL$ for all $k\in S$. But since the henselization is an immediate extension, we know that $vL=vK(x_1,x_2)$, so we have proved that $1/S\subset vK(x_1,x_2)$. ♢

Example 18.2 We take a field K for which the separable-algebraic closure $K^{\rm sep}$ is an infinite extension (i.e., K is neither separable-algebraically closed nor real closed). We construct a place of the rational function field $K(x_1,x_2)|K$ whose residue field is not finitely generated. We choose a sequence a_n, $n\in\mathbb{N}$ of elements which are separable-algebraic over K of degree at least n. We define an embedding of $K(x_1,x_2)$ in $K^{\rm sep}((t))$ by setting

$$x_1 := t \quad\text{and}\quad x_2 := \sum_{n\in\mathbb{N}} a_n t^n . \tag{57}$$

Further, we take the valuation v on $K(x_1,x_2)$ to be the restriction of the valuation of $K^{\rm sep}((t))$. We wish to show that $a_n\in K(x_1,x_2)v$ for all $n\in\mathbb{N}$, so that $K(x_1,x_2)v|K$ cannot be finitely generated. If x_2 were algebraic over $K(x_1)$, we would know by Corollary 7.3 that $K(x_1,x_2)v|K$ is finitely generated. So if it is not, then x_2 must be transcendental over $K(x_1)$, so that $K(x_1,x_2)$ is indeed the rational function field over K in two variables. By a modification of the construction, one can also generate infinite inseparable extensions of K. If K is countable, one can generate every algebraic extension of K as a residue field of $K(x_1,x_2)$.

We take again (L,v) to be the henselization of $(K(x_1,x_2),v)$. We are going to show that $a_n\in L$ for all $n\in\mathbb{N}$. Suppose we have shown this for all $n<k$, where $k\in\mathbb{N}$. Then also $s_k:=\sum_{n=1}^{k-1}a_nt^n\in L$. We write

$$\frac{x_2-s_k}{t^k} \;=\; \frac{1}{t^k}\sum_{n=k}^{\infty} a_k t^k \;=\; a_k(1+c) \tag{58}$$

where $c\in K^{\rm sep}((t))$ with $vc>0$. Take $f\in K[X]$ to be the minimal polynomial of a_k over K and note that $f=fv$. Since $a_k\in K^{\rm sep}$, we know that a_k is a simple root of f. On the other hand, $a_k=a_k(1+c)v\in Lv$.

Hence by Hensel's Lemma (Simple Root Version) there is a root a of f in L such that $av = a_k$. As we may assume that the place associated with v is the identity on K, this will give us that $a = a_k$; so $a_k \in L$.

We have now proved that $a_n \in L$ for all $n \in \mathbb{N}$. Hence, $a_n \in Lv = K(x_1, x_2)v$ for all $n \in \mathbb{N}$. $\diamondsuit$

19 The role of the transcendence basis and the dimension

In our approach described in Section 17, we have obtained the subfunction field F_0 on which the restriction of P is an Abhyankar place by choosing the elements $x_1, \ldots, x_\rho, y_1, \ldots, y_\tau$. But then we have made no effort to improve our choice. With this "stiff" approach (which in fact gives additional information), one can prove Theorem 17.9, but it can be shown that in general one cannot get local uniformization without an extension of the function field. I want to show why not. The following example is particularly interesting since it is also a key example in the model theory of valued fields of positive characteristic (cf. Section 21).

Example 19.1 We denote by $\mathbb{F}_p$ the field with p elements. We consider the following function field of transcendence degree 3 over $\mathbb{F}_p$:

$$F = \mathbb{F}_p(x_1, x_2, y, z) \quad \text{with } z^p - z = x_1 - x_2 y^p \,. \tag{59}$$

Since $x_1 \in \mathbb{F}_p(x_2, y, z)$, F is a rational function field. However, in [K1] (cf. also [K2], [K4]) we have shown that there is a rational place P of $F|\mathbb{F}_p$ such that $v_P F = \mathbb{Z}v_P x_1 \times \mathbb{Z}v_P x_2$ (ordered lexicographically) and that the valued function field $(F|\mathbb{F}_p(x_1, x_2), P)$ is not henselian generated. It follows that F cannot lie in the henselization of $(\mathbb{F}_p(t_1, t_2, t_3), P)$ if $t_1, t_2 \in F$ are algebraic over $\mathbb{F}_p(x_1, x_2)$ (and hence lie in $\mathbb{F}_p(x_1, x_2)$ since this is relatively algebraically closed in F). Therefore, Theorem 6.4 shows that $(F|\mathbb{F}_p, P)$ admits no local uniformization with t_1, t_2 algebraic over $K(x_1, x_2)$. A function field having such a local uniformization must have degree at least p over F. And indeed, degree p suffices, as

$$(F(x_2^{1/p})|\mathbb{F}_p(x_1, x_2^{1/p}), P)$$

is a rational function field. (It is an interesting fact that there is also a Galois extension of $\mathbb{F}_p(x_1, x_2)$ of degree p such that the function field becomes henselian generated.) $\diamondsuit$

The proof that $(F|\mathbb{F}_p(x_1, x_2), P)$ is not henselian generated is based on showing that $(\mathbb{F}_p(x_1, x_2), P)$ is not existentially closed in (F, P). I have not found an algebraic proof.

Open Problem 8: Develop a method to prove algebraically that a given valued function field $(F|K, v)$ (v not necessarily trivial on K) is *not* henselian generated or inertially generated.

A variant of the example (cf. [K7]) shows: *There are immediate transcendental extensions of valued fields which are not uniformizable.* The example teaches us that relative uniformization will not always hold without an extension of the function field. Hence, in general we will have to optimize our choice of the transcendence basis for F_0 or even for F in order to obtain local uniformization for $(F|K, P)$. Given a transcendence basis T of F, it is easy to measure how far F is from lying in the absolute inertia field $K(T)^i$ of $(K(T), P)$: we just have to take the degree

$$\mathrm{ig}(F, T) := [F.K(T)^i : K(T)^i] \,. \tag{60}$$

This raises the problem:

Open Problem 9: Develop a method to change T in such a way that $\mathrm{ig}(F, T)$ decreases.

In our above example, this is very easy since $F|K$ is actually a rational function field. But one can modify the example in such a way that $F|K$ is not rational. Instead of doing this for the above example, let us look at a slightly simpler example, which will also show that already a valued rational function field can have an immediate extension of degree p with defect p.

Example 19.2 We take an arbitrary field K of characteristic $p > 0$ and work in the power series field $K((\frac{1}{p^\infty}\mathbb{Z}))$ with its canonical valuation v. Recall that $\frac{1}{p^\infty}\mathbb{Z}$ is the p-divisible hull $\{m/p^n \mid m \in \mathbb{Z},\, n \in \mathbb{N}\}$ of $\mathbb{Z}$. We have that $K((t)) \subset K((\frac{1}{p^\infty}\mathbb{Z}))$. For every $i \in \mathbb{N}$, we set $\nu_i := \sum_{j=1}^{i} j$, and we define:

$$z := \sum_{i=1}^{\infty} t^{p^{\nu_i} - p^{-\nu_i}} \in K\left(\left(\frac{1}{p^\infty}\mathbb{Z}\right)\right) . \tag{61}$$

We show that $(K((\frac{1}{p^\infty}\mathbb{Z})), v)$ is an immediate extension of $(K(t,z), v)$. Since both fields have residue field K, we only have to show that $\frac{1}{p^\infty}\mathbb{Z} \subseteq vK(t,z)$. For $k \in \mathbb{N}$, we compute:

$$\begin{aligned} z^{p^{\nu_k}} - \sum_{i=1}^{k} t^{p^{\nu_k+\nu_i} - p^{\nu_k-\nu_i}} &= \sum_{i=1}^{\infty} \left(t^{p^{\nu_i} - p^{-\nu_i}}\right)^{p^{\nu_k}} - \sum_{i=1}^{k} \left(t^{p^{\nu_i} - p^{-\nu_i}}\right)^{p^{\nu_k}} \\ &= \sum_{i=k+1}^{\infty} t^{p^{\nu_k+\nu_i} - p^{\nu_k-\nu_i}} = t^{p^{\nu_k+\nu_{k+1}} - p^{\nu_k-\nu_{k+1}}} + \ldots \end{aligned}$$

So

$$vt^{p^{\nu_k+\nu_{k+1}}} \left(z^{p^{\nu_k}} - \sum_{i=1}^{k} t^{p^{\nu_k+\nu_i} - p^{\nu_k-\nu_i}} \right)^{-1} = p^{\nu_k - \nu_{k+1}} = \frac{1}{p^{k+1}} . \tag{62}$$

As the element on the left hand side is in $K(t,z)$, this shows that $p^{-k}\mathbb{Z} \subset vK(t,z)$ for every $k \in \mathbb{N}$. Consequently, $\frac{1}{p^\infty}\mathbb{Z} \subseteq vK(t,z)$, as desired. This also proves that z is transcendental over $K(t)$ since otherwise, $(vK(t,z) : \mathbb{Z})$ would be finite.

From Section 8 we know that $\vartheta = \sum_{i\in\mathbb{N}} t^{-1/p^i} \in K((\mathbb{Q}))$ is a root of the Artin–Schreier polynomial $X^p - X - 1/t$. We see that this power series already lies in the subfield $K((\frac{1}{p^\infty}\mathbb{Z}))$ of $K((\mathbb{Q}))$. Hence, $(K(t,z,\vartheta)|K(t,z),v)$ is a subextension of $(K((\frac{1}{p^\infty}\mathbb{Z}))|K(t,z),v)$ and thus, it is immediate too. In order to show that it has non-trivial defect, we have to show that it has a unique extension of the valuation, or equivalently, that it is linearly disjoint from the henselization of $(K(t,z),v)$. Since it is a Galois extension of prime degree, it suffices to show that it does not lie in this henselization.

We take the subfield $K(t^{1/p^k} \mid k \in \mathbb{N})$ of $K((\frac{1}{p^\infty}\mathbb{Z}))$. By definition, z lies in the completion of $(K(t^{1/p^k} \mid k \in \mathbb{N}), v)$ (since the values of the summands form a sequence which is cofinal in the value group $\frac{1}{p^\infty}\mathbb{Z}$). Since this value group is archimedean, that is, v has rank 1, we know that the henselization of $(K(t,z),v)$ lies in the completion of $(K(t,z),v)$, which by what we have just shown lies in the completion of $(K(t^{1/p^k} \mid k \in \mathbb{N}), v)$. On the other hand, we have seen in Section 8 that ϑ does not lie in the completion of $(K(t^{1/p^k} \mid k \in \mathbb{N}), v)$. Hence, it does not lie in the henselization of $(K(t,z),v)$.

We have now shown that the function field $K(t,z,\vartheta)|K$ admits a place with a value group which is not finitely generated and such that the extension $(K(t,z,\vartheta)|K(t,z),v)$ is immediate of degree p and defect p. Now you will point out that our function field $K(t,z,\vartheta)$ is again rational: since $1/t = \vartheta^p - \vartheta$, we have that $K(t,z,\vartheta) = K(z,\vartheta)$. So let's change something. We take a polynomial $f(t) \in K[t]$ and note that $vf(t) \geq 0$. Now we replace ϑ by a root ϑ_f of the polynomial $X^p - X - (\frac{1}{t} + f(t))$. It can be shown that the new extension $(K(t,z,\vartheta_f)|K(t,z),v)$ will again be immediate of degree p and defect p. In fact, this is obvious if we choose f without constant term. In that case, $vf(t) > 0$ and we know from Example 4.4 that the polynomial $X^p - X - f(t)$ has a root in $K(t,z)^h$. By the additivity of $X^p - X$ it follows that the two polynomials $X^p - X - \frac{1}{t}$ and $X^p - X - (\frac{1}{t} + f(t))$ define the same extension of degree p and defect p over $K(t,z)^h$. Consequently, also $(K(t,z,\vartheta_f)|K(t,z),v)$ must be immediate of degree p and defect p.

Now we have that

$$\vartheta_f^p - \vartheta_f = \frac{1}{t} + f(t) = \frac{1 + tf(t)}{t} . \tag{63}$$

So the minimal polynomial of t over $K(z,\vartheta_f)$ will be

$$Xf(X) - (\vartheta_f^p - \vartheta_f)X + 1 . \tag{64}$$

The transition from the representation $F = K(t, z, \vartheta_f)$ with ϑ_f algebraic over $K(t, z)$ to the representation $F = K(z, \vartheta_f, t)$ with t algebraic over $K(z, \vartheta_f)$ may be called **Artin–Schreier inversion**. With a suitable choice of f, the function field $F = K(t, z, \vartheta_f)|K$ will not be rational. However, whatever choice of f I computed, I found that after a little Artin–Schreier surgery on $X^p - X - (\frac{1}{t} + f(t))$ (which replaces f by a better polynomial), Artin–Schreier inversion will yield a tame extension $(K(z, \vartheta_f, t)|K(z, \vartheta_f), v)$. So at least we got rid of the defect, probably even of the ramification. After all, this is what we expect since we know from Abhyankar's work (cf. [A10]) that $(F|K, P)$ always admits local uniformization for trdeg $F|K$ up to 3 (with the possible exception of characteristic 2, 3, 5). $\diamondsuit$

Getting rid of defect and ramification by Artin–Schreier inversion seems to be the algebraic kernel of local uniformization and, in particular, of Abhyankar's proofs. However, the following questions should be answered without a restriction of the dimension (i.e., the transcendence degree of $F|K$):

Open Problem 10: Prove (or disprove) that by Artin–Schreier inversion in connection with Artin–Schreier surgery one can always get rid of the defect. How about ramification?

The only case where I know that the answer is positive is the case of Theorem 17.5, the Henselian Rationality of Immediate Function Fields. There, it is the crucial part of the proof. There seems to be no reason why the answer to the above problem should depend on the transcendence degree of $F|K$ or on the particular value of the positive characteristic. Actually, what I am saying is not quite true since the proof of Theorem 17.5 so far only works under a strong assumption about the base field, and a generalization may more easily be achieved if the restriction of P to that base field is an Abhyankar place. That might indicate that there is more hope for function fields of transcendence defect at most 1 than for those with higher transcendence defect. In dimension 2 (trdeg $F|K = 2$), every place P of $F|K$ will have transcendence defect at most 1 since there is always a subfunction field of transcendence degree at least 1 on which the restriction of P is an Abhyankar place. This seems to separate the case of dimension ≤ 2 from the case of dimension ≥ 3. But as Abhyankar was able to tackle dimension 3 (where the transcendence defect may well be 2), there seems to be no reason why all this shouldn't work for even higher dimensions.

By looking at these crucial Artin–Schreier extensions, we have considered the kernel of the problem. But are we really sure that we can always reduce to this kernel (for instance, by passing to ramification fields as described in Section 14)?

Open Problem 11: Is it always (in all dimensions) possible to pull down local uniformization through tame extensions? That is, if $(\mathcal{F}|K, P)$ is uniformizable where $(\mathcal{F}^h|F^h, P)$ is a tame extension, will then also $(F|K, P)$ be uniformizable? Which additional assumptions do we possibly need? What answers can be extracted from Abhyankar's work? Is there some generalization of his "Going Up" and "Coming Down" techniques to all dimensions?

Again, there is no hint why the dimension should have an influence on this problem. Possibly it can be found in Abhyankar's work.

If we look at our examples, we see that bad places and defect extensions in the generation of a function field already appear in dimension 2, so from this point of view, the dividing line seems to lie between dimension 1 and dimension 2. Our consideration concerning the transcendence defect seems to suggest a dividing line between dimension 2 and dimension 3. Also Example 19.1 goes in this direction, although it is not clear what it actually means for local uniformization and whether there possibly is an analogue of transcendence degree 2.

There is, however, another point which we have not yet mentioned. If our place P has rank > 1, can we then always proceed by induction on the rank? If we are ready to extend our function field F, then the answer is: yes ([K5], [K6], [K7]). But what if we want to prove local uniformization without extension of F and we have $P = Q\overline{Q}$ such that $FQ|K$ is not finitely generated, hence not a function field? In this situation, Q consumes already transcendence degree 2, and if we assume that also $\overline{Q}$ is non-trivial, then we have trdeg $F|K \geq 3$. Analogously, if we find that critical things happen in dimension 3 but not in dimension 2, these things might only develop their destructive influence in connection with composition of places, which would lift the critical dimension up to 4. (But to be true, I do not believe that this could happen. I believe, if we have local uniformization in dimension 3, then ultimately there will be a proof which works for dimension 3 in the same way as for all higher dimensions.)

20 The space of all places of $F|K$

The set $S(F|K)$ of all places of $F|K$ (where equivalent places are identified) is called the **Zariski–Riemann manifold** or just the **Zariski space** of $F|K$. See [V] for the definition of the Zariski topology on $S(F|K)$, its compactness and other properties of this space. Here, we will consider yet another property. We have seen in Section 18 that the Zariski space even of very simple function fields can contain bad places. On the other hand, we have seen that there are good places (e.g., Abhyankar places) for which local uniformization is easier than in the general case. But do good places or Abhyankar places exist in every Zariski space?

An ad hoc method to prove that this is true is to construct places of maximal rank. Take a transcendence basis $t_1, \ldots, t_k$ of $F|K$ and set $K_{k+1} := K$ and $K_i := K(t_i, \ldots, t_k)$ for $1 \leq i < k$. Take P_1 to be any place of $F|K_2$ such that $t_1 P_1 = 0$. By Corollary 7.3, $FP_1|K_2$ is finite; hence, $FP_1|K_3$ is a function field. So we can choose a place P_2 of $FP_1|K_3$ such that $t_2 P_2 = 0$. Again, $FP_1 P_2|K_3$ is finite. By induction, we construct places (in fact, prime divisors) $P_1, \ldots, P_k$. Their composition $P = P_1 \ldots P_k$ is a place of $F|K$ of rank $k = \operatorname{trdeg} F|K$. Hence, P is a place of maximal rank and thus an Abhyankar place of $F|K$.

We wish to give a much more sophisticated method which shows that good places not only exist, but even are "very representative" in every Zariski space. The general result reads as follows (cf. [K3]; the special case of $\operatorname{char} K = 0$ was proved in [KP]):

Theorem 20.1 *Let $F|K$ be a function field in k variables. Let Q be a place of $F|K$ and $a_1, \ldots, a_m$, $b_1, \ldots, b_n \in F$. Then there exists a place P of $F|K$ such that:*

1) $v_P F$ is a finitely generated group and extends the subgroup of $v_Q F$ generated by $v_Q b_1, \ldots, v_Q b_n$,

2) FP is finitely generated over K and extends $K(a_1 Q, \ldots, a_m Q)$,

3) the following holds:

$$\begin{array}{rcll} a_i P &=& a_i Q & \textit{for} \quad 1 \leq i \leq m\,, \\ v_P b_j &=& v_Q b_j & \textit{for} \quad 1 \leq j \leq n\,. \end{array}$$

Moreover, P can be chosen such that $v_P F$ is a subgroup of the p-divisible hull $\frac{1}{p^\infty} v_Q F$ of $v_Q F$ if $\operatorname{char} K = p > 0$, and $v_P F \subseteq v_Q F$ otherwise, and that FP is a subfield of the perfect hull of FQ. Alternatively, if $r, d \in \mathbb{N} \cup \{0\}$ satisfy

$$\dim Q \leq d \leq k - \operatorname{rr} Q\,, \qquad \operatorname{rr} Q \leq r \leq k - d\,, \tag{65}$$

then P may be chosen such that $\dim P = d$ and $\operatorname{rr} P = r$. (If $r = k - d$, then P is an Abhyankar place of $F|K$.)

The theorem tells us that for every place Q of $F|K$ and every choice of finitely many elements in F there is an Abhyankar place P which agrees with Q on these elements.

For the proof of the above theorem, see [K3]. The main ideas are the following. First, we choose an Abhyankar subfunction field F_0 of $(F|K, Q)$ as in (32), where we replace P by Q. Using the model theory of tame fields, we "pull the situation down" into a finite extension F_1 of F_0. This is done as follows.

By our choice of F_0, $v_Q F / v_Q F_0$ is a torsion group and $FQ|F_0 Q$ is algebraic. Now we take (L, Q) to be a maximal purely wild extension of the henselization of (F, Q); then by Lemma 11.4, (L, Q) is a tame field. Further, we take L' to be the relative algebraic closure of F_0 in L. Then by Lemma 11.10, (L', Q) is a tame field and $(L|L', Q)$ is an immediate extension. Hence, there are elements $a''_1, \ldots, a''_m$ and $b''_1, \ldots, b''_n$ in L' whose values or residues coincide with those of $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$. Now we choose generators $t_1, \ldots, t_k, z$ for the function field $F.L'|L'$, like for $F|K$ in Example 12.6. The elements a_i, b_i are rational functions in these generators. Similarly as in Example 12.6, we want to pull down these generators to L', preserving the values and residues of a_i, b_i. So the existential $\mathcal{L}_{\mathrm{VF}}(L')$-sentence we employ will now contain also the information that $v_Q a_i = v_Q a_i''$ and $b_i Q = b_i'' Q$ (note that a_i'', b_i'' are constants from L'). Since $(L', v_Q) \prec_\exists (L, v_Q)$ by Theorem 17.8 (because $v_Q K \prec_\exists v_Q L$ and $Kv_Q \prec_\exists Lv_Q$ trivially hold), we know that the existential sentence also holds in (L', v_Q). This gives us the elements $c_1, \ldots, c_k, d \in L'$ and thus also the new elements a_i', b_i' as rational functions in these new elements, satisfying that $v_Q a_i' = v_Q a_i'' = v_Q a_i$ and $b_i' Q = b_i'' Q = b_i Q$.

The elements $c_1, \ldots, c_k, d$ generate a finite extension (F_1, Q) of (F_0, Q) inside of (L', Q). This extension will be responsible for the extension of the value group and the residue field. But as it lies in L, we can employ Theorem 11.2 to show that $v_Q F_1 / v_Q F_0$ is a p-group and $F_1 Q | F_0 Q$ is purely inseparable, which yields the corresponding assertion in Theorem 20.1.

Adjoining enough transcendental elements and extending Q in a suitable way, we build up a function field (F_2, P) having dimension d and rational rank r and such that P is an Abhyankar place of $F_2|K$. Finally, using the Implicit Function Theorem, we embed F over K in the completion of (F_2, P) and pull back P through this embedding. We construct the embedding in such a way that the image of every a_i and every b_j is very close to a_i' and b_j', respectively. This implies that $a_i P = a_i' P = a_i' Q = a_i Q$ and $v_P b_j = v_P b_j' = v_Q b_j' = v_Q b_j$ (recall that by construction, P and Q coincide on the field F_1 which contains the elements $a_1', \ldots, a_m', b_1', \ldots, b_n'$).

In [K3], I prove several modifications of Theorem 20.1, which have various applications. Let me give an example.

Example 20.2 A modification of Theorem 20.1 (cf. [K3]) shows that one can replace Q by a discrete place P such that FP is a subfield of the perfect hull of FQ (again, one can preserve finitely many residues, but not values anymore). Hence if K is perfect and Q is a rational place, then also P will be rational. As (F, P) is discrete, F embeds over K in $K((t))$. If we assume that K is large, then by Theorem 12.9, $K \prec_\exists K((t))$. Since every existential elementary sentence holding in F will also hold in the bigger field $K((t))$, it follows that $K \prec_\exists F$. We have proved:

Theorem 20.3 *Assume that K is a large field. Assume further that K is perfect and that $F|K$ admits a rational place Q. Then K is existentially closed in F (in the language of fields).*

$\diamondsuit$

Reviewing the results on local uniformization that we have stated so far, we see that the best results can be obtained for zero-dimensional discrete or zero-dimensional Abhyankar places (and if K is assumed to be algebraically closed, then "zero-dimensional" is the same as "rational"). But the above theorem only renders places P with $\dim P \geq \dim Q$, so starting from a place which is not zero-dimensional, we will again get a place which is not zero-dimensional. This can be overcome by a modification like the one in the last example. For the formulation of the results we shall use the **Zariski patch topology** for which the basic open sets are the sets of the form

$$\{P \in S(F|K) \mid a_1 P \neq 0\,, \ldots,\, a_k P \neq 0\,;\, b_1 P = 0\,, \ldots,\, b_\ell P = 0\} \tag{66}$$

with $k, \ell \in \mathbb{N} \cup \{0\}$ and $a_1, \ldots, a_k, b_1, \ldots, b_\ell \in F \setminus \{0\}$. It is finer than the Zariski topology. But some proofs showing that the Zariski topology is compact actually show first that the Zariski patch topology is compact (cf. [SP]). Also, the compactness of the Zariski patch topology and the Zariski topology can easily be derived from the Compactness Theorem of model theory (Theorem 13.2); see [K2], [K3].

Theorem 20.4 *The following places lie dense in $S(F|K)$ with respect to the Zariski patch topology:*

a) the zero-dimensional rank 1 Abhyankar places,
b) the zero-dimensional places of maximal rank,
c) the zero-dimensional discrete places,
d) the prime divisors.

This can be proved by a combination of Theorem 20.1 and Lemma 12.7 (cf. [K3]). For the proof, one does not need the model theory of tame fields; an application of Theorem 13.7 will suffice. From Theorem 20.4 together with Theorem 17.3 or Theorem 16.1, we obtain:

Corollary 20.5 *If K is algebraically closed, then the uniformizable places P lie dense in $S(F|K)$ with respect to the Zariski patch topology.*

This result immediately generates the following questions:

Open Problem 12: If we have proved local uniformization for a set of places which lies dense in $S(F|K)$ with respect to the Zariski patch topology, can we patch the local solutions together to obtain the global resolution of singularities? If this doesn't work, how about finer topologies? What other properties of $S(F|K)$ can be deduced from dense subsets?

Certainly, it doesn't follow directly from Corollary 20.5 that all places in $S(F|K)$ are uniformizable. But it would follow if the next open problem had a positive answer. Observe that local uniformization is an open property, that is, if P is uniformizable, then there is an open neighborhood of P in which every place admits (the same) local uniformization.

Open Problem 13: Can we define something like a "radius" of these "local uniformization neighborhoods" and show that there is a lower bound for this radius?

After all, being a finitely generated field extension, a function field only contains "finite algebraic information". On the other hand, it should be clear from the examples of bad places that there are infinitely many ways of being bad... So it would be nice if we could forget about bad places. However, the badness expresses itself already on a transcendence basis, and the lower bound for the radius might only depend on the algebraic extension above that transcendence basis.

Now let's have a look at the main open problem of the model theory of valued fields. It has been around since the work of Ax and Kochen, and several excellent model theorists have tried their luck on it, in vain.

21 $\mathbb{F}_p((t))$

May I introduce to you my dearest friend and scariest enemy: $\mathbb{F}_p((t))$. Recall that it appeared on the right hand side of (47). On the left hand side, there were the fields $\mathbb{Q}_p$ of p-adic numbers. In a second paper [AK2], Ax and Kochen gave a nice ("recursive") complete axiom system for $\mathbb{Q}_p$ with its p-adic valuation. This generated the problem to give a nice complete axiom system also for $\mathbb{F}_p((t))$ with its t-adic valuation. One can always give an axiom system by writing down *all* sentences which hold in a structure. But "writing down" is very optimistic: there are infinitely many such sentences, and we may not even have a procedure to generate them in some algorithmic way. In contrast to this, a finite axiom system causes no problem. Also schemes like we use for "algebraically closed" or "henselian" aren't problematic since increasingly large finite subsets of them can be produced by an algorithm. This is what "recursive" means. Now if we have a complete recursive axiom system then there is also an algorithm to decide whether a given elementary sentence holds in every model of that axiom system. This is what one means when asking the famous question:

Open Problem 14: Is the elementary theory of $(\mathbb{F}_p((t)), v_t)$ decidable? In other words, does $(\mathbb{F}_p((t)), v_t)$ admit a complete recursive axiomatization?

The complete recursive axiomatization for $(\mathbb{Q}_p, v_p)$ is not hard to state. It is essentially the following:
1) (K, v) is a valued field,
2) (K, v) is henselian,
3) $\operatorname{char} K = 0$,
4) $Kv = \mathbb{F}_p$,
5) vK is an ordered abelian group which is elementarily equivalent to $\mathbb{Z}$.

We can write $Kv = \mathbb{F}_p$ since every field which is elementarily equivalent to $\mathbb{F}_p$ is already equal to $\mathbb{F}_p$ (because $\mathbb{F}_p$ has finitely many elements, and their number can thus be expressed by an elementary sentence). In contrast to this, we cannot write $vK = \mathbb{Z}$; since $\mathbb{Z}$ has infinitely many elements, Theorem 13.3

implies that there are many other ordered abelian groups which are elementarily equivalent to $\mathbb{Z}$. These are called **$\mathbb{Z}$-groups**. An ordered abelian group is a $\mathbb{Z}$-group if and only if $\mathbb{Z}$ is a convex subgroup of G and $G/\mathbb{Z}$ is divisible.

Observe that $(\mathbb{F}_p((t)), v_t)$ looks very much like $(\mathbb{Q}_p, v_p)$. Indeed, the only axiom that does not hold is axiom 3). So if we replace it by 3′): "$\mathrm{char}\, K = p$", will we get a complete axiom system (which by our above remarks would be recursive)? We have stated in Section 8 that every henselian discretely valued field of characteristic 0 is a defectless field. From this it follows that in the presence of the other axioms (including 3)!), axiom 2) implies that (K, v) is defectless. If we change 3) to "$\mathrm{char}\, K = p$", this is not any longer true, and we have to replace axiom 2) by 2′): "(K, v) is henselian and defectless".

For a long time, many model theorists believed that the axiom system 1), 2′), 3′), 4), 5) could be complete. But based on an observation by Lou van den Dries, I was able to show in [K1] that this is not the case (cf. [K2]). It is precisely Example 19.1 which proves the incompleteness. The point is that if K has positive characteristic, then we have non-linear additive polynomials which we can use to express additional elementary properties. In characteristic 0, the only additive polynomials are of the form cx, so there is nothing interesting about them. But in positive characteristic, the image $f(K)$ of an additive polynomial is a subgroup of the additive group of K. If $f_1, \ldots, f_n$ are additive polynomials, then one can consider the subgroup $f_1(K) + \ldots + f_n(K)$. For certain choices of the f's, these subgroups have nice elementary properties if K is elementarily equivalent to $\mathbb{F}_p((t))$. This implies that for certain choices, one can even show that $K = f_1(K) + \ldots + f_n(K)$. To some extent, this has the same flavour as Hensel's Lemma, but the incompleteness result shows that all this doesn't follow from Hensel's Lemma, or to be more precise, doesn't even follow from the axiom system 1), 2′), 3′), 4), 5). See [K4] for details.

The subgroups of the form $f_1(K) + \ldots + f_n(K)$ are definable by an elementary sentence using constants from K (as coefficients of the polynomials f_i). This fact leads to the following questions:

Open Problem 15: Does the axiom system 1), 2′), 3′), 4), 5) become complete if we add the elementary properties of the subgroups of the form $f_1(K) + \ldots + f_n(K)$? What other subgroups of $\mathbb{F}_p((t))$ are elementarily definable, and what are their elementary properties?

As we have seen already that additive polynomials play a crucial role for local uniformization in positive characteristic, it makes sense to ask:

Open Problem 16: What is the relation between the elementary properties of $\mathbb{F}_p((t))$ expressible by use of additive polynomials and algebraic geometry in positive characteristic?

On the valuation theoretical side, I can say that work in progress indicates that these elementary properties have a crucial meaning for the structure theory of valued function fields. For example, it seems that the Henselian Rationality of Immediate Function Fields can be generalized to the case of base fields (K, v) which are not tame but have these properties. By the way, these properties don't play a role in the model theory of tame fields because all tame fields are perfect (like all other fields of positive characteristic for which we know that Ax–Kochen–Ershov principles hold). In contrast to this, $\mathbb{F}_p((t))$ is not perfect since t has no p-th root in $\mathbb{F}_p((t))$.

In comparison to the model theory of tame fields, that of $\mathbb{F}_p((t))$ is much more complex since some tools available for tame fields will not work anymore. As an example, we do not have an analogue of the crucial Lemma 11.10 which we used to separate extensions of tame fields into extensions without transcendence defect and immediate extensions. Thus, we cannot do this (in general) in the case of fields which are elementarily equivalent to $\mathbb{F}_p((t))$. We are also not able to "slice" immediate extension into extensions of transcendence degree 1. But then we would have to develop an analogue of Kaplansky's theory of immediate extensions for the case of higher transcendence degree, or even worse, simultaneously for all mixed extensions without a possibility of separation. However, this is more or less the generalization of the theory of approximate roots that is recently discussed.

22 Local uniformization versus Ax–Kochen–Ershov

My first encounter with Zariski's Local Uniformization Theorem was when I studied the model theoretic proof of the p-adic Nullstellensatz by Moshe Jarden and Peter Roquette [JR]. At one point, they consider the following situation. They have an extension $L|K$ of p-adically closed fields, and a function field $F|K$ inside of $L|K$. Inside of F, they have a certain subring B containing K and an element $g \in B$ which is not a unit in B. Now they wish to show that there is a rational place P of $F|K$ such that $gP = 0$. They take a maximal ideal $\mathcal{M}$ in B such that $g \in \mathcal{M}$. By the existence theorem for places (see [V], Proposition 1.2), there is a place Q of $F|K$ such that $B \subseteq \mathcal{O}_Q$ and $\mathcal{M}_Q \cap B = \mathcal{M}$. It follows that $gQ = 0$. Since $K \subset B$ we know that $K \subseteq FQ$. But Q may not be the required place since it may not be rational. If $FQ|K$ is a function field, then one can proceed as follows. By the special choice of the ring B one knows that FQ is contained in some p-adically closed extension field L'. By the work of Ax–Kochen and Ershov, one knows that $K \prec L'$. It follows that $K \prec_\exists FQ$, hence by Lemma 12.7 there is a rational place $\overline{Q}$ of $FQ|K$. So the place $Q\overline{Q}$ is a rational place of $F|K$ which satisfies $gQ\overline{Q} = 0\overline{Q} = 0$, as desired.

But it may well happen that $FQ|K$ is not finitely generated. Then we can't apply Lemma 12.7. In this situation, Jarden and Roquette use Zariski's Local Uniformization (Theorem 15.1) to show that there exists a place P of $F|K$ such that $gP = 0$ and that $FP|K$ is finitely generated. More generally, they show:

Lemma 22.1 *Take a function field $F|K$ of characteristic 0, a place Q of $F|K$ and elements $y_1, \ldots, y_n \in \mathcal{O}_Q$. Then there exists a place P of $F|K$ such that $y_iP = y_iQ$, $1 \leq i \leq n$, $FP \subseteq FQ$ and $FP|K$ is finitely generated.*

The proof works as follows. After adding elements if necessary, we can assume that $F = K(y_1, \ldots, y_n)$. By Zariski's Local Uniformization Theorem, after adding further elements we can assume that $a = (y_1Q, \ldots, y_nQ)$ is a simple point of the K-variety whose generic point is $(y_1, \ldots, y_n)$. Hence, the local ring $\mathcal{O}_a$ is regular. Now Jarden and Roquette employ the following lemma from [A2]:

Lemma 22.2 *Suppose that R is a regular local ring with maximal ideal M and quotient field F. Then there exists a place P dominating R such that $FP = R/M$.*

(In fact, P is the place associated with the order valuation deduced from (R, M).)

Corollary 22.3 *Suppose that a is a simple point of V. Then there exists a place P of $F|K$ such that $a = (y_1P, \ldots, y_nP)$ and $FP = K(a)$.*

Proof: The residue field $\mathcal{O}_a/\mathcal{M}_a$ of $\mathcal{O}_a$ is isomorphic to $K(a)$. We identify both fields, so that the residue map $\mathcal{O}_a \to \mathcal{O}_a/\mathcal{M}_a = K(a)$ maps every y_i to a_i. By applying the foregoing lemma to $R = \mathcal{O}_a$, we obtain a place P of $F|K$ dominating $\mathcal{O}_a$ and such that $FP = \mathcal{O}_a/\mathcal{M}_a = K(a)$. Since P dominates $\mathcal{O}_a$, it extends the residue map, whence $a = (y_1P, \ldots, y_nP)$. □

This proves Lemma 22.1 since from the definition of a it follows that $(y_1P, \ldots, y_nP) = (y_1Q, \ldots, y_nQ)$ and $FP = K(y_1P, \ldots, y_nP) = K(y_1Q, \ldots, y_nQ) \subset FQ$, showing also that FP is a finitely generated extension of K.

Note that if $a \in K^n$ then the place P obtained from the foregoing corollary is rational. Hence we have (you may compare this with Lemma 12.7):

Corollary 22.4 *Suppose that a is a simple K-rational point of V. Then there exists a rational place P of $F|K$ such that $a = (y_1P, \ldots, y_nP)$.*

In my Masters Thesis, I showed how to avoid the use of Zariski's Local Uniformization Theorem by constructing a place of maximal rank (which by Corollary 7.3 always has a finitely generated residue field). This trick was then used by Alexander Prestel and Peter Roquette in their book [PR] for the proof of the p-adic Nullstellensatz. It also provided the first idea for the paper [PK] in which we proved a version of Theorem 20.1 for function fields of characteristic 0 by using the Ax–Kochen–Ershov Theorem. This version has interesting applications to real algebra and real algebraic geometry ([PK], [P]). Surprisingly, a paper by Ludwig Bröcker and Heinz-Werner Schülting [BS] derives about the same results and applications, using resolution of singularities in characteristic 0 (Hironaka) in the place of the Ax–Kochen–Ershov Theorem. See also the survey paper [SCH]. The first question I have in this connection is:

Open Problem 17: To which extent is it (easily) possible to replace the use of resolution of singularities by local uniformization in applications to real algebraic geometry?

Seeing that the Ax–Kochen–Ershov Theorem and Zariski's Local Uniformization Theorem can be used to deduce the same results, Roquette asked:

Open Problem 18: What is the relation between Ax–Kochen–Ershov Theorem and Zariski's Local Uniformization Theorem? Can one prove one from the other?

If that were true, then there would be some hope that a progress in positive characteristic made on one side could be transferred to the other side. For instance, as already mentioned, local uniformization or resolution of singularities in positive characteristic could possibly help to solve problems in the model theory of valued fields of positive characteristic.

In view of the details I have told you about, my own preliminary answer is that the relation between local uniformization and the model theory of valued fields lies in the facts and theorems from the structure theory of valued function fields which play a crucial role in both problems. Therefore, new insights in this structure theory will also be of importance for both problems.

On the other hand, there are ingredients on either side which do not appear on the other. For example, Lemma 15.4 does not seem to play any role on the model theoretic side. Further, the henselization causes a lot of serious problems for the local uniformization of places of rank > 1, whereas it is the best friend of model theorists. The need to optimize the choice of the transcendence basis (cf. Section 19) to avoid these problems and to avoid the defect has (so far) no analogue on the model theoretic side; however, this may change with a deeper insight in the theory of $\mathbb{F}_p((t))$. Conversely, model theory is forced to deal with extensions $(L|K,v)$ of valued fields where v is non-trivial on K. To some extent, we did the same when we considered relative uniformization. But in contrast to that situation, the valuations on K in the model theoretic case may be of arbitrary rank and arbitrarily nasty.

Our discussion would not be complete if we would not mention the following nice result, due to Jan Denef [DEN]. It says that "the existential theory of $\mathbb{F}_p((t))$ is decidable":

Theorem 22.5 *If resolution of singularities holds in positive characteristic, then there is an algorithm to decide whether a given existential elementary sentence in the language of valued fields holds in* $(\mathbb{F}_p((t)), v_t)$.

23 Back to local uniformization in positive characteristic

In the last section, we have seen that for certain applications it matters to know what the residue fields of our places are. In particular, when changing a bad place to a good place, we might wish to keep the residue field within a certain class of fields. For example, this could be the class of all fields in which the base field K is existentially closed. (Note that if K is perfect and existentially closed in L, then it will also be existentially closed in any purely inseparable algebraic extension of L.) If we take into the bargain an extension of the function field in order to obtain local uniformization, but require that this extension should be normal or even Galois, then we may not be able anymore to control the corresponding extension of the residue field. So it makes sense to ask for the minimal possible change of the residue field.

The key to this question is the fact that there are minimal algebraic extensions of (F,P) which are tame (or separably tame) fields; we just have to pass to the henselization of (F,P) and then choose a field W according to Theorem 11.2, cf. Lemma 11.4. (For the case of "tame", see also Corollary 11.9.) Working inside of such an extension, we can nicely apply the theory of tame and separably tame fields. In particular, we can use Lemma 11.10 and the transitivity of relative uniformization to reduce to extensions of the types discussed in **I)–IV)** in Section 15.

On the other hand, if (L,P) is such an extension of (F,P), then by Theorem 11.2, $v_P L/v_P F$ will be a p-torsion group if $\mathrm{char}\, K = p > 0$, and $LP|FP$ will be purely inseparable. If $\mathrm{char}\, K = 0$, then (L,P) is just the henselization of (F,P). As our extension $\mathcal{F}$ remains inside of L, we can show (cf. [K7]):

Theorem 23.1 *In addition to the assertion of Theorem 17.9, the finite extension $\mathcal{F}|F$ can be chosen to be separable and to satisfy:*

a) if $\mathrm{char}\, K = p > 0$, then the finite group $v_P\mathcal{F}/v_P F$ is a p-torsion group, and the finite extension $\mathcal{F}P|FP$ is purely inseparable,
b) if $\mathrm{char}\, K = 0$, then $\mathcal{F}$ can be chosen to lie in the henselization of F.

(Clearly, in the case of $\mathrm{char}\, K = 0$, our result is weaker than Zariski's Local Uniformization Theorem. However, it provides some more information since we can uniformize a finite extension of F within its henselization while keeping fixed the once chosen transcendence basis of the subfunction field $F_0|K$ on which P is an Abhyankar place.)

The following corollary illustrates the advantage of controlling the residue field extension. In fact, Theorem 20.3 can also be proved by use of this corollary.

Corollary 23.2 *Assume that K is perfect and that P is a rational place of $F|K$. Take $\zeta_1,\ldots,\zeta_m \in \mathcal{O}_F$. Then there is a finite extension $\mathcal{F}|F$ and an extension of P to $\mathcal{F}$ such that $(\mathcal{F}|K,P)$ is uniformizable with respect to $\zeta_1,\ldots,\zeta_m$ and P is still a rational place of $\mathcal{F}|K$.*

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